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On Intuitionistic Fuzzy Structure Space On Γ-Ring

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ABSTRACT. In this research article, we investigate and study the intuitionistic fuzzy structure space of a Γ-ring M set up by the class of intuitionistic fuzzy prime ideals of M called the intuitionistic fuzzy prime spectrum of Γ-ring. Apart from studying basic properties of this structure space, we explore separation axioms, compactness, irreducibility and connectedness in this structure space.

1. INTRODUCTION

Algebraic systems found to take a noteworthy role in mathematics with ample applications in numerous directions such as theoretical physics, computer sciences, control engineering, information sciences, coding theory etc. The prime spectrum of a ring with unity is a space formed by introducing Zariski topology on the set of all prime ideals in a commutative ring with unity which plays a crucial role in commutative algebra (for detail see [5, 10]).

It is well known that the concept of a Γ-ring was intially introduced and investigated by Nobusawa [14]. Barnes [4] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. Since then, many researchers have investigated various properties of this Γ-ring. Any ring can be regarded as a Γ-ring by suitably choosing Γ. Many primary results in ring theory have been broaden to Γ-rings. R. Paul [19] studied various types of ideals in Γ-ring and the corresponding operator rings.

W. E. Coppage and J. Luh [6] studied radical of Γ-ring. Y. B. Jun [12], elucidate fuzzy prime ideal of a Γ-ring and derived a number of characterization for a fuzzy ideal to be a fuzzy prime ideal. T. K. Dutta and T. Chanda [8] proved the same result in a different way and also proved handful characterization of fuzzy prime ideals. B. A. Ersoy [9] defined fuzzy semi-prime ideal and obtained some results. A. K. Aggarwal et al in [1] studied some theorems on fuzzy prime ideals of Γ-ring.

The conception of intuitionistic fuzzy set (IFS) was first launched by Atanassaov [2, 3], as an extension to the notion of fuzzy set (FS) given by Zadeh [25]. Kim et al in [13] examined the intuitionistic fuzzification of ideal of Γ-ring which were further studied by Palaniappan at al in [15, 16, 17]. The notion of IF prime ideal and IF semi-prime were studied by Palaniappan and Ramachandran in [18]. Authors in [21] studied the notion of IF characteristic ideals of a Γ-ring and obtained a one to one correlation between the set of all IF characteristic ideals of Γ-ring and that of its operator ring. Further in [22] they introduced the notion of IF prime radical and IF primary ideal of a of Γ-ring. An extension of IF ideal of Γ-ring was introduced in [23] which is used to characterise IF prime and IF semi-prime ideals. In [11] S. M. Goswami et al studied structure space of semi-ring and Γ-Semirings.

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In 2017, P. K. Sharma et al. in [20] introduced the notion of IF prime spectrum of a commutative ring with identity and studied it. Since Γ-ring is a generalization of ring, it is natural to investigate the ring theoretic analogues in these general settings. Keeping this view in mind we introduce in this paper a topology on the set of all IF prime ideals of a commutative Γ-ring M with identity and denote the resulting structure space by $IFSpec(M)$. We study separation axioms, compactness, irreducibility and connectedness in this structure space.

2. Preliminaries

In this section we recollect a few definitions and results, which are necessary for the development of the article,

Definition 2.1. ([14, 4]) If $(M, +)$ and $(\Gamma, +)$ are additive Abelian groups. Then M is called a Γ-ring (in the sense of Barnes [2]) if there exist mapping $M \times \Gamma \times M \to M$, $(m_1, \alpha, m_2) \mapsto m_1\alpha m_2$, $m_1, m_2 \in M, \gamma \in \Gamma$ holding the following circumstances:
(1) $m_1\alpha m_2 \in M$.
(2) $(m_1+m_2)\alpha m_3 = m_1\alpha m_3+m_2\alpha m_3$, $m_1(\alpha+\beta)m_2 = m_1\alpha m_2+m_1\beta m_2$, $m_1\alpha(m_2+m_3) = m_1\alpha m_2 + m_1\alpha m_3$.
(3) $(m_1\alpha m_2)\beta m_3 = m_1\alpha(m_2\beta m_3)$. for all $m_1, m_2, m_3 \in M$, and $\gamma \in \Gamma$.

A non-void subset N of M is considered as left (right) ideal of M provided N is an additive subgroup of M and $M\Gamma N \subseteq N(N\Gamma M \subseteq N)$. Also, N is called an ideal of M if N is both left and right ideal. A mapping $f : M \to M^{'}$ of Γ-rings is called a Γ-homomorphism [4] if $f(m_1 + m_2) = f(m_1) + f(m_2)$ and $f(m_1\alpha m_2) = f(m_1)\alpha f(m_2)$ for all $m_1, m_2 \in M, \alpha \in \Gamma$. When $M^{'} = M$, then a Γ-homomorphism is called a Γ-endomorphism, further a one-one and onto Γ-endomorphism is called a Γ-automorphism.

Definition 2.2. ([6]) A non-zero element m of a commutative Γ-ring M is called a unit element if for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$ there exist an element $m^{'}$ in M such that $m\gamma_1 m^{'}\gamma_2 x = x$ for all $x \in M$.

Definition 2.3. ([6, 24]) An element m of a Γ-ring M is called nilpotent if for any $\gamma \in \Gamma$ there exists a positive integer n depending on γ such that $(m\gamma)^n m = (m\gamma)(m\gamma)...(m\gamma)m = 0_M$. A subset S of M is said to be nil if each element of S is nilpotent. The nil radical of M is defined as the sum of all nil ideals of M.

In a Γ-ring the prime radical is a subset of the nil radical.

Definition 2.4. Let M be a Γ-ring and $m \in M$, then the principal ideal generated by m, denoted by $< m >$ is the intersection of all ideals containing m and is the set of all finite sums of the elements of the form $nm + a\gamma_1 m + m\gamma_2 b + c\gamma_3 m\gamma_4 d$, where n is an integer, $a, b, c, d \in M$, $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$.

Definition 2.5. ([7]) A Γ-ring M is called a Boolean Γ-ring if $\forall m \in M, m\gamma m = m$, for all $\gamma \in \Gamma$.

Theorem 2.1. ([7]) *Let M be a Boolean Γ-ring with unity **e**. Then*
(i) $m = -m, \forall m \in M$;
(ii) $m_1\gamma m_2 = m_2\gamma m_1, \forall m_1, m_2 \in M, \gamma \in \Gamma$, i.e., M is commutative. Γ-ring.
*(iii) m is idempotent element in M if and only if **e** $- m$ is idempotent element in M.*

Definition 2.6. ([10]) A topological space (X, T) is called irreducible if every pair of non-empty open subsets of the space X has a non-empty intersection.

Definition 2.7. ([2, 3]) An IFS A of a non-void set X is described by the formation $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$, where $\mu_A, \nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Remark 2.1. ([2, 3])
(i) When $\mu_A(x)+\nu_A(x) = 1$, i.e., $\nu_A(x) = 1 - \mu_A(x) = \mu_{A^c}(x)$. Then A is called a fuzzy set.
(ii) An IFS $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ is shortly denoted by $A(x) = (\mu_A(x), \nu_A(x))$, for all $x \in X$. We will write $IFS(X)$, the set of all IFSs of X.

If $A, B \in IFS(X)$, then $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$ and $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. For any subset Y of X, the IF characteristic function χ_Y is an IFS of X, defined as $\chi_Y(x) = (1, 0), \forall x \in Y$ and $\chi_Y(x) = (0, 1), \forall x \in X \backslash Y$. Let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then the crisp set $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$ is called the (α, β)-level cut subset of A. Also the IFS $x_{(\alpha,\beta)}$ of X defined as $x_{(\alpha,\beta)}(y) = (\alpha, \beta)$, if $y = x$, otherwise $(0, 1)$ is called the intuitionistic fuzzy point (IFP) in X with support x. By $x_{(\alpha,\beta)} \in A$ we mean $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. Further if $f : X \to Y$ is a mapping and A, B be respectively IFS of X and Y. Then the image $f(A)$ is an IFS of Y is defined as $\mu_{f(A)}(y) = Sup\{\mu_A(x) : f(x) = y\}$, $\nu_{f(A)}(y) = Inf\{\nu_A(x) : f(x) = y\}$, for all $y \in Y$ and the inverse image $f^{-1}(B)$ is an IFS of X is defined as $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$, for all $x \in X$, i.e., $f^{-1}(B)(x) = B(f(x))$, for all $x \in X$. Also the IFS A of X is said to be f-invariant if for any $x, y \in X$, whenever $f(x) = f(y)$ implies $A(x) = A(y)$.

Definition 2.8. ([15]) Let A and B be two IFSs of a Γ-ring M and $\gamma \in \Gamma$. Then the product $A\Gamma B$ and the composition $A \circ B$ of A and B are defined by

$$A\Gamma B(m) = \begin{cases} (\vee_{m=m_1\gamma m_2}(\mu_A(m_1) \wedge \mu_B(m_2)), \wedge_{m=m_1\gamma m_2}(\nu_A(m_1) \vee \nu_B(m_2)), & \text{if } m = m_1\gamma m_2 \\ (0,1), & \text{otherwise} \end{cases}$$

and

$$A\circ B(m) = \begin{cases} (\vee_{m=\sum_{i=1}^n y_i\gamma z_i}(\mu_A(y_i) \wedge \mu_B(z_i)), \wedge_{m=\sum_{i=1}^n y_i\gamma z_i}(\nu_A(y_i) \vee \nu_B(z_i))), & \text{if } m = \sum_{i=1}^n y_i\gamma z_i \\ (0,1), & \text{otherwise} \end{cases}$$

Remark 2.2. ([15]) If A and B be two IFSs of a Γ-ring M, then $A\Gamma B \subseteq A \circ B \subseteq A \cap B$

Definition 2.9. ([15]) Let A be an IFS of a Γ-ring M. Then A is called an intuitionistic fuzzy ideal (IFI) of M if for all $m_1, m_2 \in M, \gamma \in \Gamma$, the following circumstances holds:
(i) $\mu_A(m_1 - m_2) \geq \mu_A(m_1) \wedge \mu_A(m_2)$;
(ii) $\mu_A(m_1\alpha m_2) \geq \mu_A(m_1) \vee \mu_A(m_2)$;
(iii) $\nu_A(m_1 - m_2) \leq \nu_A(m_1) \vee \nu_A(m_2)$;
(iv)$\nu_A(m_1\alpha m_2) \leq \nu_A(m_1) \wedge \nu_A(m_2)$.

The IFS $\tilde{0}$ and $\tilde{1}$ defined by $\tilde{0}(m) = (0, 1)$ and $\tilde{1}(m) = (1, 0), \forall m \in M$ are IFIs of M. These are called trivial IFIs of M. Also if A is an IFI of M, then $\mu_A(0_M) \geq \mu_A(m)$ and $\nu_A(0_M) \leq \nu_A(m), \forall m \in M$ (See [12]).

Remark 2.3. ([15, 17, 18]) If A, B and C be IFIs of a Γ-ring M, then $A\Gamma B$, $A \circ B$, $A \cap B$ are also IFI of M. Further, $A\Gamma B \subseteq C$ if and only if $A \circ B \subseteq C$.

Definition 2.10. ([18]) Let P be an IFI of a Γ-ring M. Then P is said to be IF prime (IF semi-prime) if P is non-constant and for any IFIs A, B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ (for any IFI A of M such that $A\Gamma A \subseteq P$ implies $A \subseteq P$).

Remark 2.4. ([18]) Let $x_{(p,q)}, y_{(t,s)} \in IFP(M)$. Then $x_{(p,q)}\Gamma y_{(t,s)} = (x\Gamma y)_{(p\wedge t, q\vee s)}$

Theorem 2.2. ([18]) *Let M be a commutative Γ-ring and A be an IFI of M. Then following are equivalent*
(i) $x_{(p,q)}\Gamma y_{(t,s)} \subseteq A \Rightarrow x_{(p,q)} \subseteq A$ or $y_{(t,s)} \subseteq A$, where $x_{(p,q)}, y_{(t,s)} \in IFP(M)$.
(ii) A is an IF prime ideal of M.

Theorem 2.3. ([18]) *Let A be an IFI of Γ-ring M. Then each (p,q)-level cut set $A_{(p,q)}$ is either empty or an ideal of M, where $p \leq \mu_A(0_M)$ and $q \geq \nu_A(0_M)$. In particular $A_{(1,0)}$ which is denoted by A_*, i.e., the set $A_* = \{x \in M : \mu_A(x) = \mu_A(0_M)$ and $\nu_A(x) = \nu_A(0_M)\}$ is ideal of M. If $A \in IFPI(M)$, then A_* is a prime ideal of M.*

Theorem 2.4. ([18]) *If P is an IF prime ideal of a Γ-ring M, then the following conditions hold:*
(i) $P(0_M) = (1,0)$,
(ii) P_* is a prime ideal of M,
(iii) $Img(P) = \{(1,0),(t,s)\}$, where $t, s \in [0,1)$ such that $t + s \leq 1$.

Definition 2.11. ([20]) A non-constant IFI A of a Γ-ring M is called an IF maximal ideal if, $Img(A) = \{(1,0),(t,s)\}$, where $t,s \in [0,1)$ such that $t+s \leq 1$ and A_* is a maximal ideal of M.

Clearly every IF maximal ideal A of a Γ-ring M is an IF prime ideal of M.

3. Intuitionistic fuzzy structure space of Γ-ring

In this section, we introduce a topological structure on the collection $\mathcal{X}$ of all IF prime ideals of Γ-ring M and investigate some of its properties.

Remark 3.5.
(i) $\mathcal{X} = \{P : P$ is an IF prime ideal of Γ-ring $M\}$
(ii) $\mathcal{V}(A) = \{P \in \mathcal{X} : A \subseteq P\}$, where A is any IFS of M.
(iii) $\mathcal{X}(A) = \mathcal{X} \setminus \mathcal{V}(A)$, the complement of $\mathcal{V}(A)$ in $\mathcal{X}$, i.e., = $\{P \in \mathcal{X} : A \nsubseteq P\}$
(iv) For any IFS B of M, $< B >$ denote the IFI generated by B.

Theorem 3.5. *Let M be a Γ-ring and $\tau = \{\mathcal{X}(A) : A$ is an IFPI of $M\} = \{P \in \mathcal{X} : A \nsubseteq P\}$. Then τ is a topology on $\mathcal{X}$ and the ordered pair $(\mathcal{X}, \tau)$ is a topological space.*

Proof. Consider the trivial IFIs $A = \tilde{0}$ and $B = \tilde{1}$ of M. Then $\mathcal{V}(A) = \mathcal{V}(\tilde{0}) = \mathcal{X}$ and $\mathcal{V}(B) = \mathcal{V}(\tilde{1}) = \emptyset$, so as $\mathcal{X}(\tilde{0}) = \emptyset$ and $\mathcal{X}(\tilde{1}) = \mathcal{X}$ implies $\emptyset, \mathcal{X} \in \tau$.
Next, let A_1 and A_2 be any two IFIs of M. Then
$B \in \mathcal{V}(A_1) \cup \mathcal{V}(A_2) \Rightarrow A_1 \subseteq B$ or $A_2 \subseteq B \Rightarrow A_1 \cap A_2 \subseteq B \Rightarrow B \in \mathcal{V}(A_1 \cap A_2)$ and $B \in \mathcal{V}(A_1 \cap A_2) \Rightarrow A_1 \cap A_2 \subseteq B \Rightarrow A_1 \Gamma A_2 \subseteq B$ [As $A_1 \Gamma A_2 \subseteq A_1 \cap A_2$]
$\Rightarrow A_1 \subseteq B$ or $A_2 \subseteq B$ [As B is intuitionistic fuzzy prime ideal of M]
$\Rightarrow B \in \mathcal{V}(A_1)$ or $B \in \mathcal{V}(A_2) \Rightarrow B \in \mathcal{V}(A_1) \cup \mathcal{V}(A_2)$.
Hence $\mathcal{V}(A_1) \cup \mathcal{V}(A_2) = \mathcal{V}(A_1 \cap A_2) \Rightarrow \mathcal{X} \setminus (\mathcal{V}(A_1) \cup \mathcal{V}(A_2)) = \mathcal{X} \setminus \mathcal{V}(A_1 \cap A_2) \Rightarrow (\mathcal{X} \setminus \mathcal{V}(A_1)) \cap (\mathcal{X} \setminus \mathcal{V}(A_2)) = \mathcal{X} \setminus \mathcal{V}(A_1 \cap A_2)$, i.e., $\mathcal{X}(A_1) \cap \mathcal{X}(A_2) = \mathcal{X}(A_1 \cap A_2)$.
From this we conclude that τ is closed under finite intersections.
Now, suppose that $\{A_i : i \in \Lambda\}$ be any family of IFIs of M. It can be confirmed that $\cap\{\mathcal{V}(A_i) : i \in \Lambda\} = \mathcal{V}(< \cup\{A_i : i \in \Lambda\} >)$. In another way,
$\{\mathcal{X}(A_i) : i \in \Lambda\} = \mathcal{X}(< \cup\{A_i : i \in \Lambda\} >)$. Hence τ is closed under arbitrary unions. Hence, τ defines a topology on $\mathcal{X}$. □

Remark 3.6. The topological space (X, τ) defined in Theorem (3.5) is assigned as the IF prime spectrum of M and is denoted by $IFSpec(M)$ or , for comfort, we denote it by $\mathcal{X}$ only.

Example 3.1. *(1) Consider $M = \Gamma = \mathbb{Z}$, the ring of integers. Then M is a Γ-ring. Suppose that $p \in \mathbb{Z}$ is a prime integer. Then for every $t, s \in [0,1)$ such that $t + s \leq 1$, define $P_{t,s} \in IFS(M)$ as*

$$\mu_{P_{t,s}}(x) = \begin{cases} 1, & \text{if } x \in <p> \\ t, & \text{if otherwise} \end{cases}; \quad \nu_{P_{t,s}}(x) = \begin{cases} 0, & \text{if } x \in <p> \\ s, & \text{otherwise.} \end{cases}$$

for all $x \in M$. Then by Theorem (2.4), $P_{s,t}$ is an intuitionistic fuzzy prime ideal of M.

Thus, $IFSpec(M) = \{P_{t,s},$ where $t, s \in [0,1)$ such that $t + s \leq 1$ and p is prime element of $\mathbb{Z}\}$.

(2) Consider $M = \Gamma = \mathbf{Z_2}$, where $\mathbf{Z_2} = \{\bar{0}, \bar{1}\}$ be a boolean ring. Then M is a Γ-ring and for every $t, s \in [0,1)$ such that $t + s \leq 1$, define $P_{t,s} \in IFS(M)$ as

$$\mu_{P_{t,s}}(x) = \begin{cases} 1, & \text{if } x = \bar{0} \\ t, & \text{if } x = \bar{1} \end{cases}; \quad \nu_{P_{t,s}}(x) = \begin{cases} 0, & \text{if } x = \bar{0} \\ s, & \text{if } x = \bar{1}. \end{cases}$$

for all $x \in M$. Then by Theorem (2.4), $P_{t,s}$ is an intuitionistic fuzzy prime ideal of M.

Thus, $IFSpec(M) = \{P_{t,s},$ where $t, s \in [0,1)$ such that $t + s \leq 1\}$.

Proposition 3.1. *Let M, N be Γ-rings. If $f : M \to N$ is a surjective homomorphism, then $\forall x \in M, \alpha, \beta \in (0,1]$ such that $\alpha + \beta \leq 1$, we have*

$$f(x_{(\alpha,\beta)}) = (f(x))_{(\alpha,\beta)}$$

Proof. Let $y \in N$ be any element, then $f(x_{(\alpha,\beta)})(y) = (\mu_{f(x_{(\alpha,\beta)})}(y), \nu_{f(x_{(\alpha,\beta)})}(y))$, where

$$\mu_{f(x_{(\alpha,\beta)})}(y) = Sup\{\mu_{x_{(\alpha,\beta)}}(p) : f(p) = y\} = \begin{cases} \alpha, & \text{if } p = x \text{ (i.e., } y = f(x)); \\ 0, & \text{otherwise.} \end{cases} = \mu_{(f(x))_{(\alpha,\beta)}}(y)$$

and

$$\nu_{f(x_{(\alpha,\beta)})}(y) = Inf\{\nu_{x_{(\alpha,\beta)}}(p) : f(p) = y\} = \begin{cases} \beta, & \text{if } p = x \text{ (i.e., } y = f(x)); \\ 1, & \text{otherwise.} \end{cases} = \nu_{(f(x))_{(\alpha,\beta)}}(y)$$

Hence $f(x_{(\alpha,\beta)}) = (f(x))_{(\alpha,\beta)}$. □

Recollect that a topological space $\mathcal{Y}$ is compact if and only if every covering of $\mathcal{Y}$ by basic open sets is reducible to a finite sub covering of $\mathcal{Y}$.

Theorem 3.6. *Let M be a Γ-ring and $x, y \in M$ and $\alpha, \beta \in (0,1]$ with $\alpha + \beta \leq 1$. Then the following statements are true*
(i) $\mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}(y_{(\alpha,\beta)}) = \mathcal{X}((x\gamma y)_{(\alpha,\beta)})$, for all $\gamma \in \Gamma$.
(ii) $\mathcal{X}(x_{(\alpha,\beta)}) = \emptyset$ if and only if x is nilpotent.
(iii) $\mathcal{X}(x_{(\alpha,\beta)}) = \mathcal{X}$ if x is a unit in M.

Proof. (i) Let $x, y \in M, \gamma \in \Gamma$ and $\alpha, \beta \in (0,1]$ with $\alpha + \beta \leq 1$. Let $P \in \mathcal{X}$. Then $\mu_P(0_M) = 1, \nu_P(0_M) = 0$, $Img(P) = \{(1,0), (t,s)\}$, where $t, s \in [0,1)$ such that $t + s \leq 1$, P_* is a prime ideal of M (by Theorem (2.4)).
Suppose $P \in \mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}(y_{(\alpha,\beta)})$, then $P \in \mathcal{X}(x_{(\alpha,\beta)})$ and $P \in \mathcal{X}(y_{(\alpha,\beta)})$
$\Leftrightarrow x_{(\alpha,\beta)} \nsubseteq P$, $y_{(\alpha,\beta)} \nsubseteq P \Leftrightarrow \mu_P(x) < \alpha, \nu_P(x) > \beta$ and $\mu_P(y) < \alpha, \nu_P(y) > \beta$
$\Leftrightarrow \alpha = \mu_{x_{(\alpha,\beta)}}(x) > \mu_P(x)$, $\beta = \nu_{x_{(\alpha,\beta)}}(x) < \nu_P(x)$ and $\alpha = \mu_{y_{(\alpha,\beta)}}(y) > \mu_P(y)$, $\beta = \nu_{y_{(\alpha,\beta)}}(y) < \nu_P(y)$
$\Leftrightarrow x, y \notin P_*$, for if $x, y \in P_*$, then $\alpha > \mu_P(x) = \mu_P(y) = 1$ and $\beta < \nu_P(x) = \nu_P(y) = 0$
$\Leftrightarrow x\gamma y \notin P_*$, for all $\gamma \in \Gamma$, as P_* is a prime ideal of M.
$\Leftrightarrow \alpha > \mu_P(x\gamma y)$ and $\beta < \nu_P(x\gamma y)$, since $Img(P) = \{(1,0), (t,s)\}$, $t, s \in [0,1)$ such that $t + s \leq 1$

$\Leftrightarrow (x\gamma y)_{(\alpha,\beta)} \nsubseteq P \Leftrightarrow P \in \mathcal{X}((x\gamma y)_{(\alpha,\beta)})$.
This proves that $\mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}(y_{(\alpha,\beta)}) = \mathcal{X}((x\gamma y)_{(\alpha,\beta)})$, for all $\gamma \in \Gamma$.

(ii) Suppose J be any prime ideal of M and χ_J be the intuitionistic fuzzy characteristic function of J. Then from Theorem (2.4) we have $\chi_J \in \mathcal{X}$. Further, if $\mathcal{X}(x_{(\alpha,\beta)}) = \emptyset$ then $\mathcal{V}(x_{(\alpha,\beta)}) = \mathcal{X}$ that implies $x_{(\alpha,\beta)} \subseteq \chi_J$ and therefore, $\mu_{\chi_J}(x) \geq \alpha > 0$ and $\nu_{\chi_J}(x) \leq \beta < 1$ so that $\mu_{\chi_J}(x) = 1$ and $\nu_{\chi_J}(x) = 0$ and so $x \in J$. Thus $x \in \cap\{J :$ J is a prime ideal of M $\}$. As the prime radical is subset of the nil radical so x is nilpotent.

Conversely, assume that x is nilpotent. Then for every $\gamma \in \Gamma, \exists\, n \in \mathbb{N}$ depending on γ so that $(x\gamma)^n x = 0_M$. Let $P \in \mathcal{X}$ be any element. Then $\mu_P((x\gamma)^n x) = \mu_P(0_M) = 1$ and $\nu_P((x\gamma)^n x) = \nu_P(0_M) = 0$. Therefore $1 = \mu_P((x\gamma)^n x) \geq \mu_P(x)$ and $0 = \nu_P((x\gamma)^n x) \leq \nu_P(x)$ implies that $\mu_P(x) = 1$ and $\nu_P(x) = 0$. So $x \in P_*$. But P_* is a prime ideal of M. Hence $\alpha = \mu_{x_{(\alpha,\beta)}}(x) \leq \mu_P(x)$ and $\beta = \nu_{x_{(\alpha,\beta)}}(x) \geq \nu_P(x)$, whence $x_{(\alpha,\beta)} \subseteq P, \forall P \in \mathcal{X}$. Thus $\mathcal{V}(x_{(\alpha,\beta)}) = \mathcal{X}$, i.e., $\mathcal{X}(x_{(\alpha,\beta)}) = \emptyset$.

(iii) Suppose J and χ_J be same as in part (ii). Now if $\mathcal{X}(x_{(\alpha,\beta)}) = \mathcal{X}$ then $\mathcal{V}(x_{(\alpha,\beta)}) = \emptyset$ that implies $x_{(\alpha,\beta)} \nsubseteq \chi_J$ and thus $\mu_{\chi_J}(x) < \alpha$ and $\nu_{\chi_J}(x) > \beta$ so that $x \notin J$. Hence $x \notin \cup\{J :$ J is a prime ideal of M $\}$. This shows that x is a unit. □

The following example show that the converse of Theorem (3.6)(iii) is not true in general. This is a deviation of the result from the crisp theory (see [5], Proposition (2.2)).

Example 3.2. Consider M, Γ and $\mathcal{X} = IFSpec(M)$ as in Example (3.1)(1).
Define $A \in \mathcal{X}$ as follow

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in <2> \\ 0.6, & \text{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in <2> \\ 0.3, & \text{otherwise.} \end{cases}$$

Take $\alpha = 0.5, \beta = 0.4$ and $x = 1$. Then we see that IFP $x_{(\alpha,\beta)} \subseteq A$, hence $A \notin \mathcal{X}(x_{(\alpha,\beta)})$, and consequently $\mathcal{X} \neq \mathcal{X}(x_{(\alpha,\beta)})$.

Proposition 3.2. *The subfamily* $\{\mathcal{X}(x_{(\alpha,\beta)}) : x \in M, \alpha, \beta \in (0,1]$ *s.t.* $\alpha + \beta \leq 1\}$ *of* τ *is a base for* τ*.*

Proof. Let $\mathcal{X}(A) \in \tau$, where A is an IFI of M. Let $B \in \mathcal{X}(A)$. Then $A \nsubseteq B$. This implies that there exists $x \in M$ such that $\mu_A(x) > \mu_B(x)$ and $\nu_A(x) < \nu_B(x)$. Thus $x \notin B_*$ and hence $\mu_B(x) = t$ and $\nu_B(x) = s$, for some $t, s \in [0,1)$ with $t + s \leq 1$.
Let $\mu_A(x) = \alpha > 0, \nu_A(x) = \beta < 1$. Clearly $x_{(\alpha,\beta)} \nsubseteq B$ and so $B \in \mathcal{X}(x_{(\alpha,\beta)})$.

Now, $\mathcal{V}(A) \subseteq \mathcal{V}(x_{(\alpha,\beta)})$, because if $P \in \mathcal{V}(A)$ then $A \subseteq P$ and so $\mu_{x_{(\alpha,\beta)}}(x) = \alpha = \mu_A(x) < \mu_P(x)$ and $\nu_{x_{(\alpha,\beta)}}(x) = \beta = \nu_A(x) > \nu_P(x)$. This implies that $x_{(\alpha,\beta)} \subseteq P$ and thus $P \in \mathcal{V}(x_{(\alpha,\beta)})$. Hence $\mathcal{X}(x_{(\alpha,\beta)}) \subseteq \mathcal{X}(A)$. Thus $B \in \mathcal{X}(x_{(\alpha,\beta)}) \subseteq \mathcal{X}(A)$. Hence the subfamily $\{\mathcal{X}(x_{(\alpha,\beta)}) : x \in M, \alpha, \beta \in (0,1]$ such that $\alpha + \beta \leq 1\}$ is a base for τ. □

Proposition 3.3. *The subset* $\mathcal{Y} = \{P \in \mathcal{X} : Img(P) = \{(1,0),(t,s)\},$ *where* $t, s \in [0,1)$ *with* $t + s \leq 1\}$*, is compact with respect to the subspace topology.*

Proof. Proceeding in the same manner as in Proposition (3.2), we can easily verify that the family $\{\mathcal{X}(x_{(\gamma,\delta)}) \cap \mathcal{Y} : x \in M,$ and $\gamma \in (t,1]$ and $\delta \in [0,s)$ such that $\gamma + \delta \leq 1\}$ forms a base for $\mathcal{Y}$. Now, suppose that $\{\mathcal{X}((x_i)_{(p,q)}) \cap \mathcal{Y} : i \in \Lambda$ and $(p,q) \in K \times S \subseteq (t,1] \times [0,s)\}$ is a covering of $\mathcal{Y}$ taken from the basic open sets. Suppose $\gamma = Sup\{p : p \in K\}$ and

$\delta = Inf\{q : q \in S\}$. Then the family $\{\mathcal{X}((x_i)_{(\gamma,\delta)}) \cap \mathcal{Y} : i \in \Lambda\}$ also covers $\mathcal{Y}$. Now,

$$\begin{aligned}\mathcal{Y} &= \cup\{\mathcal{X}((x_i)_{(\gamma,\delta)}) \cap \mathcal{Y} : i \in \Lambda\} \\ &= (\cup\{\mathcal{X}((x_i)_{(\gamma,\delta)}) : i \in \Lambda\}) \cap \mathcal{Y} \\ &= (\mathcal{X} \setminus \mathcal{V}(\cup\{(x_i)_{(\gamma,\delta)} : i \in \Lambda\})) \cap \mathcal{Y} \\ &= (\mathcal{X} \cap \mathcal{Y})\setminus(\mathcal{V}(\cup\{(x_i)_{(\gamma,\delta)} : i \in \Lambda\}) \cap \mathcal{Y}) \\ &= \mathcal{Y}\setminus(\mathcal{V}(\cup\{(x_i)_{(\gamma,\delta)} : i \in \Lambda\}) \cap \mathcal{Y}).\end{aligned}$$

This show that $\mathcal{V}(\cup\{(x_i)_{(\gamma,\delta)} : i \in \Lambda\}) \cap \mathcal{Y} = \emptyset$. Further, suppose that J be any prime ideal of Γ-ring M. Consider an IFI A of M given by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in J \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in J \\ \beta, & \text{if otherwise} \end{cases}.$$

Clearly, A is an IFPI of M and $A \in \mathcal{Y}$. So $A \notin \mathcal{V}(\cup\{(x_i)_{(\gamma,\delta)} : i \in \Lambda\})$. Hence $(x_j)_{(\gamma,\delta)} \nsubseteq A$ for some $j \in \Lambda$. Thus $\gamma > \mu_A(x_j)$ and $\delta < \nu_A(x_j)$ for some $j \in \Lambda$. As a result, $x_j \notin J$. This proves that there is no prime ideal of M containing the set $\{x_i : i \in \Lambda\}$. Therefore, $< \{x_i : i \in \Lambda\} >= M$. Let $\sum_{l=1}^{n}[\delta_l, e_l]$ be the right unity of Γ-ring M, where $\delta_l \in \Gamma$, $e_l \in M$ for all $l = 1, 2, ..., n$ and $e_l = \sum_{q=1}^{n_l} m_{q_l}\gamma_{q_l}x_{q_l}$, where n_l is a finite positive integer, $m_{q_l} \in M$, $x_{q_l} \in \{x_j : J \in \Lambda\}$, $\gamma_{q_l} \in \Gamma$ for all $q = 1, 2, ..., n_l$ and $l = 1, 2, ...n$. Now we claim that $\mathcal{V}(\cup_{l=1}^{n} \cup_{q=1}^{n_l} (x_{q_l})_{(\gamma,\delta)}) \cap \mathcal{Y} = \emptyset$, as $A \in \mathcal{V}(\cup_{l=1}^{n} \cup_{q=1}^{n_l} (x_{q_l})_{(\gamma,\delta)}) \cap \mathcal{Y}$ implies $\cup_{l=1}^{n} \cup_{q=1}^{n_l} (x_{q_l})_{(\gamma,\delta)} \subseteq A$ and $Img(A) = \{(1, 0), (\alpha, \beta)\}$. This imply
$\gamma = \mu_{(x_{q_l})_{(\gamma,\delta)}}(x_{q_l}) \leq \mu_A(x_{q_l})$ and $\delta = \nu_{(x_{q_l})_{(\gamma,\delta)}}(x_{q_l}) \geq \nu_A(x_{q_l}), \forall q = 1, 2, .., n_l, l = 1, 2, .., n$.
$\Rightarrow \mu_A(x_{q_l}) = 1, \nu_A(x_{q_l}) = 0$, for all $q = 1, 2, ..., n_l, l = 1, 2, ..., n$, since $\gamma > \alpha, \delta < \beta$.
$\Rightarrow x_{q_l} \in A_*$ for all $q = 1, 2, ..., n_l, l = 1, 2, ..., n$
$\Rightarrow e_l \in A_*$ for all $l = 1, 2, ..., n$
$\Rightarrow x_j = \sum_{i=1}^{n} x_j\delta_l e_l \in A_* = J$, which is a contradiction. Thus we have

$$\begin{aligned}\mathcal{Y} &= \mathcal{Y} \setminus (\mathcal{V}(\cup_{l=1}^{n} \cup_{q=1}^{n_l} (x_{q_l})_{(\gamma,\delta)}) \cap \mathcal{Y}) \\ &= (\mathcal{X} \cap \mathcal{Y}) \setminus (\mathcal{V}(\cup_{l=1}^{n} \cup_{q=1}^{n_l} (x_{q_l})_{(\gamma,\delta)}) \cap \mathcal{Y}) \\ &= (\mathcal{X} \setminus \mathcal{V}(\cup_{l=1}^{n} \cup_{q=1}^{n_l} (x_{q_l})_{(\gamma,\delta)})) \cap \mathcal{Y} \\ &= (\cup_{l=1}^{n} \cup_{q=1}^{n_l} \mathcal{X}(x_{q_l})_{(\gamma,\delta)}) \cap \mathcal{Y} \\ &= \cup_{l=1}^{n} \cup_{q=1}^{n_l} (\mathcal{X}(x_{q_l})_{(\gamma,\delta)} \cap \mathcal{Y}).\end{aligned}$$

This proves that $\{\mathcal{X}((x_{q_l})_{(\gamma,\delta)}) \cap \mathcal{Y} : q = 1,2,...,n_l, l = 1,2,..., n\}$ covers $\mathcal{Y}$. Hence $\mathcal{Y}$ is compact. □

4. Separation axioms of IF Spec(M)

We know that a topological space $\mathcal{X}$ is called T_0, if $\forall, x \neq y \in \mathcal{X}$, $\exists$ atleast one open set containing x but not y (or $\exists$ an open set containing y but not x). Also we know that a topological space is called T_1 if and only if every subset containing one point is closed set.

Proposition 4.4. *The space $\mathcal{X}$ is T_0*

Proof. Let $A, B \in \mathcal{X}$ such that $A \neq B$. Then either $A \nsubseteq B$ or $B \nsubseteq A$. Let $B \nsubseteq A$. Then $B \in \mathcal{X}(A)$. Also, $A \notin \mathcal{X}(A)$ and $\mathcal{X}(A)$ is open. Therefore, $\mathcal{X}$ is T_0 space. □

In the following examples we show that there exists some element of basis of $\mathcal{X}$ which is not closed, and it is even possible that $\mathcal{X}$ is not T_1 and hence not T_2. These results are also deviation from the results in crisp theory (see [5], Theorem (4.12)).

Example 4.3. Consider M and Γ as in Example (3.1)(2).
Then $\mathcal{X} = \{P_{t,s},$ where $t, s \in [0, 1)$ such that $t + s \leq 1\}$, where $P_{t,s}$ is defined as

$$\mu_{P_{t,s}}(x) = \begin{cases} 1, & \text{if } x = \bar{0} \\ t, & \text{if } x = \bar{1} \end{cases}; \quad \nu_{P_{t,s}}(x) = \begin{cases} 0, & \text{if } x = \bar{0} \\ s, & \text{if } x = \bar{1}. \end{cases}$$

for all $x \in M$. Now we show that if $x = \bar{1}$ and $\alpha = 0.6, \beta = 0.3$, then $\mathcal{X}(\bar{1}_{(\alpha,\beta)})$ is not closed. Suppose on the contrary that $\mathcal{X}(\bar{1}_{(\alpha,\beta)})$ is closed. Then there exists subset $K \times S$ of $[0, 1] \times [0, 1]$ such that $\mathcal{X}(\bar{1}_{(\alpha,\beta)}) = \cap\{\mathcal{V}(y_{(p,q)}) : (p, q) \in K \times S, y \in \mathbf{Z_2}\}$. If $y = \bar{1}$ and $(p, q) \in K \times S = (\alpha, 1] \times [0, \beta)$ such that $p + q \leq 1$, then it is not difficult to check that $\mathcal{X}(\bar{1}_{(\alpha,\beta)}) \nsubseteq \mathcal{V}(\bar{1}_{(p,q)})$ and if $y = \bar{1}$ and $p = 0, q = 1$ or $y = \bar{0}, (p, q) \in [0, 1] \times [0, 1]$, then it is seen that $\mathcal{V}(y_{(p,q)}) = \mathcal{X}$. Thus $\mathcal{X}(\bar{1}_{(\alpha,\beta)})$ must be equal to $\mathcal{X}$, which is a contradiction. Therefore $\mathcal{X}(\bar{1}_{(\alpha,\beta)})$ is not closed.

Example 4.4. Consider the space $\mathcal{X}$ as in Example (4.3). Choose $P_{0.6,0.3}, P_{0.5,0.4} \in \mathcal{X}$. Let W be an open set containing $P_{0.6,0.3}$. Then $W = \cap\{\mathcal{X}(\bar{1}_{(p,q)}) : (p, q) \in K \times S\}$ for some $K \times S \subseteq (0, 1] \times (0, 1]$. Thus there exists $(p, q) \in K \times S$ such that $P_{0.6,0.3} \in \mathcal{X}(\bar{1}_{(p,q)})$. So $p > 0.6 > 0.5$ and $q < 0.3 < 0.4$. Consequently $P_{0.5,0.4} \in \mathcal{X}(\bar{1}_{(p,q)}) \subseteq W$. In other words any open neighbourhood of $P_{0.6,0.3}$ also contain $P_{0.5,0.4}$. Thus $\mathcal{X}$ is not T_1.

Proposition 4.5. *Let M be a Γ-ring and $A \in \mathcal{X}$ then $\mathcal{V}(A) = cl\{A\}$, the closure of A in $\mathcal{X}$. Further $B \in cl\{A\}$ if and only if $A \subseteq B$, where $A, B \in \mathcal{X}$.*

Proof. Since $\mathcal{V}(A)$ is a closed subset of $\mathcal{X}$ containing A. Therefore $cl\{A\} \subseteq \mathcal{V}(A)$
For the reverse inclusion, consider $B \in \mathcal{X}$ such that $B \notin cl\{A\}$. Then, $\exists$ an open set $\mathcal{X}(C)$ where C is an IFI of M containing B but not A. Therefore, $C \nsubseteq B$ but $C \subseteq A$. So $A \nsubseteq B$ and hence $B \notin \mathcal{V}(A)$. Thus $\mathcal{V}(A) \subseteq cl\{A\}$. Hence $\mathcal{V}(A) = cl\{A\}$.

Further, $B \in cl\{A\}$ if and only if $B \in \mathcal{V}(A)$, which is equivalent to $A \subseteq B$. □

Proposition 4.6. *Let $\mathcal{Y}$ be same as in Proposition (3.3). If $A \in \mathcal{Y}$, then $\{A\}$ is closed in $\mathcal{Y}$ if and only if A is an IF maximal ideal of M. (In other words, $\mathcal{Y}$ is T_1 if and only if every singleton element of $\mathcal{Y}$ is an IF maximal ideal of M.)*

Proof. Let $A \in \mathcal{Y}$ and $\{A\}$ be closed. Then $\mathcal{V}(A) = cl\{A\} = \{A\}$. Hence $\mathcal{V}(A) \cap \mathcal{Y} = \{A\}$, by Proposition (4.5). Now, we show that A is an IF maximal ideal. As $A \in \mathcal{Y}$, $Img(A) = \{(1, 0), (t, s)\}$. So it is left to prove that the ideal $A_* = \{x \in M : \mu_A(x) = 1$ and $\mu_A(x) = 0\}$ is maximal. For this, it is enough to show that there is no prime ideal of M properly containing A_*. Let J be a prime ideal of M properly containing A_*.

Let B be an IFI of M defined by

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in J \\ t, & \text{if otherwise} \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x \in J \\ s, & \text{if otherwise} \end{cases}, \text{where } t + s \leq 1.$$

Then $B \in \mathcal{Y}$ and A is properly contained in B. This contradicts the fact that $\mathcal{V}(A) \cap \mathcal{Y} = \{A\}$. This proves that A_* is a maximal ideal of M and so A is an IF maximal ideal of M.

Conversely, let $A \in \mathcal{Y}$ and A is an IF maximal ideal. Then the ideal $A_* = \{x \in M : \mu_A(x) = 1$ and $\mu_A(x) = 0\}$ is maximal ideal of M. We claim that $\mathcal{V}(A) \cap \mathcal{Y} = \{A\}$. Clearly, $\{A\} \subseteq \mathcal{V}(A) \cap \mathcal{Y}$. Next

$$B \in \mathcal{V}(A) \cap \mathcal{Y} \Rightarrow A_* \subseteq B_* \Rightarrow A_* = B_*$$

since A_* is maximal ideal. Thus we have $A = B$, since $Img(A) = Img(B) = \{(1,0),(t,s)\}$. Therefore, $\mathcal{V}(A) \cap \mathcal{Y} = \{A\}$. Consequently, $\{A\}$ is a closed subset of $\mathcal{Y}$. □

We know that a topological space $\mathcal{X}$ is Hausdorff (or T_2 space), if and only if $\forall, x \neq y \in \mathcal{X}$, $\exists$ two disjoint open sets one containing x and another containing y. As a remarkable deviation from commutative algebra, we notice that for a Γ-ring M in which each prime ideal is maximal ideal, the space $IFSpec(M)$ is not Hausdorff, but, it may, a portion of its subspaces are demonstrated to be Hausdorff.

Theorem 4.7. *Let M be a Γ-ring whose each prime ideal is a maximal ideal. Then the space $\mathcal{X} = IFSpec(M)$ is not T_2.*

Proof. For the proof we show that $\exists$ two distinct elements A, B of $\mathcal{X} = IFSpec(M)$ that cannot be separated by two disjoint basic open sets.
Consider a prime ideal J and two IF prime ideals A and B of M as follow

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in J \\ 0.1, & \text{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in J \\ 0.2, & \text{if otherwise} \end{cases};$$

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in J \\ 0.3, & \text{if otherwise} \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x \in J \\ 0.4, & \text{if otherwise} \end{cases}.$$

Consider $\mathcal{X}(x_{(\alpha,\beta)})$ and $\mathcal{X}(y_{(\alpha,\beta)})$ be two basic open sets in $\mathcal{X}$ containing A and B respectively, where $x, y \in M$ and $\alpha, \beta \in (0,1]$ s.t. $\alpha + \beta \leq 1$. Then $x_{(\alpha,\beta)} \nsubseteq A$ and $y_{(\alpha,\beta)} \nsubseteq B$ and so $x \notin A_* = J$ and $y \notin B_* = J$. Since J is prime ideal in M, so $x\gamma y \notin J$, for every $\gamma \in \Gamma$. Then $x\gamma y$ is not nilpotent and so by Theorem (3.6) (i) and (ii) we have $\mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}(y_{(\alpha,\beta)}) = X((x\gamma y)_{(\alpha,\beta)}) \neq \emptyset$. Hence $\mathcal{X}$ is not T_2. □

Theorem 4.8. *Let M be a Boolean Γ-ring with unity **e**. Let $t, s \in [0,1)$ with $t+s \leq 1$ and suppose $\mathcal{Y} = \{P \in \mathcal{X} : Img(P) = \{(1,0),(t,s)\}\}, x, y \in M$, and $\gamma, \delta \in (0,1]$ so that $\gamma + \delta \leq 1$. Then:*
(i) The set $\mathcal{X}(x_{(\gamma,\delta)}) \cap \mathcal{Y}$ is a clopen set in $\mathcal{Y}$, provided $\gamma > t$ and $\delta < s$.
(ii) $\mathcal{X}(x_{(\gamma,\delta)}) \cup \mathcal{X}(y_{(\gamma,\delta)}) = \mathcal{X}(z_{(\gamma,\delta)})$ for some $z \in M$.
(iii) The space $\mathcal{Y}$ is T_2.

Proof. (i) Since $\mathcal{X}(x_{(\gamma,\delta)})$ is open set in $\mathcal{X}$, it follows that $\mathcal{X}(x_{(\gamma,\delta)}) \cap \mathcal{Y}$ is open set in $\mathcal{Y}$. We now show that $\mathcal{X}(x_{(\gamma,\delta)}) \cap \mathcal{Y} = \mathcal{V}((\mathbf{e} - x)_{(\gamma,\delta)}) \cap \mathcal{Y}$. [This would simply implies that $\mathcal{X}(x_{(\gamma,\delta)})$ is closed set in $\mathcal{Y}$.

If $A \in \mathcal{X}(x_{(\gamma,\delta)}) \cap \mathcal{Y}$ then $\mu_A(x) < \gamma, \nu_A(x) > \delta$, but $Img(A) = \{(1,0),(t,s)\}$ so that $\mu_A(x) = t, \nu_A(x) = s$. Hence $\gamma > t$ and $\delta < s$ and $x \notin A_*$. This implies that $\gamma > t$ and $\delta < s$ and $\mathbf{e} - x \in A_*$, since $x\Gamma(\mathbf{e} - x) = x\Gamma\mathbf{e} - x\Gamma x = x - x = 0 \in A_*$ and the ideal A_* is prime implies that $(\mathbf{e} - x) \in A_*$. As a result, $\mu_A(\mathbf{e} - x) = 1$ and $\nu_A(\mathbf{e} - x) = 0$ so that $(\mathbf{e} - x)_{(\gamma,\delta)} \subseteq A$ and thus $A \in \mathcal{V}((\mathbf{e} - x)_{(\gamma,\delta)}) \cap \mathcal{Y}$.

Conversely, let $A \in \mathcal{V}((\mathbf{e}-x)_{(\gamma,\delta)}) \cap \mathcal{Y}$ then $(\mathbf{e}-x)_{(\gamma,\delta)} \subseteq A$ and $Img(A) = \{(1,0),(t,s)\}$ which implies that $\gamma \leq \mu_A(\mathbf{e}-x)$ and $\delta \geq \nu_A(\mathbf{e}-x)$. Hence $t < \mu_A(\mathbf{e}-x)$ and $s > \mu_A(\mathbf{e}-x)$ and thus $\mu_A(\mathbf{e} - x) = 1$ and $\nu_A(\mathbf{e} - x) = 0$. It follows that $\mathbf{e} - x \in A_*$ and hence $x \in A_*$ so that $\mu_A(x) = t < \gamma$ and $\nu_A(x) = s > \delta$. This means that $x_{(\gamma,\delta)} \nsubseteq A$ and thus $A \in \mathcal{X}(x_{(\gamma,\delta)}) \cap \mathcal{Y}$. Hence $\mathcal{X}(x_{(\gamma,\delta)}) \cap \mathcal{Y} = \mathcal{V}((\mathbf{e} - x)_{(\gamma,\delta)}) \cap \mathcal{Y}$.

(ii) If $A \in \mathcal{X}(x_{(\gamma.\delta)}) \cup \mathcal{X}(y_{(\gamma.\delta)})$ then $x_{(\gamma.\delta)} \nsubseteq A$ or $y_{(\gamma.\delta)} \nsubseteq A$ (which mean that $\mu_A(x) < \gamma$ and $\nu_A(x) > \delta$ or $\mu_A(y) < \gamma$ and $\nu_A(y) > \delta$). This implies that $x \notin A_*$ or $y \notin A_*$ and thus

$\mathbf{e} - x \notin A_*$ or $\mathbf{e} - y \notin A_*$. As a result, $(\mathbf{e} - x)\Gamma(\mathbf{e} - y) = \mathbf{e} - x - y + x\Gamma y \notin A_*$, so that $x + y - x\Gamma y \notin A_*$. Hence $A \in \mathcal{X}(z_{(\gamma.\delta)})$, where $z = x + y - x\Gamma y$.

(iii) Let $A, B \in \mathcal{X}, A \neq B$. Then A and B are IF prime ideals of M and $Img(A) = Img(B) = \{(1, 0), (t, s)\}$. As we know that every prime ideal in a Boolean Γ-ring is maximal ideal. It follows that A_*, B_* are maximal ideals of M. So $A_* \nsubseteq B_*$, since $A \neq B$. Choose $x \in A_*$ and $x \notin B_*$. Then $\mathbf{e} - x \in B_*$ and $\mathbf{e} - x \notin A_*$. Now, $\mu_B(x) = \mu_A(\mathbf{e} - x) = t$ and $\nu_B(x) = \nu_A(\mathbf{e} - x) = s$ and $\mu_A(x) = 1 = \mu_B(\mathbf{e} - x)$ and $\nu_A(x) = 0 = \nu_B(\mathbf{e} - x)$. Let $\alpha \in (t, 1)$ and $\beta \in (0, s)$ such that $\alpha + \beta \leq 1$. Then $\mu_{x_{(\alpha,\beta)}}(x) = \alpha > t = \mu_B(x)$ and $\nu_{x_{(\alpha,\beta)}}(x) = \beta < s = \nu_B(x)$ so that $x_{(\alpha,\beta)} \nsubseteq B$. Hence $B \in \mathcal{X}(x_{(\alpha,\beta)})$. Also, $\mu_{(\mathbf{e}-x)_{(\alpha,\beta)}}(\mathbf{e} - x) = \alpha > t = \mu_A(\mathbf{e} - x)$ and $\nu_{(\mathbf{e}-x)_{(\alpha,\beta)}}(\mathbf{e} - x) = \beta < s = \nu_A(\mathbf{e} - x)$, so that $(\mathbf{e} - x)_{(\alpha,\beta)} \nsubseteq A$. Hence $A \in \mathcal{X}((\mathbf{e} - x)_{(\alpha,\beta)})$. Then, by Theorem (3.6)(i), we have $\mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}((\mathbf{e} - x)_{(\alpha,\beta)}) = \mathcal{X}((x\Gamma(\mathbf{e} - x))_{(\alpha,\beta)}) = \mathcal{X}((0)_{(\alpha,\beta)}) = \emptyset$ [As M is Boolean Γ-ring]. Consequently, $\mathcal{Y}$ is Hausdorff. □

Theorem 4.9. *If M is Boolean Γ-ring, $t, s \in [0, 1)$ with $t + s \leq 1$ and $\mathcal{Y} = \{P \in \mathcal{X} : Img(P) = \{(1, 0), (t, s)\}\}$, then the space $\mathcal{Y}$ is compact, Hausdorff.*

Proof. Follows immediately from Proposition (3.3) and Theorem (4.8)(i),(iii). □

5. Intuitionistic fuzzy prime radical and algebraic nature of intuitionistic fuzzy prime ideal under Γ-homomorphism

Definition 5.12. ([22]) Let M be a Γ-ring. For any IFI A of M. The IFS $\sqrt{A}$ defined by

$$\mu_{\sqrt{A}}(x) = \vee\{\mu_A((x\gamma)^{n-1}x) : n \in \mathbf{N}\} \text{ and } \nu_{\sqrt{A}}(x) = \wedge\{\nu_A((x\gamma)^{n-1}) : n \in \mathbf{N})\}$$

is called the IF prime radical of A, where $(x\gamma)^{n-1}x = x$, for $n = 1, \gamma \in \Gamma$.
Further, $\sqrt{A}$ is the smallest IF semi-prime ideal of M containing A.

Proposition 5.7. ([22]) *For every IFIs A and B of Γ-ring M, we have*
(i) $A \subseteq \sqrt{A}$;
(ii) $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$;
(iii) $\sqrt{\sqrt{A}} = \sqrt{A}$.

Proposition 5.8. ([22]) *Let A be an $IFPI$ of a Γ-ring M. Then $\sqrt{A} = A$ and hence every $IFPI$ is IF semi prime ideal.*

Theorem 5.10. *Let A be any IFI of a Γ-ring M. Then*
(i) $\mathcal{V}(A) = \mathcal{V}(\sqrt{A})$
(ii) $\mathcal{X}(x_{(\alpha,\beta)}) = \mathcal{X}(y_{(\alpha,\beta)})$ if and only if $\sqrt{< x_{(\alpha,\beta)} >} = \sqrt{< y_{(\alpha,\beta)} >}$, where $\alpha, \beta \in (0, 1]$ with $\alpha + \beta \leq 1$.

Proof. (i) Suppose $B \in \mathcal{V}(A)$ be any element. Then $A \subseteq B$, where B is an IFPI of M, then from Proposition (5.8) we have $\sqrt{B} = B$, therefore we have $A \subseteq \sqrt{B}$. Hence $B \in \mathcal{V}(\sqrt{A})$, so that $\mathcal{V}(A) \subseteq \mathcal{V}(\sqrt{A})$. The reverse inclusion is clear-cut.

(ii)If $\mathcal{X}(x_{(\alpha,\beta)}) = \mathcal{X}(y_{(\alpha,\beta)})$, then $\mathcal{V}(x_{(\alpha,\beta)}) = \mathcal{V}(y_{(\alpha,\beta)})$ which implies $\mathcal{V}(< x_{(\alpha,\beta)} >) = \mathcal{V}(< y_{(\alpha,\beta)} >)$. This mean $\cap\{B : B \in \mathcal{V}(< x_{(\alpha,\beta)} >)\} = \cap\{B : B \in \mathcal{V}(< y_{(\alpha,\beta)} >)\}$ and therefore, $\sqrt{< x_{(\alpha,\beta)} >} = \sqrt{< y_{(\alpha,\beta)} >}$.

Conversely, let $\sqrt{< x_{(\alpha,\beta)} >} = \sqrt{< y_{(\alpha,\beta)} >}$. Then

$$\begin{aligned} B \in \mathcal{V}(x_{(\alpha,\beta)}) &\Leftrightarrow x_{(\alpha,\beta)} \subseteq B \\ &\Leftrightarrow < x_{(\alpha,\beta)} > \subseteq B \\ &\Leftrightarrow \sqrt{< x_{(\alpha,\beta)} >} \subseteq B \\ &\Leftrightarrow \sqrt{< y_{(\alpha,\beta)} >} \subseteq B \\ &\Leftrightarrow y_{(\alpha,\beta)} \subseteq B \text{ as before} \\ &\Leftrightarrow B \in \mathcal{V}(y_{(\alpha,\beta)}). \end{aligned}$$

Hence $\mathcal{V}(x_{(\alpha,\beta)}) = \mathcal{V}(y_{(\alpha,\beta)})$ so that $\mathcal{X}(x_{(\alpha,\beta)}) = \mathcal{X}(y_{(\alpha,\beta)})$.

It is prompt from above Theorem (5.10) that the topology τ is exactly the collection of all open sets $\mathcal{X}(A)$, where A runs over IF semi-prime ideals of M. □

Now we recall the following results for immediate use

Definition 5.13. ([18]) Let $f : M \to N$ be a function. An IFS A of M is called an f - invariant if $f(x) = f(y) \Rightarrow A(x) = A(y)$, i.e., $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$, where $x, y \in M$.

If A be any f - invariant IFS of M, then $f^{-1}(f(A)) = A$.

Theorem 5.11. ([18]) *Let $f : M \to N$ is a surjective Γ-homomorphism and A be any f - invariant IF prime ideal of M and B be any IF prime ideal of N. Then $f(A)$ and $f^{-1}(B)$ are IF prime ideal of N and M respectively.*

Theorem 5.12. *Let $f : M \to N$ is a surjective Γ-homomorphism and $\mathcal{X} = IFSpec(M)$, $\mathcal{X}' = IFSpec(N)$, $\mathcal{X}^* = \{A \in \mathcal{X} : A \text{ is } f\text{- invariant }\}$, $\mathcal{X}'(B) = \mathcal{X}' \setminus \mathcal{V}(B)$, where B is any IFI of N, and h be a map from $\mathcal{X}'$ to $\mathcal{X}^*$ defined by $h(A') = f^{-1}(A'), A' \in \mathcal{X}'$. Then the following considerations are equivalent*
(i) h is continuous
(ii) h is open, and
(iii) h is a homeomorphism of $\mathcal{X}'$ onto $\mathcal{X}^*$ in other words the map h is an embedding of $\mathcal{X}'$ onto $\mathcal{X}^*$.

Proof. (i) Let $A' \in \mathcal{X}'$. It follows from Theorem(5.11) that $f^{-1}(A') \in \mathcal{X}$.
Also, $f^{-1}(A')$ is f -invariant, since for all $a, b \in M$, if $f(a) = f(b)$, then $\mu_{A'}(f(a)) = \mu_{A'}(f(b))$ and $\nu_{A'}(f(a)) = \nu_{A'}(f(b)) \Rightarrow \mu_{f^{-1}(A')}(a) = \mu_{f^{-1}(A')}(b)$ and $\nu_{f^{-1}(A')}(a) = \nu_{f^{-1}(A')}(b)$, i.e., $f^{-1}(A')(a) = f^{-1}(A')(b)$. Hence $h(A') = f^{-1}(A') \in \mathcal{X}^*$.
Next we show that $h^{-1}(\mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}^*) = \mathcal{X}'((f(x))_{(\alpha,\beta)})$.
Since $A' \in h^{-1}(\mathcal{X}(x_{(\alpha,\beta)}) \Leftrightarrow h(A') \in \mathcal{X}(x_{(\alpha,\beta)})$
$\Leftrightarrow x_{(\alpha,\beta)} \nsubseteq h(A') = f^{-1}(A')$
$\Leftrightarrow (f(x))_{(\alpha,\beta)} = f(x_{(\alpha,\beta)}) \nsubseteq A'$, by Proposition (3.1)
$\Leftrightarrow A' \in \mathcal{X}'((f(x))_{(\alpha,\beta)})$.

This shows that the pre-image of any basic open set in $\mathcal{X}^*$ is open set in $\mathcal{X}'$. Hence h is continuous.

(ii) Let $\mathcal{X}'((f(x))_{(\alpha,\beta)}), x \in M$ and $\alpha, \beta \in (0, 1]$ with $\alpha + \beta \leq 1$, be any basic open set in $\mathcal{X}'$. Let $B \in \mathcal{X}'((f(x))_{(\alpha,\beta)})$. Then $B = h(A') = f^{-1}(A')$ for some $A' \in \mathcal{X}'$ such that $(f(x))_{(\alpha,\beta)} \nsubseteq A'$. As in part (1) we can show that B is f - invariant.

Next, $h(\mathcal{X}'((f(x))_{(\alpha,\beta)})) = \mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}^*$, because
$A \in h(X'((f(x))_{(\alpha,\beta)})) \Leftrightarrow h^{-1}(A) \in X'((f(x))_{(\alpha,\beta)})$ and A is f -invariant
$\Leftrightarrow f(x_{(\alpha,\beta)}) = (f(x))_{(\alpha,\beta)} \nsubseteq h^{-1}(A) = f(A)$
$\Leftrightarrow x_{(\alpha,\beta)} \nsubseteq f^{-1}(f(A)) = A$, since A is f-invariant
$\Leftrightarrow A \in \mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}^*$.
Thus the direct image of each basic open set in $\mathcal{X}'$ is open in $\mathcal{X}^*$ and so h is open.

(iii) In the light of part (i) and part (ii), it is enough to prove that h is one-one and onto. Let $A', B' \in \mathcal{X}'$. Then $h(A') = h(B') \Rightarrow f^{-1}(A') = f^{-1}(B') \Rightarrow f(f^{-1}(A')) = f(f^{-1}(B'))$. As f is onto, therefore, we get $A' = B'$. Thus f is one-one. Finally, let $A \in \mathcal{X}^*$. Then A is an f -invariant IF prime ideal of M and Therefore by Theorem (5.11), $f(A)$ is an IF prime ideal of N. Further, $h(f(A)) = f^{-1}(f(A)) = A$. Since A is f -invariant. Therefore h is onto. □

6. Irreducibility and Connectedness of IF Spec(M)

Recollect that a space is an irreducible if and only if the intersection of any two non-empty basic open sets is non-empty. Also it is disconnected if and only if it can be written as the union of two non-empty disjoint closed subsets.

Definition 6.14. The intersection of all IF prime ideals of M is called the IF nil radical of Γ-ring M and is written as $IFnil(M)$.

Theorem 6.13. *The space $\mathcal{X}$ is irreducible if and only if $IFnil(M) \in \mathcal{X}$.*

Proof. Let $\mathcal{X}$ be irreducible and let $\mathcal{N}$ be the nil radical of Γ-ring M. Then

$$\mu_{IFnil(M)}(x) = \begin{cases} 1, & \text{if } x \in \mathcal{N} \\ 0, & \text{if } M \setminus \mathcal{N} \end{cases}; \quad \nu_{IFnil(M)}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{N} \\ 1, & \text{if } M \setminus \mathcal{N} \end{cases}.$$

Next, let $x, y \in M$ and let $\alpha, \beta \in (0, 1]$ with $\alpha + \beta \le 1$. Then $x\gamma y \in \mathcal{N} \Rightarrow x\gamma y$ is nilpotent and thus $\mathcal{X}((x\gamma y)_{(\alpha,\beta)}) = \emptyset$ by Theorem (3.6)(ii). Therefore,$\mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}(y_{(\alpha,\beta)}) = \emptyset$, since $\mathcal{X}$ is irreducible. Hence either x or y is nilpotent, and thus $x \in \mathcal{N}$ or $y \in \mathcal{N}$. Consequently, $\mathcal{N}$ is prime ideal of M, whence it follows from Theorem (2.4) that $IFnil(M) \in \mathcal{X}$.

Conversely, assume that $IFnil(M) \in \mathcal{X}$. Then $\mathcal{N}$ is prime ideal of M. Let $x, y \in M$ and let $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta \le 1$. Then $\mathcal{X}(x_{(\alpha,\beta)}) \cap \mathcal{X}(y_{(\alpha,\beta)}) = \emptyset$ implies that $\mathcal{X}((x\Gamma y)_{(\alpha,\beta)}) = \emptyset$, by Theorem (3.6)(i), and thus $x\gamma y$ is nilpotent for every $\gamma \in \Gamma$, by Theorem (3.5)(ii). Then $x\gamma y \in \mathcal{N}$ and so $x \in \mathcal{N}$ or $y \in \mathcal{N}$, which means x is nilpotent or y is nilpotent. Hence $\mathcal{X}(x_{(\alpha,\beta)}) = \emptyset$ or $\mathcal{X}(y_{(\alpha,\beta)}) = \emptyset$, by Theorem (3.6)(ii). This shows that the intersection of any two non-empty basic open sets is non-empty. Hence, $\mathcal{X}$ is irreducible. □

Theorem 6.14. *The space $\mathcal{X}$ is disconnected if and only if M has a non-trivial idempotent element.*

Proof. Let $\mathcal{X}$ be disconnected. Then there exist IFIs A and B of M such that $\mathcal{X} = \mathcal{V}(A) \cup \mathcal{V}(B), \mathcal{V}(A), \mathcal{V}(B) \neq \emptyset, \mathcal{V}(A) \cap \mathcal{V}(B) = \emptyset$.

Now, $\mathcal{V}(A) \cap \mathcal{V}(B) = \emptyset$ implies $\mathcal{V}(A \oplus B) = \emptyset$ so that $\mu_{A\oplus B}(x) = 1$ and $\nu_{A\oplus B}(x) = 0$; for all $x \in M$. So, $Sup_{\mathbf{e}=m+n}\{max\{\mu_A(m), \mu_B(n)\}\} = 1$ and $Inf_{\mathbf{e}=m+n}\{min\{\nu_A(m), \nu_B(n)\}\} = 0$, where **e** is the unity of $M \Rightarrow \mu_A(m) = \mu_B(n) = 1$ and $\nu_A(m) = \nu_B(n) = 0$, for all $m, n \in M$ such that $\mathbf{e} = m + n$. Let $I = A_*$ and $J = B_*$. Let K be the prime ideal of M and χ_K be its intuitionistic fuzzy characteristic function. Then $\chi_K \in \mathcal{X}$. Since

$\mathcal{X} = \mathcal{V}(A) \cup \mathcal{V}(B) = \mathcal{V}(A \cap B)$, it follows that $A \cap B \subseteq \chi_K$.

Next, if $x \in I \cap J$, then $\mu_{A\cap B}(x) = 1$ and $\nu_{A\cap B}(x) = 0 \Rightarrow \mu_{\chi_K}(x) = 1$ and $\nu_{\chi_K}(x) = 0$ and then $x \in K$. Thus $x \in \cap\{K : K$ is a prime ideal of $M\}$. This implies that x is a nilpotent element. This shows that every element of $I \cap J$ is nilpotent.

Clearly, $M/(I\cap J) = I/(I\cap J) \oplus J/(I\cap J)$, Therefore, $\mathbf{e}+(I\cap J) = i+(I\cap J)+j+(I\cap J)$, for some $i \in I, j \in J$. So that $i\gamma(\mathbf{e} - i) \in (I \cap J)$ for every $\gamma \in \Gamma$ and hence $i\gamma(\mathbf{e} - i)$ is nilpotent. Thus $(i\gamma(\mathbf{e} - i)\gamma)^m i\gamma(\mathbf{e} - i) = 0$ for some $m \in Z^+$. Consequently, $(i\gamma(\mathbf{e} - i)\gamma)^m = (i\gamma(\mathbf{e} - i)\gamma)^{m+1}Q((i\gamma(\mathbf{e} - i)))$, for some polynomial $Q(i\gamma(\mathbf{e} - i))$ in $(i\gamma(\mathbf{e} - i))$. Let $x = (i\gamma(\mathbf{e} - i)\gamma)^m Q(i\gamma(\mathbf{e} - i))$. It is now simple matter to verify that $x \neq 0, x \neq \mathbf{e}$, and $x\gamma x = x$.

Conversely, for any non-trivial idempotent element x of M, it can be easily verified that $\mathcal{X} = \mathcal{V}(x_{(\alpha,\beta)}) \cup \mathcal{V}((\mathbf{e} - x)_{(\alpha,\beta)}), \mathcal{V}(x_{(\alpha,\beta)}) \neq \emptyset, \mathcal{V}((\mathbf{e} - x)_{(\alpha,\beta)}) \neq \emptyset$,
$\mathcal{V}(x_{(\alpha,\beta)}) \cap \mathcal{V}((\mathbf{e} - x)_{(\alpha,\beta)}) = \emptyset$, where $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta \leq 1$.
This establishes that $\mathcal{X}$ is disconnected. □

Corollary 6.1. *The space $\mathcal{X}$ is connected if and only if 0_M and **e** are the only idempotent in M.*

7. Conclusions

In this paper we have constituted a topology on $\mathcal{X} = IFSpec(M)$, the collection of all intuitionistic fuzzy prime ideals of a commutative Γ-ring M with unity, which is called Zariski topology. By using the bases for the Zariski topology, it is shown that the subspace $\mathcal{Y}$ of $\mathcal{X}$ is compact. Further the space $\mathcal{X}$ is always T_0 but not T_1 and hence not T_2, however when M is a Boolean Γ-ring, then we have constructed a subspace which is T_2 space. We have also shown that subspace $\mathcal{Y}$ is T_1 if and only if every singleton element of $\mathcal{Y}$ is IF maximal ideal of M. Further for a homomorphism f from a Γ-ring M onto a Γ-ring N, it is shown that $\mathcal{X}' = IFSpec(N)$ is homeomorphic to the subset $\mathcal{X}^* = \{A \in \mathcal{X} :$ A is f- invariant $\}$ consisting of f-invariant elements of $\mathcal{X} = IFSpec(M)$. Also, the space $\mathcal{X}$ is irreducible if and only if the intersection of all the elements of $\mathcal{X}$ is also an element of $\mathcal{X}$. However the space $\mathcal{X}$ is connected if and only if 0_M and **e** are the only idempotent elements in M.

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