CREAT. MATH. INFORM.
Volume **32** (2023), No. 1,
Pages 01 - 11

Online version at `https://semnul.com/creative-mathematics/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2023.01.01

Fixed point results of $\mathfrak{R}$-Enriched Interpolative Kannan pair in $\mathfrak{R}$-convex metric spaces

MUJAHID ABBAS, RIZWAN ANJUM and SHAKEELA RIASAT

ABSTRACT. The purpose of this paper is to introduce the class of $\mathfrak{R}$-enriched interpolative Kannan pair and proved a common fixed point result in the context of $\mathfrak{R}$-complete convex metric spaces. Some examples are presented to support the concepts introduced herein. Moreover, we study the well-posedness, limit shadowing property and Ulam-Hyers stability of the mappings introduced herein. Our result extend and generalize several comparable results in the existing literature.

1. INTRODUCTION AND PRELIMINARIES

The study of the existence and approximation of the solutions of nonlinear functional equations such as differential equations, integral equations, integro-differential equations is always been a source of great interest for mathematicians. Fixed point theory plays a vital role in this aspect. The problem of finding the solutions of a functional equation can be shifted to finding the fixed point of a suitable mapping defined on a set endowed with a certain structure, that is, finding the solution of the fixed point equation given by

$$x = Tx, \tag{1.1}$$

where T is defined on a certain space X. An equation (1.1) is solved by applying some appropriate fixed point results.

Banach contraction mapping principle [10] is one of the most useful results for approximation of the solution of (1.1), if a mapping $T : X \to X$ on a complete metric space X, satisfies the following condition:

$$d(Tx, Ty) \leq cd(x, y), \text{ for all } x, y \in X, \tag{1.2}$$

where $c \in [0, 1)$. The mapping satisfying the above condition by taking $c = 1$ is called nonexpansive.

Note that, a mapping which satisfies the above condition is uniformly continuos mapping. Kannan [25] proved a fixed point result which is applicable to the equation (1.1) where the mapping defined on a complete metric space need not be continuous. In this way, Kannan fixed point result extends the scope of Banach contraction mapping principle [10]. Kannan [25] replaced the condition (1.2) with the following:

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)], \tag{1.3}$$

for all $x, y \in X$, where $a \in [0, 0.5)$. It was shown that, an equation (1.3) involving the Kannan mapping on a complete metric space, has a unique solution. This paper was a genesis for a multitude of fixed point papers over the next two decades, see ([24], [25], [26], [32], [37], [38]) and references mentioned therein.

Note that Kannan fixed point result [25] is based on the generalization of the condition

Received: 09.04.2022. In revised form: 26.09.2022. Accepted: 04.10.2022
2000 *Mathematics Subject Classification.* 47H09, 47H10, 47J25, 54H25.
Key words and phrases. *Fixed point, enriched Kannan contraction, interpolative Kannan type contraction, convex metric spaces, well-posedness, limit shadowing property, Ulam-Hyers stability.*
Corresponding author: Rizwan Anjum; rizwan.anjum@riphah.edu.pk

(1.2). The same applies to the most of results in metric fixed point theory. One of such results proved by Karapinar [28], who employed the technique of interpolation and introduced a new class of interpolative Kannan type mappings defined on a complete metric space. Let us recall the following definition:
A mapping $T : X \to X$ is called an interpolative Kannan type if there exists $a \in [0, 1)$ such that for all $x, y \in X \setminus Fix(T)$, we have

$$d(Tx, Ty) \leq a[d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha}, \tag{1.4}$$

where $\alpha \in (0, 1)$ and $Fix(T) = \{x \in X; \; x = Tx\}$. For more results see ([20],[26], [27],[34]) and references mentioned therein.
The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. Common fixed point results are useful in finding the simultaneous solution of a certain system of two nonlinear functional equations. Noorwali [36] extended the condition (1.4) to two mappings and proved a common fixed point result for such mappings given as: Let (X, d) be a complete metric space and $T, S : X \to X$ be the self mappings. If there exists $a \in [0, 1)$ such that for all $x, y \in X \setminus \{Fix(T), Fix(S)\}$, we have

$$d(Tx, Sy) \leq a[d(x, Tx)]^{\alpha}[d(y, Sy)]^{1-\alpha},$$

where $\alpha \in (0, 1)$. Then, T and S has a unique common fixed point in X.
For more results in this direction, we refer to [5], [8], [9], [19], [21], [22], [29], [30], [31], [36] and references mentioned therein.

On the other hand, authors [2], [3], [4], [7], [12], [13], [14], [15], [16],[17] used the technique of enrichment of contractive mappings in the setting of Banach spaces.
In 2021, Rizwan and Abbas [6] introduced the class of (a, b, c)-modified enriched Kannan pair (T, S) in the setting of Banach space. A pair of self mappings (T, S) is called $(a, b, 0)$-modified enriched Kannan pair on a normed space $(X, \|\cdot\|)$, if there exist $b \in [0, +\infty)$ and $a \in [0, 0.5)$ such that

$$\|b(x - y) + Tx - Sy\| \leq a\big[\, \|x - Tx\| + \|y - Sy\| \,\big],$$

holds for all $x, y \in X$. It was proven that any $(a, b, 0)$-modified enriched Kannan pair (T, S) in the setting of Banach space X admits a unique common fixed point.
An equation (1.1) involving a nonexpansive mapping defined on a complete metric space may have no solution or have infinitely many solutions. A fundamental problem in fixed point theory of nonexpansive mappings is to find conditions under which an equation (1.1) has a solution. It is intimately connected with differential equations and with the geometry of the Banach spaces. The techniques used to prove the existence of the solution of non expansive fixed point equations are different than those considered in metric fixed point theory for mappings satisfying certain contraction conditions. The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. Geometrical properties of Banach spaces which mainly use convexity hypothesis, play key role in solving an equation (1.1) involving a nonexpansive mapping. The results obtained in this direction were starting point of a new mathematical field : the application of the geometric theory of Banach spaces to fixed point theory.

In 1970, Takahashi [41] introduced a notion of convexity structure in a metric space with the aim of studying the fixed point problem for nonexpansive mappings in such spaces.

Definition 1.1. [41] Let (X, d) be a metric space. A continuous function $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if, for all $x, y \in X$ and for any $\lambda \in [0, 1]$, we have

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y), \text{ for all } u \in X.$$

A metric space (X, d) endowed with a convex structure W is called a Takahashi convex metric space and is usually denoted by (X, d, W). Obviously, any linear normed space and each of its convex subsets are convex metric spaces with the natural convex structure given by

$$W(x, y; \lambda) = \lambda x + (1-\lambda) y, \text{ for all } x, y \in X.$$

However, the converse does not hold in general. There are many examples of convex metric spaces which cannot be embedded in any normed space. Let us consider the following example.

Example 1.1. [1] Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X and $\alpha \in [0, 1]$. Define a mapping $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \alpha) = \left(\alpha x_1 + (1-\alpha) y_1, \frac{\alpha x_1 x_2 + (1-\alpha) y_1 y_2}{\alpha x_1 + (1-\alpha) y_1}\right),$$

and a metric $d : X \times X \to [0, +\infty)$ by $d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|$. It can be verified that X is a convex metric space but not a normed space.

By using the above idea of Takahashi [41], Berinde and Păcurar [15], in 2021, proposed a new class of enriched contractions in a more generalized setting of convex metric spaces which is major inspiration for this manuscript.
We need the following lemma in the sequel.

Lemma 1.1. [15] *Let (X, d, W) be a convex metric space and $T : X \to X$. Define the mapping $T_\lambda : X \to X$ by*

$$T_\lambda x = W(x, Tx; \lambda). \tag{1.5}$$

Then, for any $\lambda \in [0, 1)$, we have

$$Fix(T) = Fix(T_\lambda).$$

Remark 1.1. Since, convex structure W is a continuous mapping on $X \times X \times [0, 1]$, then by (1.5), we can say that T_λ is also a continuous mapping on X.

It is a matter of great interest to study the necessary conditions which ensure the existence and uniqueness of the solution of an equation (1.1) where the mapping is defined on a space equipped with some order structure in addition to the distance structure. Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [39] and then by Nieto and López [35]. These results weakened the underlying spaces and added a new dimension to metric fixed point theory: the bridging between to metric and ordered fixed point theories.
In 2017, Eshaghi et al. [23], gave the idea of orthogonal relation on metric spaces which was generalized by Rahimi et al. [33], in 2020, by introducing the concept of $\mathfrak{R}$-metric spaces, where $\mathfrak{R}$ is an arbitrary relation on a metric space X. Formally, any relation $\mathfrak{R}$ on two sets E and F is the subset of the cartesian product between E and F.
We now give the notion of $\mathfrak{R}$-convex metric space as follows:

Definition 1.2. Suppose (X, d, W) is a convex metric space and $\mathfrak{R}$ is an arbitrary relation on X. Then $(X, d, W, \mathfrak{R})$ is called an $\mathfrak{R}$-convex metric space.

Example 1.2. Let $X = \mathbb{R} \times \mathbb{R}$, $a = (\alpha, \beta)$, and $b = (\zeta, \eta)$ be two arbitrary points in X. Define a relation $\mathfrak{R} := \leq$ on X as $a \leq b$ if and only if $\alpha \leq \zeta$ and $\beta \leq \eta$. Let $d : X \times X \to \mathbb{R}$ be the metric given by

$$d(a, b) = \begin{cases} \sqrt{(\alpha-\zeta)^2 + (\beta-\eta)^2} & ;\text{if } a \mathfrak{R} b \\ \max\{|\alpha-\zeta|, |\beta-\eta|\} & ;\text{otherwise.} \end{cases}$$

Convex structure W on X is given by $W(a, b; \lambda) = (\lambda\alpha + (1-\lambda)\zeta, \lambda\beta + (1-\lambda)\eta)$. Then one can easily see that X is an $\mathfrak{R}$-convex metric space.

Definition 1.3. A sequence $\{x_n\}$ in an $\mathfrak{R}$-convex metric space is called an $\mathfrak{R}$-sequence if $x_n\mathfrak{R}x_{n+k}$ for each $n, k \in \mathbb{N}$.

Definition 1.4. An $\mathfrak{R}$-sequence $\{x_n\}$ is said to converges to $x \in X$ if for every $\epsilon > 0$ and $n \in \mathbb{N}$, there is an integer n_0 such that $d(x_n, x) < \epsilon$ for $n \geq n_0$.

Definition 1.5. An $\mathfrak{R}$-sequence $\{x_n\}$ is said to be an $\mathfrak{R}$-Cauchy sequence if for every $\epsilon > 0$ and $m, n \in \mathbb{N}$, there is an integer n_0 such that $d(x_n, x_m) \leq \epsilon$ for $n, m \geq n_0$.

Definition 1.6. Let $(X, d, W, \mathfrak{R})$ be an $\mathfrak{R}$-convex metric space. Then it is called an $\mathfrak{R}$-complete convex metric space if every $\mathfrak{R}$-Cauchy sequence converges in X.

Employing the ideas of Berinde and Păcurar [15], Rizwan and Abbas [6], Karapinar [28] and Rahimi [33], we introduce the concept of $\mathfrak{R}$-enriched interpolative Kannan pair and prove some fixed point results in the setting of $\mathfrak{R}$-complete convex metric spaces. Moreover, we study the well-posedness, limit shadowing property and Ulam-Hyers stability of the mappings introduced herein.

2. Main Results

We first introduce the following concept.

Definition 2.7. Let $(X, d, W, \mathfrak{R})$ be an $\mathfrak{R}$-convex metric space. A pair of mappings $T, S : X \to X$ is said to be enriched interpolative Kannan pair if there exist $a, \lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(W(x, Tx; \lambda), W(y, Sy; \lambda)) \leq a[d(x, W(x, Tx; \lambda))]^{\alpha}[d(y, W(y, Sy; \lambda))]^{1-\alpha}, \tag{2.6}$$

holds for all $x\mathfrak{R}y$, where $x, y \in X$.
To specify the parameters a, λ and α and relation $\mathfrak{R}$ in (2.6), we also call (T, S) a $(a, \alpha, \lambda, \mathfrak{R})$-enriched interpolative Kannan pair.

We start with the following common fixed point result.

Theorem 2.1. *Suppose that $(X, d, W, \mathfrak{R})$ is an $\mathfrak{R}$-complete convex metric space and (T, S) is $(a, \alpha, \lambda, \mathfrak{R})$-enriched interpolative Kannan pair. Suppose the following conditions hold:*

i) there exists $x_0 \in X$ such that $x_0\mathfrak{R}W(x_0, Tx_0; \lambda)$,
ii) if for all $x, y \in X$ such that $x\mathfrak{R}y$,

$$\Rightarrow \quad W(x, Tx; \lambda)\mathfrak{R}W(y, Sy; \lambda) \text{ or } W(x, Sx; \lambda)\mathfrak{R}W(y, Ty; \lambda). \tag{2.7}$$

Then S and T have common fixed point $p \in X$. Moreover, the iterative sequence $\{x_n\}_{n=0}^{+\infty}$ defined by

$$x_{2n+1} = W(x_{2n}, Tx_{2n}; \lambda), \quad x_{2n+2} = W(x_{2n+1}, Sx_{2n+1}; \lambda) \ \ n \geq 0, \tag{2.8}$$

converges to the point p.

Proof. Using Lemma 1.1, the condition (2.6) is transformed to following equivalent form

$$d(T_\lambda x, S_\lambda y) \leq a[d(x, T_\lambda x)]^{\alpha}[d(y, S_\lambda y)]^{1-\alpha}, \tag{2.9}$$

holds for all $x\mathfrak{R}y$. That is, a pair (T_λ, S_λ) becomes an interpolative Kannan pair. Note that the equation (2.8) becomes

$$x_{2n+1} = T_\lambda x_{2n},$$
$$x_{2n+2} = S_\lambda x_{2n+1}, \quad n \geq 0.$$

Without any loss of generality, we assume that the successive terms of $\{x_n\}$ are distinct. Otherwise, we are done. Since, $x_0 \Re T_\lambda x_0 = x_1$ by (2.7), we have

$$(x_1 = T_\lambda x_0)\Re(S_\lambda x_1 = x_2).$$

Continuing this way, we obtain that

$$x_1 \Re x_2 \Re \cdots \Re x_n \Re x_{n+1} \Re \cdots$$

Thus, $\{x_n\}_{n=0}^{+\infty}$ is an $\Re$-sequence in $(X, d, W, \Re)$.
Take $x = x_{2n}$ and $y = x_{2n+1}$ in (2.9), we get

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(T_\lambda x_{2n}, S_\lambda x_{2n+1}) \\
&\leq a[d(x_{2n}, T_\lambda x_{2n})]^\alpha [d(x_{2n+1}, S_\lambda x_{2n+1})]^{1-\alpha} \\
&\leq a[d(x_{2n}, x_{2n+1})]^\alpha [d(x_{2n+1}, x_{2n+2})]^{1-\alpha} \\
[d(x_{2n+1}, x_{2n+2})]^\alpha &\leq a[d(x_{2n}, x_{2n+1})]^\alpha \\
&\leq ad(x_{2n}, x_{2n+1}) \\
&\leq a^{2n+1} d(x_0, x_1). \qquad (2.10)
\end{aligned}$$

Similarly, for $x = x_{2n}$ and $y = x_{2n-1}$ in (2.9), we obtain that

$$\begin{aligned}
d(x_{2n+1}, x_{2n}) &= d(T_\lambda x_{2n}, S_\lambda x_{2n-1}) \\
&\leq a[d(x_{2n}, T_\lambda x_{2n})]^\alpha [d(x_{2n-1}, S_\lambda x_{2n-1})]^{1-\alpha} \\
&\leq a[d(x_{2n}, x_{2n+1})]^\alpha [d(x_{2n-1}, x_{2n})]^{1-\alpha} \\
[d(x_{2n+1}, x_{2n})]^{1-\alpha} &\leq a[d(x_{2n}, x_{2n-1})]^{1-\alpha} \\
&\leq ad(x_{2n}, x_{2n-1}) \\
&\leq a^{2n} d(x_0, x_1). \qquad (2.11)
\end{aligned}$$

It follows from(2.10) and (2.11) that

$$d(x_n, x_{n+1}) \leq a^n d(x_0, x_1).$$

For $r > 0$, we have

$$\begin{aligned}
d(x_n, x_{n+r}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \ldots + \\
&\quad\; d(x_{n+r-1}, x_{n+r}) \\
&\leq (a^n + a^{n+1} + a^{n+2} + \ldots + a^{n+r-1}) d(x_0, x_1) \\
&\leq \frac{a^n}{1-a} d(x_0, x_1).
\end{aligned}$$

On taking limit as $n \to +\infty$, we can deduce that $\{x_n\}_{n=0}^{+\infty}$ is an $\Re$-Cauchy sequence. Since X is $\Re$-complete, there exists $p \in X$ such that $\lim_{n\to+\infty} x_n = p$ which implies

$$\lim_{n\to+\infty} x_{2n} = p. \qquad (2.12)$$

By using (2.12) and the $\Re$-continuity of T_λ (which follows from Remark 1.1 and Definition 1.1), we immediately obtain

$$\lim_{n\to+\infty} T_\lambda x_{2n} = T_\lambda p.$$

Therefore,

$$T_\lambda p = T_\lambda(\lim_{n\to+\infty} x_{2n}) = \lim_{n\to+\infty} T_\lambda(x_{2n}) = \lim_{n\to+\infty} x_{2n+1} = p,$$

i.e.,

$$p \in Fix(T_\lambda) = Fix(T).$$

By the similar arguments p is the fixed point of S as well.

$\square$

For more illustration, consider the following examples.

Example 2.3. Let $X = \mathbb{R}^2$ and $D = \{(x, y) \in \mathbb{R}^2 : \ x = y\}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X and $\alpha \in [0, 1]$. Define a mapping $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \alpha) = (\alpha x_1 + (1 - \alpha)y_1, \alpha x_2 + (1 - \alpha)y_2),$$

and a metric $d : X \times X \to [0, +\infty)$ by $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$.
Define a relation $\mathfrak{R}$ on X as $x\mathfrak{R}y$ if and only if $x, y \in D$. It can be easily seen that the $(X, d, W, \mathfrak{R})$ is an $\mathfrak{R}$-complete convex metric space. Define the mappings $T, S : X \to X$ by

$$T(x) = \begin{cases} (x_1, 2x_1 - x_2) & \text{if } x = (x_1, x_2) \notin D \\ (-x_1, -x_2) & \text{if } x = (x_1, x_2) \in D, \end{cases}$$

and

$$S(x) = \begin{cases} (6x_2 - x_1, 5x_2) & \text{if } x = (x_1, x_2) \notin D \\ (-x_1, -x_2) & \text{if } x = (x_1, x_2) \in D. \end{cases}$$

On the other hand, we have

$$W(x, Tx; 0.5) = \begin{cases} (x_1, x_1) & \text{if } x = (x_1, x_2) \notin D \\ (0, 0) & \text{if } x = (x_1, x_2) \in D, \end{cases} \tag{2.13}$$

and

$$W(x, Sx; 0.5) = \begin{cases} (3x_2, 3x_2) & \text{if } x = (x_1, x_2) \notin D \\ (0, 0) & \text{if } x = (x_1, x_2) \in D. \end{cases} \tag{2.14}$$

If we choose $x_0 = (0, 0)$, then

$$(0, 0)\mathfrak{R}W(x_0, Tx_0, 0.5) = (0, 0).$$

By using (2.13), (2.14) and $x\mathfrak{R}y$ we have

$$\begin{aligned} (0, 0) = W(x, Tx, 0.5) \quad &\mathfrak{R} \quad W(y, Sy, 0.5) = (0, 0), \text{ or} \\ (0, 0) = W(x, Sx, 0.5) \quad &\mathfrak{R} \quad W(y, Ty, 0.5) = (0, 0). \end{aligned}$$

As a result, the assumptions i) and ii) of Theorem 2.1 hold.

Note that (T, S) is an $(0.5, 0.5, 0.5, \mathfrak{R})$-enriched interpolative Kannan pair.
Indeed, for $x = (x_1, x_2)\mathfrak{R}y = (y_1, y_2)$, we have

$$\begin{aligned} d(W(x, Tx; 0.5), W(y, Sy; 0.5)) &= 0, \\ d(x, W(x, Tx; 0.5)) &= 2|x|, \text{ and} \\ d(y, W(y, Sy; 0.5)) &= 2|y|. \end{aligned}$$

Therefore, for all $x\mathfrak{R}Sy$ we have

$$\begin{aligned} d(W(x, Tx; \lambda), W(y, Sy; \lambda)) &= 0 \\ &\leq a[d(x, W(x, Tx; \lambda))]^{0.5}[d(y, W(y, Sy; \lambda))]^{0.5} \\ &= 0.5(2|x|)^{0.5}(2|y|)^{0.5}. \end{aligned}$$

Thus, all the conditions of Theorem 2.1 are satisfied. Hence, $x = (0, 0)$ is the common fixed point of T and S.

Corollary 2.1. *Let (X, d, W) be a complete convex metric space and (T, S) be (a, α, λ)-enriched interpolative Kannan pair. Then the pair (T, S) has a unique common fixed point.*

Proof. Define the relation $\mathfrak{R}$ on X as $\mathfrak{R} = X \times X$ in Theorem 2.1, then the result follows from Theorem 2.1 . To prove the uniqueness, let p and q be two distinct common fixed points of the mappings T and S and by Lemma 1.1, also of T_λ and S_λ, then by (2.9)

$$d(p,q) = d(T_\lambda p, S_\lambda q) \leq a[d(p, T_\lambda p)]^\alpha [d(q, S_\lambda q)]^{1-\alpha} = 0$$

Hence, $p = q$. □

Corollary 2.2. *Let $(X, \|\cdot\|)$ be Banach space and $T, S : X \to X$ be a mappings satisfying*

$$\|b(x-y) + (Tx - Sy)\| \leq a \|x - Tx\|^\alpha \|y - Sy\|^{1-\alpha}, \tag{2.15}$$

is satisfied for all $x, y \in X$ such that $Tx \neq x$ whenever $Sy \neq y$, with $b \in [0, +\infty)$, $a \in [0, 1)$ and $\alpha \in (0, 1)$. Then S and T have a unique common fixed point.

Proof. We choose $\lambda = \frac{1}{b+1}$, then condition (2.15) becomes

$$\|(1-\lambda)(x-y) + \lambda(Tx - Sy)\| \leq a\lambda \|x - Tx\|^\alpha \|y - Sy\|^{1-\alpha} \quad \text{for all } x, y \in X.$$

which can be written in an equivalent form as:

$$\|T_\lambda x - S_\lambda y\| \leq a \|x - T_\lambda x\|^\alpha \|y - S_\lambda y\|^{1-\alpha}, \quad \text{for all } x, y \in X. \tag{2.16}$$

If we define convex structure W as $W(x, y; \alpha) = (1-\alpha)x + \alpha y$ on a Banach space X equipped with a relation $\mathfrak{R} = X \times X$. Since X is Banach space, it is $\mathfrak{R}$-Banach convex space. The inequality (2.16) and we have that the mappings $S_\lambda, T_\lambda : X \to X$ defined by (1.5) suggest that (T, S) a $(a, \alpha, \lambda, \mathfrak{R})$-enriched interpolative Kannan pair. Corollary 2.1 leads to the conclusion.

□

As a corollary of our result (Corollary 2.2), we can obtain Theorem 2.1 of [12].

Corollary 2.3. [12] *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be an (b, a)-enriched Kannan contraction, that is a mapping satisfying:*

$$||b(x-y) + Tx - Ty|| \leq a\big[||x - Tx|| + ||y - Ty||\big] \text{ for all } x, y \in X, \tag{2.17}$$

with $b \in [0, +\infty)$ and $a \in [0, 1/2)$. Then, T has a unique fixed point.

Proof. By taking $\lambda = \frac{1}{b+1}$, in (2.17) we get,

$$\|(1-\lambda)(x-y) + \lambda(Tx - Ty)\| \leq a\lambda\big[\, \|x - Tx\| + \|y - Ty\|\,\big] \quad \text{for all } x, y \in X,$$

which can be written in an equivalent form as:

$$\|T_\lambda x - T_\lambda y\| \leq a\big[\, \|x - T_\lambda x\| + \|y - T_\lambda y\|\,\big], \text{ for all } x, y \in X. \tag{2.18}$$

Therefore, by (2.18), T_λ is a Kannan contraction (1.3). It follows from [36], mapping T_λ satisfying (2.18) also satisfies (2.16). Since for value $\lambda = \frac{1}{b+1}$, the inequality (2.16) is equivalent to (2.15), hence result follows from Corollary 2.2.

□

As a corollary of our result (Corollary 2.2), we can obtain Theorem 2.1 of [36].

Corollary 2.4. [36] *Let $(X, \|\cdot\|)$ be a Banach space and $T, S : X \to X$ be a mappings satisfying*

$$\|Tx - Sy\| \leq a \|x - Tx\|^\alpha \|y - Sy\|^{1-\alpha},$$

for all $x, y \in X$ such that $Tx \neq x$ whenever $Ty \neq y$, with $a \in [0, 1)$ and $\alpha \in (0, 1)$. Then S and T have a unique common fixed point.

Proof. For $b = 0$, Corollary 2.2 leads to the conclusion. □

3. Well-Posedness, Limit Shadowing Property and Ulam-Hyers Stability

Now we will present the well-posedness, limit shadowing property and Ulam-Hyers stability results for the Corollary 2.1.

3.1. **Well-Posedness.** Let us start with the following definition.

Definition 3.8. Let (X, d, W) be a convex metric space and $T, S : X \to X$. The common fixed point problem of a pair (T, S) is said to be well-posed if $Fix(T) = Fix(S) = \{p\}$ (say) and for any sequence $\{x_n\}_{n=0}^{+\infty}$ in X satisfying

$$\begin{aligned}\lim_{n\to+\infty} d(W(x_n, Tx_n, \lambda), x_n) &= 0 \text{ and}\\ \lim_{n\to+\infty} d(x_n, W(x_n, Sx_n, \lambda)) &= 0,\end{aligned}$$

we have $\lim_{n\to+\infty} x_n = p$.

Since by Lemma 1.1 $Fix(T) = Fix(T_\lambda)$, and $Fix(S) = Fix(S_\lambda)$, we conclude that the common fixed point problem of (T, S) is well-posed if and only if the fixed point problem of (T_λ, S_λ) is well-posed.
If we take $S = T$ in the above definition, we have a well-posedness of a fixed point problem of a mapping T. Well-posedness of certain fixed point problems has been studied by several mathematicians, see for example, [18], [40] and references mentioned therein.
We now study the well-posedness of a common fixed point problem of mappings in Theorem 2.1.

Theorem 3.2. *Let (X, d, W) be a complete convex metric space. Suppose that T, S are mappings on X as in the Corollary 2.1. Then, common fixed point problem of a pair (T, S) is well-posed.*

Proof. It follows from Corollary 2.1, that $Fix(T) = Fix(S) = \{p\}$. Let $\{x_n\}_{n=0}^{+\infty}$ be any sequence in X such that $\lim_{n\to+\infty} d(W(x_n, Tx_n, \lambda), x_n) = 0$ and $\lim_{n\to+\infty} d(x_n, W(x_n, Sx_n, \lambda)) = 0$. Then, from triangular inequality and (2.9), we have

$$\begin{aligned}d(x_n, p) &\le d(x_n, W(x_n, Sx_n, \lambda)) + d(W(x_n, Sx_n, \lambda), p)\\ &= d(x_n, S_\lambda x_n) + d(T_\lambda p, S_\lambda x_n)\\ &\le d(x_n, S_\lambda x_n) + a[d(p, T_\lambda p)]^\alpha [d(x_n, S_\lambda x_n)]^{1-\alpha}.\end{aligned}$$

On taking limit as $n \to +\infty$ on both sides of the above inequality, we obtain that $d(x_n, p) = 0$. Hence the given common fixed point problem of a pair (T, S) is well-posed □

3.2. **Limit Shadowing Property.**

Definition 3.9. Let (X, d, W) be a convex metric space and $T, S : X \to X$. The common fixed point problem of a pair (T, S) is said to have the limit shadowing property if for any sequence $\{x_n\}_{n=0}^{+\infty}$ in X with $\lim_{n\to+\infty} d(W(x_n, Tx_n, \lambda), x_n) = 0$ and $\lim_{n\to+\infty} d(x_n, W(x_n, Sx_n, \lambda)) = 0$, there exists $z \in X$ such that

i) $\lim_{n\to+\infty} d(W(z, T^n z, \lambda), x_n) = 0$
ii) $\lim_{n\to+\infty} d(x_n, W(z, S^n z, \lambda)) = 0$

Theorem 3.3. *Let (X, d, W) be a complete convex metric space. Suppose that T, S are mappings on X as in the Corollary 2.1. Then, The common fixed point problem of a pair (T, S) has limit shadowing property.*

Proof. For any $z \in X$, consider

$$\begin{aligned} d(W(z, T^n z, \lambda), x_n) &= d(T_\lambda^n z, x_n) \\ &\leq d(T_\lambda^n z, S_\lambda x_n) + d(S_\lambda x_n, x_n) \\ &\leq a[d(T_\lambda^{n-1} z, T_\lambda^n z)]^\alpha [d(x_n, S_\lambda x_n)]^{1-\alpha}. \end{aligned}$$

On taking limit as $n \to +\infty$ on both sides of the above inquality, we get $d(T_\lambda^n z, x_n) = 0$. Similarly, we have $\lim_{n\to+\infty} d(x_n, W(z, S^n z, \lambda)) = 0$. Hence the given common fixed point problem of a pair (T, S) has the limit shadowing property. □

3.3. **Ulam-Hyers Stability.** Let (X, d, W) be a convex metric space, $T, S : X \to X$ and $\epsilon > 0$. A point $w^* \in X$ called an ϵ-solution of the common fixed point problem for a pair (T, S), if w^* satisfies the following inequalities

$$d(w^*, W(w^*, Tw^*, \lambda)) \leq \epsilon.$$

and,

$$d(w^*, W(w^*, Sw^*, \lambda)) \leq \epsilon.$$

Now we will give the notion of Ulam-Hyers stability in convex metric space.

Definition 3.10. Let (X, d, W) be a convex metric space, $T, S : X \to X$ and $\epsilon > 0$. The common fixed point problem of a pair (T, S) is called Ulam-Hyers stable if and only if for each ϵ-solution $w^* \in X$ of a pair (T, S), there exists a common solution x^* of the pair (T, S) in X such that

$$d(x^*, w^*) \leq c\epsilon.$$

Theorem 3.4. *Let (X, d, W) be a complete convex metric space. Suppose that T, S be mappings on X as in the Corollary 2.1. Then, the common fixed point problem of a pair (T, S) is Ulam-Hyers stable.*

Proof. Since $Fix(T) = Fix(T_\lambda)$ and $Fix(S) = Fix(S_\lambda)$, it follows that the common fixed point problem for the pair (T, S) is equivalent to the common fixed point problem for pair (T_λ, S_λ). Let w^* be ϵ-solution of the common fixed point problem of a pair (T, S) , that is,

$$d(w^*, W(w^*, Tw^*, \lambda)) \leq \epsilon,$$

and,

$$d(w^*, W(w^*, Sw^*, \lambda)) \leq \epsilon,$$

which is equivalent to,

$$d(w^*, T_\lambda w^*) \leq \epsilon.$$

and,

$$d(w^*, S_\lambda w^*) \leq \epsilon. \tag{3.19}$$

respectively. Using (2.9) and (3.19), we get

$$\begin{aligned} d(x^*, w^*) = d(T_\lambda x^*, w^*) &\leq d(T_\lambda x^*, S_\lambda w^*) + d(S_\lambda w^*, w^*) \\ &\leq a[d(x^*, T_\lambda x^*)]^\alpha [d(w^*, S_\lambda w^*)]^{1-\alpha} + d(S_\lambda w^*, w^*) \\ &\leq \epsilon \end{aligned}$$

Hence the common fixed point problem of a pair (T, S) is Ulam-Hyers stable. □

4. Conclusions

i) We introduced a more generalized class of contractive mappings, called $\mathfrak{R}$-enriched interpolative Kannan pair, and proved a common fixed point result in the setup of $\mathfrak{R}$-complete convex metric space.
ii) We presented examples to support the concepts introduced and results proved herein.
iii) Moreover, we studied well-posedness, limit shadowing property and Ulam-Hyers stability for the results (Corollary 2.1) introduced herein.

Acknowledgments. Authors are thankful to the reviewers for their useful comments and constructive remarks that helped to improve the presentation of the paper.

References

[1] Abbas, M. Common fixed point results with applications in convex metric space. *Fasc. Math.* **39** (2008), 5–15.
[2] Abbas, M.; Anjum, R.; Berinde, V. Enriched multivalued contractions with applications to differential inclusions and dynamic programming. *Symmetry* **13** (2021),, no. 8, 1350. https://doi.org/10.3390/sym13081350.
[3] Abbas, M., Anjum, R.; Berinde, V. Equivalence of Certain Iteration Processes Obtained by Two New Classes of Operators. *Mathematics* **9** (2021), no. 18, 2292. https://doi.org/10.3390/math9182292.
[4] Abbas, M.; Anjum, R.; Iqbal, H. Generalized enriched cyclic contractions with application to generalized iterated function system. *Chaos, Solitons and Fractals* **154** (2022). https://doi.org/10.1016/j.chaos.2021.111591.
[5] Agarwal, R. P.; Karapinar, E. Interpolative Rus-Reich-Ćirić Type Contractions Via Simulation Functions. *An. St. Univ. Ovidius Constanta, Ser. Mat.* **27** (2019), no. 3
[6] Anjum, R.; Abbas, M. Common Fixed point theorem for modified Kannan enriched contraction pair in Banach spaces and its Applications. *J. Filomat* **35** (2021), no. 8, 2485–2495. https://doi.org/10.2298/FIL2108485A.
[7] Anjum, R.; Abbas, M. Fixed point property of a nonempty set relative to the class of friendly mappings. *RACSAM* **116** (2022), no. 1, 1-10. https://doi.org/10.1007/s13398-021-01158-5.
[8] Aydi, H.; Karapinar, E.; Roldán López de Hierro, A. F. w-Interpolative Ćirić-Reich-Rus-Type Contractions. *Mathematics* **7** (2019), no. 57, https://doi.org/10.3390/math7010057.
[9] Aydi, H.; Chen, C. M.; Karapinar, E. Interpolative Ćirić-Reich-Rus types via the Branciari distance. *Mathematics* **7** (2019), no. 1, 84. https://doi.org/10.3390/math7010084.
[10] Banach, S. Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales. *Fund. Math.* **3** (1922), 133–181.
[11] Berinde, V. General constructive fixed point theorems for Ćirić type almost contractions in metric spaces. *Carpathian J. Math.* (2008), 10–19.
[12] Berinde, V.; Păcurar, M. Kannan's fixed point approximation for solving split feasibility and variational inequality problems. *J. Comput. Appl.* (2020), 377–427.
[13] Berinde, V.; Păcurar, M. Approximating fixed points of enriched contractions in Banach spaces. *J. Fixed Point Theory Appl.* **22** (2020), no. 2, 1–10
[14] Berinde, V. Approximating fixed points of enriched nonexpansive mappings by Krasnoselskii iteration in Hilbert spaces. *Carpathian J. Math.* **35** (2019), no. 3, 293–304. http://dx.doi.org/10.37193/CJM.2019.03.04.
[15] Berinde, V.; Păcurar, M. Existence and Approximation of Fixed Points of Enriched Contractions and Enriched φ-Contractions. *Symmetry* **13** (2021), 498. https://doi.org/10.3390/sym13030498.
[16] Berinde, V.; Păcurar, M. Approximating fixed points of enriched Chatterjea contractions by Krasnoselskii iterative algorithm in Banach spaces. *J. Fixed Point Theory Appl.* **23**(2021), no. 4, 1–16.
[17] Berinde, V.; Păcurar, M. Fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces and convex metric spaces. *Carpathian J. Math.* **37** (2021), 173–184.
[18] De Blasi, F. S.; Myjak, J. Sur la porosité de l'ensemble des contractions sans point fixe. *C. R. Acad. Sci. Paris* **308** (1989), 51–54.
[19] Debnath, P.; de La Sen, M. Set-valued interpolative Hardy-Rogers and set-valued Reich-Rus-Ćirić-type contractions in b-metric spaces. *Mathematics* **7** (2019), no. 9, 849. https://doi.org/10.3390/math7090849.
[20] Fulga, A.; Yesilkaya, S. S. On Some Interpolative Contractions of Suzuki Type Mappings. *J. Funct. Spaces* (2021), 7. https://doi.org/10.1155/2021/6596096.
[21] Gaba, Y. U.; Karapinar, E. A New Approach to the Interpolative Contractions. *Axioms* **8** (2019), no. 4, 110. https://doi.org/10.3390/axioms8040110.

[22] Gaba, Y. U.; Aydi, H.; Mlaik, N. (ρ, η, μ)-Interpolative Kannan Contractions I. *Axioms* **10** (2021), no. 3, 212. https://doi.org/10.3390/axioms10030212.

[23] Gordji, M. E.; Eshaghi, M.; Ramezani, M. M.; De La Sen; Cho, Y. J. On orthogonal sets and Banach fixed point theorem. *Fixed Point Theory* **18** (2017), no. 2, 569–578.

[24] Kannan, R. Some results on fixed points. *Bull. Calcutta Math. Soc.* **60** (1968), 71–76.

[25] Kannan, R. Some results on fixed points. *II, Amer. Math. Monthky* **76** (1969), 405–408.

[26] Karapinar, E.; Aydi, H.; Mitrovic, Z. D. On Interpolative Boyd-Wong And Matkowski Type Contractions. *J. Pure Appl. Math.* **11** (2020), no. 2, 204–212.

[27] Karapinar, E.; Fulga, A.; Yesilkaya, S. S. New Results on Perov-Interpolative Contractions of Suzuki Type Mappings. *J. Funct. Spaces* (2021), 7. https://doi.org/10.1155/2021/9587604

[28] Karapinar, E. Revisiting the Kannan Type Contractions via Interpolation. *Adv. Theory Nonlinear Anal. Appl.* **2** (2018), no. 2, 85–87. http://dx.doi.org/10.31197/atnaa.431135

[29] Karapinar, E.; Alqahtani, O.; Aydi, H. On Interpolative Hardy-Rogers Type Contractions. *Symmetry* **11** (2019), no. 1, 8. https://doi.org/10.3390/sym11010008.

[30] Karapinar, E.; Agarwal R.; Aydi, H. Interpolative Reich-Rus-Ćirić Type Contractions on Partial Metric Spaces. *Mathematics* **6** (2018), no. 11, 256. https://doi.org/10.3390/math6110256.

[31] Karapinar, E.; Fulga, A. New Hybrid Contractions on b-Metric Spaces. *Mathematics* **7** (2019), no. 7, 578. https://doi.org/10.3390/math7070578

[32] Kikkawa, M.; Suzuki, T. Some similarity between contractions and Kannan mappings II. *Bull. Kyushu Inst. Technol. Pure Appl. Math.* **55** (2008), 1–13.

[33] Khalehoghli, S.; Rahimi, H.; Gordji, M. E. Fixed point theorems in R-metric spaces with applications. *AIMS Mathematics* **5** (2020), no. 4, 3125–3137.

[34] Mohammad, S.; Khan, Y.; Singh, M.; Karapinar, E. On the interpolative (varphi, psi)-type z-contraction. *Sci. Bull., Series A* **83** (2021), no. 2, 1223–7027.

[35] Nieto, J. J.; López, R. R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **22** (2005), 223–239.

[36] Noorwali, M. Common Fixed Point for Kannan type via Interpolation *J. Math. Anal.* **9** (2018), 92–94.

[37] Noorwali, M.; Yesilkaya, S. S. New Fixed Point Theorems for Admissible Hybrid Maps. *J. Funct. Spaces* (2022), 8. https://doi.org/10.1155/2022/5800790.

[38] Noorwali, M.; Yesilkaya, S. S. On Jaggi-Suzuki-Type Hybrid Contraction Mappings. *J. Funct. Spaces* (2021), 7. https://doi.org/10.1155/2021/6721296.

[39] Ran, A. C. M.; Reurings, M. C. B. A fixed point theorem in partially ordered sets and some application to matrix equations. *Proc. Amer. Math.Soc.* **132** (2004), 1435–1443.

[40] Reich, S.; Zaslavski, A. J. Well-posedness of fixed point problems. *Far East J. Math. Sci* (FJMS), (2001), 393–401.

[41] Takahashi, W. A. Convexity in metric space and nonexpansive mappings. *I. Kodai Math. Sem. Rep.* **22** (1970), 142–149.

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS
UNIVERSITY OF PRETORIA
HATFIELD 002, PRETORIA, SOUTH AFRICA
Email address: mujahid.abbas@up.ac.za

DEPARTMENT OF MATHEMATICS
RIPHAH INTERNATIONAL UNIVERSITY
RIPHAH INSTITUTE OF COMPUTING AND APPLIED SCIENCES
LAHORE 54000, PAKISTAN
Email address: rizwan.anjum@riphah.edu.pk

GOVERNMENT COLLEGE UNIVERSITY
ABDUS SALAM SCHOOL OF MATHEMATICAL SCIENCES
LAHORE 54600, PAKISTAN
Email address: shakeelariasat@sms.edu.pk