CREAT. MATH. INFORM.
Volume **31** (2022), No. 2,
Pages 267 - 271

Online version at https://semnul.com/creative-mathematics/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2022.02.12

Some results on star-line graphs

SEEMA VARGHESE

ABSTRACT. Let H be a connected graph with at least three vertices. The H-line graph, $HL(G)$, of a graph G has all the edges of G as its vertices, two vertices of $HL(G)$ are adjacent if the corresponding edges in G are adjacent and belong to a common copy of H. In this paper we investigate some properties of the star-line graph $K_{1,n}L(G)$ of a graph G. We also obtain a Krausz type characterization for star-line graphs. Traversability of star-line graphs is also studied.

1. INTRODUCTION

The notion of 'line graph' as a 'graph operator' was introduced by Krausz [11]. The line graph $L(G)$ of a graph G has all the edges (ie K_2 subgraphs) of G as its vertices and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. Since then, many other graph operators such as clique graph, total graph etc. and their dynamics were studied [14].

Jarrett [8] defined the triangular line graph $\triangle(G)$ of a nonempty graph G as that graph whose vertices are edges of G and two vertices of $\triangle(G)$ are adjacent if the corresponding edges belong to a common triangle of G . Triangular line graph is also referred as the anti-Gallai graph by some authors [12]. The triangular line graph was introduced to model a metric space defined on the edge set of a graph. These concepts were generalized in [2].

H-line graphs were introduced by Chartrand in [5], as a generalization of line graphs and also of triangular line graphs. For a connected graph H of order at least 3, the H-line graph of a graph is defined as that graph whose vertices are the edges of G and where two vertices are adjacent if and only if the corresponding edges of G are adjacent and belong to a common copy of H. In particular, when $H = P_3$, the H-line line graph $HL(G)$ is the standard line graph $L(G)$. The dynamics of iterated H-line graphs were studied by different authors in [4, 13, 10]. The behaviour of the sequence $\{HL^k(G)\}$ when $H = K_3$, $H = P_4, P_5$ or $K_{1,n}, n \geq 3$ and $H = C_4$ is analyzed in [8], [6], [4] and [3]. A sufficient condition for each component of $C_4L(G)$ to be Eulerian is obtained in [3]. The dynamics of iterated P_6-line graphs were studied in [10]. Limits of H-line graphs are studied and some results on $K_nL(G)$ are obtained in [1].

A Krausz-type characterization for star-line graphs is obtained in this paper. Traversability of star-line graphs is also studied. All the graphs considered here are undirected and simple. For all basic concepts and notations not mentioned in this paper we refer [16].

Received: 10.09.2021. In revised form: 24.01.2022. Accepted: 01.02.2022
2000 *Mathematics Subject Classification.* 05C.
Key words and phrases. *Star-line graphs, Krausz-type Characterization.*

2. Star-line graph

Definition 2.1. The star-line graph of H, denoted by $K_{1,n}L(H)$ has all the edges of H as its vertices, two vertices of $K_{1,n}L(H)$ are adjacent if the corresponding edges in H are adjacent and belong to a common copy of $K_{1,n}$.

An example of $K_{1,5}L(H)$ is given in Figure: 1.

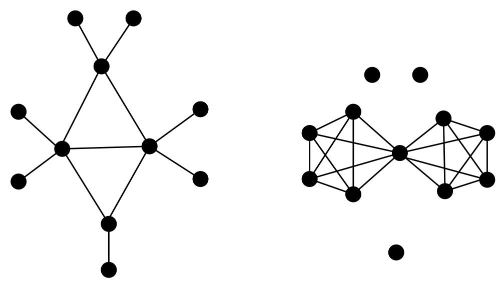

Figure 1. A graph H and its $K_{1,5}L(H)$

Definition 2.2. A graph H is a star-line graph, if there exists a graph H' such that $K_{1,n}L(H') \cong H$.

Remark 2.1. $K_{1,n}L(H)$ is a spanning subgraph of $L(H)$.

Remark 2.2. $K_{1,n}L(H)$ has an edge iff H has a copy of $K_{1,n}$.ie, $deg(v_i) \geq n$, for some i.

We now prove a necessary and sufficient condition for a $K_{1,n}L(H)$ to be connected. For this proof we need the following definition.

Definition 2.3. A graph H is said to be n-star connected if, for every x, y in H there is a path from x to y such that degree of every internal vertex is greater than or equal to n.

Theorem 2.1. *$K_{1,n}L(H)$ of a graph G is connected if and only if H is n-star connected.*

Proof. It is clear that if H is n-star connected then $K_{1,n}L(H)$ is connected. Conversely if $K_{1,n}L(H)$ is connected, then any two vertices in it are connected by a path. But a vertex in $K_{1,n}L(H)$ corresponds to an edge in H and an edge in $K_{1,n}L(H)$ comes from a copy of $K_{1,n}$, which means that the degree of the internal vertices is greater than or equal to n.

□

3. Krausz-type characterization for star-line graphs.

It is known that a class of graphs $\mathcal{C}$ has forbidden subgraph characterization if and only if $\mathcal{C}$ has induced hereditary property [9]. Using this, it is proved in [15], that H-line graphs admit a forbidden subgraph characterization if and only if $H = P_3$, ie, when $HL(G) = L(G)$. As the class of graphs $K_{1,n}L(G)$ does not admit a forbidden subgraph characterization, it is natural to look for a Krausz-type characterization for star-line graphs.

In this section, we view the line graph as $K_{1,2}L(G)$ and we try to generalize this theme. We will observe some nice properties of the star-line graphs, $K_{1,n}L(G)$. A Krausz-type characterization is obtained for star-line graphs. The line graphs have the following characterization:

Theorem 3.2. [7] *A graph H is a line graph if and only if its edges can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of the subgraphs.*

Analogous to this characterization, we have the following for star-line graphs. Here we have an additional condition that the order of the cliques in the partition is bounded below by n.

Theorem 3.3. *A graph H is a star-line graph, $K_{1,n}L(H'), n \geq 3$ if and only if its edges can be partitioned into complete subgraphs of order at least n in such a way that no vertex lies in more than two of the subgraphs.*

Proof. Suppose H is a star-line graph, say $H \cong K_{1,n}L(H'), n \geq 3$. Corresponding to every edge of H', there is a vertex in H. Let $v \in H'$ be such that $deg(v) \geq n$. The edges incident to v will form a clique of order at least n in H. Let it be denoted by C_v. Then, $\mathcal{E} = \{C_v/v \in H'$ and $deg(v) \geq n\}$ will form a clique cover of edges of H, since every edge of H lies in a copy of $L(K_{1,n}) \cong K_n$. Take any vertex u in H. It corresponds to an edge $u = u_1u_2$ in H'. u will belong to C_{u_i} if $deg(u_i) \geq n$ for $i = 1, 2$. Also u cannot belong to any other clique of $\mathcal{E}$. Thus every vertex of H is in at most two members of $\mathcal{E}$. An edge of H corresponds to two adjacent vertices of H' which lie in a $K_{1,n}$. Clearly, two adjacent vertices H' cannot lie in two different copies of $K_{1,n}$. Hence, every edge of H is in exactly one member of $\mathcal{E}$.

Conversely, Suppose H has an edge clique partition $\mathcal{E}$ satisfying the condition of the theorem. Take the intersection graph $I(\mathcal{E})$. Corresponding to every vertex of H, which belong to exactly one clique C of $\mathcal{E}$, draw a pendant vertex to the vertex corresponding C in $I(\mathcal{E})$. Corresponding to every isolated vertex of H, draw an isolated edge. Let the new graph thus constructed be H'.
Now we will show that $K_{1,n}L(H') \cong H$. Define $\phi : V(H) \longrightarrow V(K_{1,n}L(H'))$ as follows:
If $v \in V(H)$ is such that $v \in C_i \cap C_j$, then C_i and C_j are adjacent in $I(\mathcal{E})$ and define $\phi(v)$ to be the edge in H' joining C_i and C_j. If $v \in C_i$ only, then there will be a pendant vertex in H' corresponding to v and define $\phi(v)$ to be the pendant edge attached to C_i. If v is an isolated vertex in H, define $\phi(v)$ to be the isolated edge in H' corresponding to v.
It is clear that ϕ is well defined. Since $\mathcal{E}$ is a clique partition, $|C_i \cap C_j| \leq 1$. Hence ϕ is injective. Let $v \in V(K_{1,n}L(H'))$. Then $v = v_1v_2$ is an edge in H'. If it is a pendant edge, then it will correspond to some vertex of H which belongs to exactly one clique. If it is an isolated edge, then it will correspond to some isolated vertex in H. Otherwise it corresponds to an edge which is in two cliques. Therefore ϕ is surjective.
Let u and v be adjacent vertices in H. Then u and v belong to a clique C_i of the partition. Since every clique of the partition is of order at least n, there are vertices $w_1, w_2 \ldots w_{n-2}$ in C_i. The construction of H' is such that edges corresponding to these vertices $u, v, w_1, \ldots w_{n-2}$ will have a common vertex forming a $K_{1,n}$ in H'. Thus the edges corresponding to u and v are adjacent and lie in common copy of $K_{1,n}$ in H' and hence u and v are adjacent in $K_{1,n}L(H')$. Therefore, ϕ is an isomorphism.

□

Remark 3.3. Note that the graph constructed H' is such that $H = L(H')$, which shows that any $K_{1,n}L(H')$ is a line graph also. In fact, $K_{1,n}L(H')$ is a line graph in which every edge lie in a K_n.

4. TRAVERSABILITY OF STAR-LINE GRAPHS

In this section, we derive a necessary and sufficient condition for a star-line graph to be eulerian.

Theorem 4.4. *Let H' be a connected graph with $\delta(H') \geq n$. Then $K_{1,n}L(H')$ is eulerian if and only if $d(v), v \in H'$ are of same parity.*

Proof. Suppose H' is a connected graph with $\delta(H') \geq n$ and $H = K_{1,n}L(H')$ is eulerian. Then $d_H(u)$ is even for all $u \in H$. Also every vertex $u \in H$ belongs to at most 2 cliques of the edge partition of H described in theorem 3.3. We consider two cases.

Case 1: All vertices of H belong to exactly one clique. In this case, there would be only one clique in the partition. For otherwise H would be disconnected which is a contradiction. Also degree of every vertex in H' is the order of the only clique in the partition of H and hence all the vertices in H' are of same degree and thus of same parity.

Case 2: There is at least one vertex $u \in H$ which belong to two cliques, say C_i, C_j of the edge partition of H. Since two cliques in an edge partition of a graph cannot share an edge, we have $N_{C_i}(u) \cap N_{C_j}(u) = \phi$. This implies, $d_H(u) = d_{C_i}(u) + d_{C_j}(u)$.

Now, since the LHS is even, both the terms on the RHS are of same parity. This would mean that the order of the cliques are of the same parity which would further imply that all the vertices in H' are of same parity.

Conversely, suppose H' is a connected graph with $\delta(H') \geq n$ and $d(v), v \in H'$ are of same parity. Then, for any edge $e = v_1v_2$ in H', we have
$d_{H'}(v_1) + d_{H'}(v_2) =$ even. Hence the vertex in H which corresponds to e will have even degree which implies H is eulerian.

□

REFERENCES

[1] Britto-Pacumio, R. A. Limits of iterated H-line graphs. *Discrete Math.* **199** (1999) 1–10.
In this paper the length of the sequence $\{HL^k(G)\}$ when $H = K_n$ is determined and it is shown that $K_nL(G)$ is eulerian. A class of limits of such sequences for arbitrary H is characterized.
[2] Chartrand, G.; Gavlas, H.; Schultz, M. H-distance in graphs. *Congr. Numer.* **101** (1994), 3–15
[3] Chartrand, G. Henning, M. A.; Jarrett, E. B.; Wall, C. E. Quadrilateral line graphs. *Utilitas Mathematica* **50** (1996), 3–17.
[4] Chartrand, G.; Gavlas, H.; Schultz, M. Convergent sequences of iterated H-line graphs. *Discrete Math.* **147** (1995), 73–86.
In this paper, necessary conditions for $\{HL^k(G)\}$ to converge to a connected limit graph and sufficient conditions for the sequence $\{HL^k(G)\}$ to diverge are presented. Graphs G for which the sequence $\{HL^k(G)\}$ converges, when H is P_4 or P_5 are characterized.
[5] Chartrand, G. *Graphs and their associated line graphs*. Ph. D. Thesis, Michigan State University, (1964.
[6] Dorrough, D. Convergent sequences of iterated triangular line graphs. *Discrete Math.* **161** (1996) 79–86.
In this paper the number of iterations required for convergence and the number of disjoint copies of K_3s are obtained.
[7] Harary, F. Graph Theory *Addison-Wesley Publishing Company, Inc.* (1969).
[8] Jarrett, E. B. *On iterated triangular line graphs*, in: Y. Alavi, A. Schwenk (Eds), Proc. 7th Internat. Conf. in Graph Theory, Combinatorics, Algorithms and Applications, Wiley, New York, (1994).

[9] Greenwell, D. L.; Hemminger, R. L.; Klerlein, J. *Forbidden Subgraphs*, Proceedings of Fourth South-eastern Conference on Combinatorics, Graph Theory and Computing (1973).

[10] Kathiresan, K. M.; Devi, S. A. A study on H-line graphs. *Australian Journal of Combinatorics* **58** (2014), 358–364.
In this paper,those graphs G for which the sequence $\{HL^k(G)\}$ converges, when H is P_6 is characterized.

[11] Krausz, J. *Démonstration nouvelle d'un théorème de Whitney sur les résaux (Hungarian with French summary)*, Mat. Fiz Lapok, **50** (1943).

[12] Le, Van Bang Gallai and anti-Gallai graphs. *Discrete Math.* **159** (1996), 179–189.
In this paper it is shown that: The Four Color Theorem can be equivalently stated in terms of anti-Gallai graphs; the problems of determining the clique number, and the chromatic number of a Gallai graph are NP-complete. Furthermore, the relation of Gallai graphs to the theory of perfect graphs is discussed. A characterization of Gallai graphs and anti-Gallai graphs is also given.

[13] Manjula, K. Some results on H-line graphs. *Indian J. Pure Appl. Math.* **34** (2003), no. 9, 1307–1314.

[14] Prisner, E. *Graph dynamics*, Longman, (1995).

[15] Varghese, S. *On H-line graphs*, (communicated).

[16] West, D. B. *Introduction to Graph Theory*, PHI (2003).

DEPARTMENT OF MATHEMATICS
GOVT. ENGINEERING COLLEGE
APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY
THRISSUR-680581, INDIA
Email address: `seemavarghese@gectcr.ac.in`