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An algorithm for automorphisms of infinite dimensional Grassmann algebras

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ABSTRACT. Let G be the infinite dimensional Grassmann algebra. In this study, we determine a subgroup of the automorphism group $\mathrm{Aut}(G)$ of the algebra G which is of an importance in the description of the group $\mathrm{Aut}(G)$. We give an infinite generating set for this subgroup and suggest an algorithm which shows how to express each automorphism as compositions of generating elements.

1. INTRODUCTION

Let K be a field of characteristic zero and let A_m be the free unitary associative algebra of rank m generated by $f_1, \dots, f_m$. Then the m-generated Grassmann algebra G_m is defined as the factor algebra A_m/I_m such that I_m is the ideal of G_m generated by all elements of the form $f_i f_j + f_j f_i$, $1 \leq i, j \leq m$. We see that G_m is generated by $e_i = f_i + I_m$, $i = 1, \dots, m$. Clearly the Grassmann algebra G_m is of the canonical basis elements of the form

$$e_{i_1} \cdots e_{i_k},\ i_1 \leq \cdots \leq i_k,\ k = 1, \dots, m$$

and 1. Note that $e_i e_j = -e_j e_i$ for all $i, j = 1, \dots, m$, since $e_i^2 = 0$ as a consequence of characteristic of K. The algebra G_m satisfies the identity

$$[[x, y], z] = (xy - yx)z - z(xy - yx) = 0 \tag{1.1}$$

for all $x, y, z \in G_m$. In particular one has $\mathrm{ad}^2(x) = 0$ for $x \in G_m$.

The Grassmann algebra has become an important tool in many fields of mathematics as well as physics. One may see the book by Bourbaki [5] for a background. Working on the automorphism group of a given algebra has always become a remarkable approach in order to recognize and characterize the algebra. One of the works about the group of automorphisms of the Grassmann algebra is done by Berezin. Let U_m be the group of linear automorphisms and let B_m be the group of automorphisms of the form $T(e_p) = e_p + f_p(e_1, \cdots, e_m)$, where f_p does not have a linear component. Berezin [4] determined the group of automorphisms of G_m as the semidirect product of the subgroups B_m and U_m when K is the field of complex numbers. Djoković [6] showed that when $\mathrm{char}K \neq 2$, the group of automorphisms of G_m can be written as the semidirect product of the group of inner automorphisms of G_m and the subgroup of $\mathrm{Aut}(G_m)$ which preserves the $\mathbb{Z}_2$-grading of G_m. The description of the automorphism group $\mathrm{Aut}(G_m)$ of the Grassmann algebra G_m can be explicitly found in the literature (see e.g. Laszlo [9]).

Theorem 1.1. *The group* $\mathrm{Aut}(G_m)$ *of* K*-automorphisms of* G_m *is isomorphic to a semidirect product of those three subgroups.*

$$\mathrm{Aut}(G_m) = \mathrm{Inn}(G_m) \rtimes \mathrm{A}_v \rtimes \mathrm{Gl}_m(K)$$

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where

i) $\mathrm{Gl}_m(K)$ *is the group of automorphisms sending* e_i *to a linear combination of* $e_1, \ldots, e_m$ *such that the determinant of the coefficients is nonzero,*

ii) $\mathrm{Inn}(G_m)$ *is the group of inner automorphisms of* G_m*. Each automorphism in this class is of the form* $1 + \mathrm{ad}x$*, where* $\mathrm{ad}x(e_i) = [x, e_i] = xe_i - e_ix$*,*

iii) A_v *is the group of automorphisms sending* e_i *to* $e_i + v_i$ *such that* v_i *is a linear combination of monomials of odd length* ≥ 3*.*

Bavula [1] showed that the group of automorphisms of G_m can be written similarly when K is a commutative ring. In this study, following the results on the automorphism group of G_m and extending the idea of the finite dimensional Grassmann algebra to the infinite generated (or equivalently infinite dimensional) Grassmann algebra G over the field K of characteristic zero, we give a class of automorphisms of G, which consists of a subgroup H of the group $\mathrm{Aut}(G)$ of automorphisms of the Grassmann algebra G.

The group H corresponds to a subgroup of A_v in the third class of Theorem 1.1 in the setting of infinite generation, which can be considered as an important approach in the description of the group $\mathrm{Aut}(G)$ of K-automorphisms of the Grassmann algebra G. In this study, we suggest an algorithm which expresses each automorphism in H in terms of automorphisms defined in a certain set. This is also to show that this set provides an infinite list of generators for the group H.

2. Preliminaries

Let K be a field of characteristic zero. Let A stand for the free unital associative K-algebra generated by an infinite countable set $F = \{f_1, f_2, \ldots\}$. We define the quotient algebra

$$G = A/I$$

where I is the ideal of A generated by all elements of the form

$$f_if_j + f_jf_i,\ f_i, f_j \in F.$$

Then G is the infinite dimensional unitary Grassmann algebra generated by the set

$$E = \{e_j = f_j + I : j = 1, 2, \ldots\}$$

over the field K. For each $e_i \in E$, we have $e_i^2 = 0$ and G has the following canonical basis.

$$B = \{e_{i_1} \cdots e_{i_k} :\ k \geq 1,\ i_1 < \cdots < i_k\} \cup \{1\}.$$

The algebra G satisfies the identity $[[x, y], z] = 0$ for all $x, y, z \in G$. Hence, the Grassmann algebra G is a PI-algebra. Krakowski and Regev [8] showed that the T-ideal of the infinite dimensional unitary Grassmann algebra over a field of characteristic zero is generated by $[[x, y], z]$. It is still valid in the case of positive characteristic by Giambruno and Koshlukov [7]. The T-ideal of the infinite dimensional unitary Grassmann algebra over a finite field is completely defined by Bekh-Ochir and Rankin [2]; moreover, they [2, 3] describe the T-space of this algebra over a field of arbitrary characteristic .

Let $G^{(0)}$ be the subvector space of G consisting of linear combinations of monomials of even length and let $G^{(1)}$ be the subvector space containing the linear combinations of monomials of odd length. Then obviously $G^{(0)}$ is the center of G, and as a vector space

$$G = G^{(0)} \oplus G^{(1)}.$$

This gives also a $\mathbb{Z}_2$-grading of the vector space G. It is straightforward to show that if

$$f : E \longrightarrow G$$

is a function such that $f(e_i)f(e_j)+f(e_j)f(e_i)=0$ for all $e_i, e_j \in E$, then f can be uniquely extended to a homomorphism of the infinite dimensional Grassmann algebra G.

Now consider the augmentation ideal $\omega(G)$ of G consisting of elements $p(e_1, e_2, \dots) \in G$ such that $p(0,0,\dots)=0$; i.e, if $x \in \omega(G)$ then

$$x = \sum c_j y_j, \ \ y_j \in B\backslash\{1\}, \ \ c_j \in K,$$

such that only a finite number of c_j's are nonzero. Let ϕ be an automorphism of G. Then we have following observation: $\phi(1)=1$ and $\phi(e_i) \in \omega(G)$ for each $e_i \in E$. Since $[[x,y],z]=0$ is an identity for G, we naturally obtain by identity (1.1) that $\mathrm{ad}^2 x = 0$, $x \in G$, and the map defined as

$$(\exp(\mathrm{ad}x))(y) = (1+\mathrm{ad}x)(y) = \psi_x(y) = y + [x,y], \ \ y \in G$$

for a fixed $x \in G$, is an automorphism called *inner automorphism*. Note that $\psi_{x+y} = \psi_x\psi_y$, and the set

$$\mathrm{Inn}(G) = \{\psi_x : x \in G\}$$

is an abelian group called the inner automorphism group. Thus, the question on other automorphism classes arises naturally. Let x be an element in $G^{(1)} \cap \omega^3(G)$; i.e., the linear combination of the monomials of odd length at least 3, and let us define the map $f_x : E \longrightarrow G$ such that $f_x(e_i) = e_i + x$. Then

$$f_x(e_i)f_x(e_j) + f_x(e_j)f_x(e_i) = (e_ie_j + e_je_i) + (e_ix + xe_i) + (e_jx + xe_j) + 2x^2 = 0.$$

Hence f_x can be extended uniquely to a K-homomorphism $\phi_x : G \longrightarrow G$. In particular, the inverse of ϕ_x is ϕ_{-x}, when x is a monomial. In the sequel we give some technical lemmas which are to be utilized in the main results.

Lemma 2.1. *Let $x, y \in G^{(1)} \cap \omega^3(G)$. Then $\phi_x\phi_y = \phi_{x+\phi_x(y)}$.*

Proof. $\phi_x\phi_y(e_i) = \phi_x(e_i + y) = e_i + x + \phi_x(y) = \phi_{x+\phi_x(y)}(e_i)$. Note that $x + \phi_x(y) \in G^{(1)} \cap \omega^3(G)$. □

Now let us define some notations. Let $y = \beta e_{j_1} \cdots e_{j_{2n+1}}$ be a monomial in $G^{(1)} \cap \omega^3(G)$, $\beta \in K$. We define $y_{(i)} = (-1)^{i+1}\beta e_{j_1} \cdots e_{j_{i-1}} e_{j_{i+1}} \cdots e_{j_{2n+1}}$ for $i = 1, \dots, n$, and $\bar{y} = y_{(1)} + \cdots + y_{(2n+1)}$. As a consequence of this notation we have that $y\bar{y} = 0$ and $y_{(i)}\bar{y} = 0$ for $i = 1, \dots, n$. Additionally, by easy computations $\phi_x(\bar{y}) = \bar{y}$ for all $x \in G^{(1)} \cap \omega^3(G)$.

Lemma 2.2. *Let $x \in G^{(1)} \cap \omega^3(G)$ be an element and let $y \in G^{(1)} \cap \omega^3(G)$ be a monomial. Then*

$$\phi_x(y) = y + x\bar{y}.$$

Proof. Let $y = \beta e_{j_1} \cdots e_{j_{2n+1}}$. Using the fact that $x^n = 0$, $n \geq 2$, we have

$$\begin{aligned}
\phi_x(y) &= \beta(e_{j_1} + x)(e_{j_2} + x) \cdots (e_{j_{2n+1}} + x) \\
&= \beta e_{j_1} \cdots e_{j_{2n+1}} + x(\beta e_{j_2} e_{j_3} \cdots e_{j_{2n+1}} + \cdots + \beta e_{j_1} \cdots e_{j_{2n}}) \\
&= y + x(y_{(1)} + \cdots + y_{(2n+1)}) \\
&= y + x\bar{y}
\end{aligned}$$

□

The proof of following lemma is straightforward.

Lemma 2.3. *Let $x_1, \dots, x_k, x_{k+1} \in G^{(1)} \cap \omega^3(G)$ be monomials. If $\phi_{x_1+\cdots+x_k}$ is an automorphism, then*

$$\phi^{-1}_{x_1+\cdots+x_k}(x_{k+1}) = x_{k+1} - X\bar{x}_{k+1}$$

where

$$
\begin{aligned}
X =& x_1 + \cdots + x_k - x_1(\bar{x}_2 + \cdots + \bar{x}_k) - \cdots - x_k(\bar{x}_1 + \cdots + \bar{x}_{k-1}) \\
& x_1(\bar{x}_2\bar{x}_3 + \cdots + \bar{x}_k\bar{x}_{k-1}) + \cdots + x_k(\bar{x}_1\bar{x}_2 + \cdots + \bar{x}_{k-1}\bar{x}_{k-2}) \\
& \vdots \\
& + (-1)^{k-1}x_1(\bar{x}_2\bar{x}_3\cdots\bar{x}_k + \cdots + \bar{x}_k\bar{x}_{k-1}\cdots\bar{x}_2) + \cdots \\
& + (-1)^{k-1}x_k(\bar{x}_1\bar{x}_2\cdots\bar{x}_{k-1} + \cdots + \bar{x}_{k-1}\bar{x}_{k-2}\cdots\bar{x}_1).
\end{aligned}
$$

Let $\operatorname{supp}(x)$ be the set of generators appearing in the expression of a given monomial x. For instance $\operatorname{supp}(x) = \{e_{i_1}, \ldots, e_{i_k}\}$ if $x = \alpha e_{i_1}\cdots e_{i_k}$ for some $\alpha \in K$.

Lemma 2.4. *Let $x, y \in G^{(1)} \cap \omega^3(G)$ be monomials such that $|\operatorname{supp}(x) \cap \operatorname{supp}(y)| \geq 2$. Then $\phi_x(y) = y$, and thus $x\bar{y} = 0$.*

Proof. Let $y = \beta e_{j_1}\cdots e_{j_{2n+1}}$. Then

$$
\begin{aligned}
\phi_x(y) &= \beta(e_{j_1} + x)(e_{j_2} + x)\cdots(e_{j_{2n+1}} + x) \\
&= \beta e_{j_1}\cdots e_{j_{2n+1}} + \beta x(e_{j_2}\cdots e_{j_{2n+1}} - e_{j_1}e_{j_3}\cdots e_{j_{2n+1}} + \cdots + e_{j_1}\cdots e_{j_{2n}}) \\
&= y
\end{aligned}
$$

Hence $x\bar{y} = 0$ by Lemma 2.2. $\square$

As a consequence of Lemma 2.4 we obtain the following corollary.

Corollary 2.1. *Let $x_1, \ldots, x_n, y \in G^{(1)} \cap \omega^3(G)$ be monomials such that $|\operatorname{supp}(x_i) \cap \operatorname{supp}(y)| \geq 2$ for each $i = 1, \ldots, n$. Then $\phi_{x_1+\cdots+x_n}(y) = y$.*

Lemma 2.5. *Let $x_1, \ldots, x_n, y \in G^{(1)} \cap \omega^3(G)$ be monomials such that $|\operatorname{supp}(x_i) \cap \operatorname{supp}(y)| \geq 2$, for each $i = 1, \ldots, n$. If $\phi_{x_1+\cdots+x_n}$ is an automorphism, then $\phi^{-1}_{x_1+\cdots+x_n}(y) = y$.*

Proof. Let $x = x_1 + \cdots + x_n$. By Corollary 2.1 we have $\phi_x(y) = y$. Thus $\phi_x^{-1}(\phi_x(y)) = \phi_x^{-1}(y)$. Finally, $\phi_x^{-1}(y) = y$. $\square$

Lemma 2.6. *Let $x_1, \ldots, x_n \in G^{(1)} \cap \omega^3(G)$ be monomials where $n \geq 2$. Then*

$$\phi_{(-1)^{n-1}x_1\bar{x}_2\cdots\bar{x}_n}$$

is an automorphism. Furthermore,

$$\phi_{(-1)^{n-1}x_1\bar{x}_2\cdots\bar{x}_n} = \phi_{(-1)^{n-1}x_1(x_2)_{(1)}\bar{x}_3\cdots\bar{x}_n}\cdots\phi_{(-1)^{n-1}x_1(x_2)_{(2t+1)}\bar{x}_3\cdots\bar{x}_n}$$

where $\bar{x}_2 = (x_2)_{(1)} + \cdots + (x_2)_{(2t+1)}$.

Proof. We make induction on n. Let us check the statement of the lemma for $n = 2$. Let $x_2 = v$. Then

$$
\begin{aligned}
\phi_{-x_1\bar{x}_2} &= \phi_{-x_1\bar{v}} \\
&= \phi_{-x_1(v_{(1)}+\cdots+v_{(2t+1)})} \\
&= \phi_{-x_1v_{(1)}-\cdots-x_1v_{(2t+1)}} \\
&= \phi_{-x_1v_{(1)}+\phi_{-x_1v_{(1)}}\phi^{-1}_{-x_1v_{(1)}}(-x_1v_{(2)}-\cdots-x_1v_{(2t+1)})} \\
&= \phi_{-x_1v_{(1)}}\phi_{\phi^{-1}_{-x_1v_{(1)}}(-x_1v_{(2)}-\cdots-x_1v_{(2t+1)})} \\
&= \phi_{-x_1v_{(1)}}\phi_{\phi_{x_1v_{(1)}}(-x_1v_{(2)}-\cdots-x_1v_{(2t+1)})} \\
&= \phi_{-x_1v_{(1)}}\phi_{-\phi_{x_1v_{(1)}}(x_1v_{(2)})-\cdots-\phi_{x_1v_{(1)}}(x_1v_{(2t+1)})}.
\end{aligned}
$$

Since $|\mathrm{supp}(w_1 v_{(1)}) \cap \mathrm{supp}(w_1 v_i)| \geq 2$ for $i = 2, \ldots, 2t+1$, by Lemma 2.4 we have $\phi_{-x_1\bar{x}_2} = \phi_{-x_1 v_{(1)}} \phi_{-x_1 v_{(2)} - \cdots - x_1 v_{(2t+1)}}$. Similarly, by easy computations after $2t$-steps we get that $\phi_{-x_1\bar{x}_2} = \phi_{-x_1 v_{(1)}} \phi_{-x_1 v_{(2)}} \cdots \phi_{-x_1 v_{(2t+1)}}$.

We know that $-x_1 v_{(l)}$ is a monomial for $l = 1, \ldots, 2t+1$, and $\phi_{-x_1 v_l}$ is an automorphism. Therefore, $\phi_{-x_1\bar{x}_2}$ is an automorphism as being a composition of automorphisms.

Assume that the statement hold for $n = k$; i.e., $\phi_{(-1)^{k-1} x_1 \bar{x}_2 \cdots \bar{x}_k}$ be an automorphism and $\phi_{(-1)^{k-1} x_1 \bar{x}_2 \cdots \bar{x}_k} = \phi_{(-1)^{k-1} x_1 v_{(1)} \bar{x}_3 \cdots \bar{x}_k} \cdots \phi_{(-1)^{k-1} x_1 v_{(2t+1)} \bar{x}_3 \cdots \bar{x}_k}$ by substituting $x_2 = v$. Now let us check the statement of lemma for $n = k+1$. Let use the notation $\xi(i,k) = (-1)^k x_1 v_{(i)} \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}$, where $i = 1, \ldots, 2n+1$, $\bar{v} = v_{(1)} + \cdots + v_{(2t+1)}$.

$$\begin{aligned}
\phi_{(-1)^k x_1 \bar{x}_2 \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}} &= \phi_{(-1)^k x_1 \bar{v} \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}} \\
&= \phi_{(-1)^k x_1 (v_{(1)} + \cdots + v_{(2t+1)}) \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}} \\
&= \phi_{(-1)^k x_1 v_{(1)} \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1} + \cdots + (-1)^k x_1 v_{(2t+1)} \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}} \\
&= \phi_{\xi(1,k) + \cdots + \xi(2t+1,k)} \\
&= \phi_{\xi(1,k) + \phi_{\xi(1,k)} \phi^{-1}_{\xi(1,k)} \left(\xi(2,k) + \cdots + \xi(2t+1,k)\right)} \\
&= \phi_{\xi(1,k)} \phi_{\phi^{-1}_{\xi(1,k)} \left(\xi(2,k) + \cdots + \xi(2t+1,k)\right)}.
\end{aligned}$$

Making use of Lemma 2.5 we have that

$$\phi_{(-1)^k x_1 \bar{x}_2 \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}} = \phi_{\xi(1,k)} \phi_{\xi(2,k) + \cdots + \xi(2t+1,k)}.$$

Similarly, after $2t$-steps we have that

$$\begin{aligned}
\phi_{(-1)^k x_1 \bar{x}_2 \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}} &= \phi_{\xi(1,k)} \phi_{\xi(2,k)} \cdots \phi_{\xi(2t+1,k)} \\
&= \phi_{(-1)^k x_1 v_{(1)} \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}} \cdots \phi_{(-1)^k x_1 v_{(2t+1)} \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}}.
\end{aligned}$$

$\phi_{(-1)^k x_1 v_{(l)} \bar{x}_3 \cdots \bar{x}_k \bar{x}_{k+1}}$ is an automorphism for $l = 1, \ldots, 2t+1$, because of induction hypothesis, $\phi_{(-1)^k x_1 \bar{x}_2 \cdots \bar{x}_k \bar{x}_{k+1}}$ is an automorphism being a composition of several automorphisms. Thus, the induction statement holds for all $n \geq 2$. □

Lemma 2.7. *Let $x_1, \ldots, x_n, y \in G^{(1)} \cap \omega^3(G)$ be monomials. Then*

$$\phi_{(x_1 + \cdots + x_n)\bar{y}} = \phi_{x_1 \bar{y}} \cdots \phi_{x_n \bar{y}}.$$

Proof.

$$\phi_{(x_1 + \cdots + x_n)\bar{y}} = \phi_{x_1 \bar{y} + \cdots + x_n \bar{y}} = \phi_{x_1 \bar{y}} \phi_{\phi^{-1}_{x_1 y}(x_2 \bar{y} + \cdots + x_n \bar{y})}$$

By Lemma 2.5 we have $\phi_{(x_1 + \cdots + x_n)\bar{y}} = \phi_{x_1 \bar{y}} \phi_{x_2 \bar{y} + \cdots + x_n \bar{y}}$. Similarly, after $n-1$ steps we have that $\phi_{(x_1 + \cdots + x_n)\bar{y}} = \phi_{x_1 \bar{y}} \cdots \phi_{x_n \bar{y}}$. □

Lemma 2.8. *Let $x, y \in G^{(1)} \cap \omega^3(G)$ and let x, y be monomials. Then $\phi_{x+y} = \phi_x \phi_y \phi_{-x\bar{y}}$.*

Proof. By Lemma 2.1 and Lemma 2.2 we obtain the followings:

$$\begin{aligned}
\phi_{x+y} &= \phi_{x + \phi_x \phi_x^{-1}(y)} = \phi_x \phi_{\phi_x^{-1}(y)} = \phi_x \phi_{\phi_{-x}(y)} = \phi_x \phi_{y - x\bar{y}} = \phi_x \phi_{y + \phi_y \phi_y^{-1}(-x\bar{y})} \\
&= \phi_x \phi_y \phi_{\phi_y^{-1}(-x\bar{y})} = \phi_x \phi_y \phi_{\phi_{-y}(-x\bar{y})} = \phi_x \phi_y \phi_{-\phi_{-y}(x)\phi_{-y}(\bar{y})} = \phi_x \phi_y \phi_{-(x - y\bar{x})\bar{y}} \\
&= \phi_x \phi_y \phi_{-x\bar{y}}
\end{aligned}$$

□

3. Main results

Theorem 3.2. *Let $x_i, x_{j_1}, \ldots, x_{j_k} \in G^{(1)} \cap \omega^3(G)$ be monomials for $i = 1, \ldots, n$, $k = 1, \ldots, n-1$, $i \neq j_k$. The homomorphism $\phi_{x_1+\cdots+x_n}$ can be expressed as a composition of automorphisms of the form $\phi_{x_i}, \phi_{x_i \bar{x}_{j_1} \cdots \bar{x}_{j_k}}$.*

Proof. We make induction on n. The statement of the theorem is clear for $n = 2$ by Lemma 2.8:

$$\phi_{x_1+x_2} = \phi_{x_1}\phi_{x_2}\phi_{-x_1\bar{x}_2}.$$

Assume that the statement hold for $n = k$. Now let us check the statement for $n = k+1$.

$$\begin{aligned}\phi_{x_1+\cdots+x_k+x_{k+1}} &= \phi_{x_1+\cdots+x_k+\phi_{(x_1+\cdots+x_k)}\phi^{-1}_{x_1+\cdots+x_k}(x_{k+1})} \\ &= \phi_{x_1+\cdots+x_k}\phi_{\phi^{-1}_{x_1+\cdots+x_k}(x_{k+1})}\end{aligned}$$

By Lemma 2.3 we get that

$$\begin{aligned}\phi_{x_1+\cdots+x_k+x_{k+1}} &= \phi_{x_1+\cdots+x_k}\phi_{x_{k+1}-X\bar{x}_{k+1}} \\ &= \phi_{x_1+\cdots+x_k}\phi_{x_{k+1}}\phi_{-\phi^{-1}_{x_{k+1}}(X\bar{x}_{k+1})} \\ &= \phi_{x_1+\cdots+x_k}\phi_{x_{k+1}}\phi_{-\phi_{-x_{k+1}}(X\bar{x}_{k+1})}\end{aligned}$$

where X is the same as in Lemma 2.3, and note that

$$\phi_{-x_{k+1}}(X\bar{x}_{k+1}) = \phi_{-x_{k+1}}(X)\bar{x}_{k+1} = (X + x_{k+1}Y)\bar{x}_{k+1} = X\bar{x}_{k+1}$$

for some $Y \in G^{(0)}$. Hence $\phi_{x_1+\cdots+x_k+x_{k+1}} = \phi_{x_1+\cdots+x_k}\phi_{x_{k+1}}\phi_{-X\bar{x}_{k+1}}$.

By Lemma 2.3 and Lemma 2.7 taking the structure of X into account, $\phi_{-X\bar{x}_{k+1}}$ is a composition of automorphism appearing in the statement of Lemma 2.6 which completes the proof. □

Corollary 3.2. *Let $x \in G^{(1)} \cap \omega^3(G)$. Then*

$$\phi_x : e_i \longrightarrow e_i + x$$

is an automorphism.

Example 3.1. Let $x = e_1e_2e_3 + e_1e_4e_5$.

$$\begin{aligned}\phi_{e_1e_2e_3+e_1e_4e_5} &= \phi_{e_1e_2e_3+\phi_{e_1e_2e_3}\phi^{-1}_{e_1e_2e_3}(e_1e_4e_5)} = \phi_{e_1e_2e_3}\phi_{\phi^{-1}_{e_1e_2e_3}(e_1e_4e_5)} \\ &= \phi_{e_1e_2e_3}\phi_{\phi_{-e_1e_2e_3}(e_1e_4e_5)} = \phi_{e_1e_2e_3}\phi_{e_1e_4e_5-e_1e_2e_3(\overline{e_1e_4e_5})} \\ &= \phi_{e_1e_2e_3}\phi_{e_1e_4e_5+\phi_{e_1e_4e_5}\phi^{-1}_{e_1e_4e_5}(-e_1e_2e_3(\overline{e_1e_4e_5}))} \\ &= \phi_{e_1e_2e_3}\phi_{e_1e_4e_5}\phi_{\phi^{-1}_{e_1e_4e_5}(-e_1e_2e_3(\overline{e_1e_4e_5}))} \\ &= \phi_{e_1e_2e_3}\phi_{e_1e_4e_5}\phi_{\phi_{-e_1e_4e_5}(-e_1e_2e_3(\overline{e_1e_4e_5}))} \\ &= \phi_{e_1e_2e_3}\phi_{e_1e_4e_5}\phi_{\phi_{-e_1e_4e_5}(-e_1e_2e_3)\phi_{-e_1e_4e_5}(\overline{e_1e_4e_5}))} \\ &= \phi_{e_1e_2e_3}\phi_{e_1e_4e_5}\phi_{-(e_1e_2e_3-e_1e_4e_5(\overline{e_1e_2e_3}))(\overline{e_1e_4e_5})} \\ &= \phi_{e_1e_2e_3}\phi_{e_1e_4e_5}\phi_{-(e_1e_2e_3)\overline{e_1e_4e_5}} = \phi_{e_1e_2e_3}\phi_{e_1e_4e_5}\phi_{-e_1e_2e_3e_4e_5}\end{aligned}$$

$$\begin{aligned}
\phi_{e_1e_2e_3+e_1e_4e_5}(e_i) &= (\phi_{e_1e_2e_3}\phi_{e_1e_4e_5}\phi_{-e_1e_2e_3e_4e_5})(e_i) = \phi_{e_1e_2e_3}(\phi_{e_1e_4e_5}(\phi_{-e_1e_2e_3e_4e_5}(e_i)))\\
&= \phi_{e_1e_2e_3}(\phi_{e_1e_4e_5}(e_i - e_1e_2e_3e_4e_5))\\
&= \phi_{e_1e_2e_3}(\phi_{e_1e_4e_5}(e_i) - \phi_{e_1e_4e_5}(e_1e_2e_3e_4e_5))\\
&= \phi_{e_1e_2e_3}(e_i + e_1e_4e_5 - (e_1e_2e_3e_4e_5 + e_1e_4e_5(\overline{e_1e_2e_3e_4e_5})))\\
&= \phi_{e_1e_2e_3}(e_i + e_1e_4e_5 - e_1e_2e_3e_4e_5)\\
&= \phi_{e_1e_2e_3}(e_i) + \phi_{e_1e_2e_3}(e_1e_4e_5) - \phi_{e_1e_2e_3}(e_1e_2e_3e_4e_5)\\
&= e_i + e_1e_2e_3 + e_1e_4e_5 + e_1e_2e_3(\overline{e_1e_4e_5}) - e_1e_2e_3e_4e_5 - e_1e_2e_3(\overline{e_1e_2e_3e_4e_5})\\
&= e_i + e_1e_2e_3 + e_1e_4e_5 + e_1e_2e_3e_4e_5 - e_1e_2e_3e_4e_5 = e_i + e_1e_2e_3 + e_1e_4e_5
\end{aligned}$$

Remark 3.1. Note that, the inverse of an automorphism of the form ϕ_x and the composition $\phi_x\phi_y$ of two automorphisms ϕ_x and ϕ_y indicated in Corollary 3.2 are of the same form by Lemma 2.1 and Theorem 3.2. Thus, we have the following result.

Corollary 3.3. *The set H of automorphisms of the form ϕ_x, $x \in G^{(1)} \cap \omega^3(G)$ forms a subgroup of* $\mathrm{Aut}(G)$. *Furthermore, the group H is generated by the infinite set*

$$\{\phi_x \mid x \in G^{(1)} \cap \omega^3(G) \text{ is monomial}\}.$$

4. Conclusions

In this paper, a special subgroup H of the group $\mathrm{Aut}(G)$ of automorphisms of the infinite dimensional Grassmann algebra G is characterized, similar to the subgroup A_v of the group of automorphisms $\mathrm{Aut}(G_m)$ as indicated in Theorem 1.1. We also give an infinite generating set for the subgroup H, suggesting a canonical way to express an arbitrary automorphism in H in terms of the generating elements.

The next step of the main result of this paper might be the determination of the automorphisms of the form $\phi : e_i \to e_i + x_i$, for each nonnecessarily equal $x_i \in G^{(1)} \cap \omega^3(G)$, $i \geq 1$. This will solve an important component of the group $\mathrm{Aut}(G)$. A special case of these automorphisms was suggested by Vesselin Drensky in the next theorem.

Theorem 4.3. *An endomorphism ϕ of the form*

$$\phi : e_i \to e_i + x_i \ , \ \ x_i \in G_m^{(1)} \cap \omega^3(G) \subset G$$

is an automorphism of G.

Proof. Consider the *triangular* automorphism of G

$$\tau_x(e_i) = e_i \ , \ \ i = 1, \ldots, m,$$

$$\tau_x(e_i) = e_i + x_i \ , \ \ i = m+1, m+2, \ldots,$$

with inverse automorphism τ_{-x}. Then

$$\tau_{-x}\phi(e_i) = e_i + x_i \ , \ \ i = 1, \ldots, m,$$

$$\tau_{-x}\phi(e_i) = e_i \ , \ \ i = m+1, m+2, \ldots,$$

Clearly, $\tau_{-x}\phi$ sends G_m to G_m and is an automorphism of G_m if and only if its restriction on G_m is an automorphism of G_m. But this holds in virtue of the known results on automorphisms of G_m; i.e, its restriction is an element of A_v. □

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