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Non-existence of forbidden subgraph characterization of H-line graphs

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ABSTRACT. H-line graph, denoted by $HL(G)$, is a generalization of line graph. Let G and H be two graphs such that H has at least 3 vertices and is connected. The H-line graph of G, denoted by $HL(G)$, is that graph whose vertices are the edges of G and two vertices of $HL(G)$ are adjacent if they are adjacent in G and lie in a common copy of H. In this paper, we show that H-line graphs do not admit a forbidden subgraph characterization.

1. INTRODUCTION

Intersection graphs have been receiving attention in graph theory for some time. The line graph $L(G)$ was the first intersection graph to be defined in literature. The notion of 'line graph' as a 'graph operator' was introduced by Krausz [10]. The line graph, $L(G)$, of a graph G has all the edges (ie K_2 subgraphs) of G as its vertices and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. Since then, many other graph operators such as clique graph, total graph etc. and their dynamics were studied [13].

Jarrett [8] defined the triangular line graph, $\triangle(G)$, of a nonempty graph G as that graph whose vertices are edges of G and two vertices of $\triangle(G)$ are adjacent if the corresponding edges belong to a common triangle of G . Triangular line graph was also referred as anti-gallai graph by some authors [11]. The triangular line graph was introduced to model a metric space defined on the edge set of a graph. These concepts were generalized in [2].

H-line graphs were introduced by Chartrand in [5], as a generalization of line graphs and also of triangular line graphs. Let G and H be two graphs such that H has at least 3 vertices and is connected. The H-line graph of G, denoted by $HL(G)$, is that graph whose vertices are the edges of G and two vertices of $HL(G)$ are adjacent if they are adjacent in G and lie in a common copy of H. In particular, when $H = P_3$, the H-line line graph $HL(G)$ is the standard line graph $L(G)$. For $k \geq 2$, the $k-$th iterated H-line graph $HL^k(G)$ is defined as $HL(HL^{k-1}(G))$, where $HL^1(G) = HL(G)$ and $HL^{k-1}(G)$ is assumed to be non-empty. A sequence $\{G^k\}$ of graphs is said to converge to a graph G if there exists a positive integer N such that $G^k \cong G$ for every integer $k \geq N$. If the sequence $\{G^k\}$ is finite, it is said to terminate. If $\{G^k\}$ neither converges nor terminates, then the sequence is said to diverge. Chartrand et al. characterized those graphs for which the sequence $\{HL^k(G)\}$ converges, when H is P_4, P_5 or $K_{1,n}$ in [4]. Manjula [12], discussed the behavior of the sequence $\{HL^k(G)\}$ for a unicyclic graph G, which consists of a cycle C_t and a path P_m originating from a vertex v_i on the cycle such that C_t and P_m have only one vertex v_i in common, when $H \cong P_n$. In [9], Kathiresan et al. proved a necessary and sufficient condition for the convergence of $\{HL^k(G)\}$ when H is isomorphic to P_6, as an extension of the result in [4]. The behaviour of the sequence $\{HL^k(G)\}$ when $H = K_3$ were analyzed in [8] and [6] . Quadrilateral line graph, $C_4L(G)$ was introduced in [3].

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They obtained a sufficient condition for each component of $C_4L(G)$ to be Eulerian. Limits of K_n-line graphs were studied and some results on $K_nL(G)$ were obtained in [1].

In this paper, H-line graphs are viewed as spanning subgraphs of line graphs. It is shown that H-line graphs do not admit forbidden subgraph characterization, even though they are spanning subgraphs of line graphs which admit forbidden subgraph characterization. All the graphs considered here are undirected, finite and simple. For all basic concepts and notations not mentioned in this paper we refer [14].

2. H-LINE GRAPHS

Definition 2.1. Let G and H be two graphs such that H has at least 3 vertices and is connected. The H-line graph of G, denoted by $HL(G)$, is that graph whose vertices are the edges of G and two vertices of $HL(G)$ are adjacent if they are adjacent in G and lie in a common copy of H.

A graph G and its $C_4L(G)$ is shown in Figure 1. Here, the vertices v_i of $C_4L(G)$ corresponds to the edges e_i of G. The vertices v_1 and v_2 of $C_4L(G)$ are adjacent since the edges e_1 and e_2 are adjacent in G and they lie in a copy of C_4, whereas the vertices v_2 and v_3 of $C_4L(G)$ are not adjacent since the the edges e_2 and e_3 do not lie in a copy of C_4.

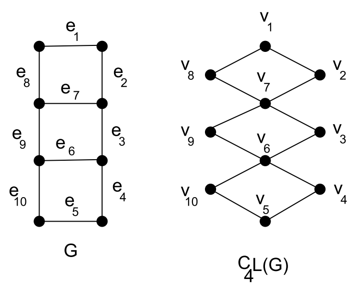

FIGURE 1. G and $C_4L(G)$

Remark 2.1. For any connected graph G on at least three vertices, $GL(G) = L(G)$.

Remark 2.2. $HL(G)$ is a spanning subgraph of $L(G)$.

Remark 2.3. $HL(G) = L(G)$ when $H = P_3$ or $K_{1,2}$.

3. NON-EXISTENCE OF FORBIDDEN SUBGRAPH CHARACTERIZATION

The main theorem of this paper is proved in this section. We employ the following definitions for the same.

Definition 3.2. A subset $F \subseteq E$ of edges is said to be an independent set of edges or a matching if no two edges in F have a vertex in common. The maximum cardinality of a matching set of edges is the matching number or edge-independence number and is denoted by $\alpha'(G)$.

Definition 3.3. The graph $H = (V', E')$ is an induced subgraph of $G = (V, E)$ if E' is the collection of all edges in G which has both its end vertices in V'. The induced subgraph with vertex set V' is denoted by$\langle V' \rangle$.

Definition 3.4. A graph G is $H-$free if it does not contain H as an induced subgraph. Given a nonempty class $\mathcal{C}$ of graphs, a graph G is said to be $\mathcal{C}-$free, if none of the induced subgraphs of G belong to $\mathcal{C}$. The class of graphs which are $\mathcal{C}-$free is denoted by $G(\mathcal{C})$.

Definition 3.5. For any class of graphs $\mathcal{H}$, we say that F is a forbidden subgraph for $\mathcal{H}$ if no element of $\mathcal{H}$ has F as an induced subgraph.

Definition 3.6. If $\mathcal{H} = G(\mathcal{C})$, for some class $\mathcal{C}$ of graphs, we say that $\mathcal{H}$ has a forbidden subgraph characterization.

Definition 3.7. A class $\mathcal{C}$ of graphs has the induced hereditary property if $G \in \mathcal{C}$ implies that every induced subgraph of G also belongs to $\mathcal{C}$.

In [7], Greenwell et al. give a necessary and sufficient condition for a class of graphs to admit a forbidden subgraph characterization.

Theorem 3.1. [7] *A class of graphs $\mathcal{C}$ has a forbidden subgraph characterization if and only if $\mathcal{C}$ has the induced hereditary property.*

We use this result to show that H-line graphs admit forbidden subgraph characterization, if and only if $H = P_3$. In what follows, we take H to be a connected graph of order at least three.

Lemma 3.1. *If H is a graph with the edge-independence number, $\alpha'(H) > 1$ and G is a graph with at least one edge, then $HL(G)$ cannot be complete.*

Proof. Suppose that $\alpha'(H) > 1$. Let e_1, e_2 be two independent edges in H. Since $HL(G)$ has an edge if and only if G contains a copy of H, e_1, e_2 will be independent in G also. Clearly the vertices corresponding to e_1 and e_2 will not be adjacent in $HL(G)$ and hence $HL(G)$ cannot be complete. □

Theorem 3.2. *$HL(G)$ do not admit forbidden subgraph characterization if $\alpha'(H) > 1$.*

Proof. By Lemma 3.1, if $\alpha'(H) > 1$, then $HL(G)$ cannot be complete. ie. $HL(G) \ncong K_n$, for any n. In other words, K_n is not a H-line graph for any H with $\alpha'(H) > 1$. Now, since H is a connected graph of order at least 3, it will have at least two adjacent edges. The following construction shows that K_n can be embedded in $HL(G)$ as an induced subgraph: Take $K_{1,n}$ and with each pair of adjacent edges $\{vv_i, vv_j\}$ construct a copy of H. Then $\{vv_1, vv_2, \ldots vv_n\}$ will induce a K_n in $HL(G)$. Therefore, $HL(G)$ do not have induced hereditary property and hence do not admit a forbidden subgraph characterization.

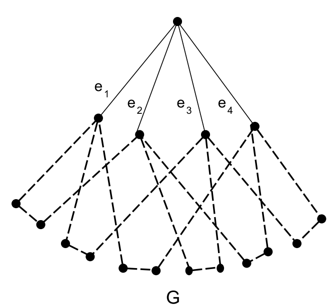

FIGURE 2. The edges $\{e_1, e_2, e_3, e_4\}$ induces a K_4 in $C_5L(G)$

An illustration of the construction with $H = C_5$ is given in Figure 2 . Here a graph G is constructed by taking $K_{1,4}$ and with each pair of edges, making a copy of C_5. Then, the edges $\{e_1, e_2, e_3, e_4\}$ will induce a K_4 in $C_5L(G)$. But, by Lemma 3.1, K_4 is not a C_5-line graph of any graph. Hence, the class of C_5-line graphs do not have induced herditary property and they cannot have a forbidden subgraph characterization. □

Remark 3.4. When $\alpha'(H) = 1$. Then H is either a star, $K_{1,n}, n \geq 2$ or a triangle, K_3. $K_{1,2}L(G)$ is the line graph which admits induced hereditary property and hence forbidden subgraph characterization. $K_3L(G)$ is the triangular line graph of G which is also known as the anti-gallai graph. In [11], it is proved that anti-gallai graphs do not have induced hereditary property and hence do not admit forbidden subgraph characterization.

Now we will show that the star-line graphs, $K_{1,n}L(G), n \geq 3$ do not have induced hereditary property.

Lemma 3.2. *If G is a H-line graph, then every edge of G lies in a copy of $L(H)$.*

Proof. Let G be a H-line graph, ie $G = HL(G')$ for some G'. The vertices of G corresponds to edges of G' and the edges of G corresponds to adjacent edges of G' which lie in a copy of H. If there is an edge in G, then there will be a copy of H in G'. Then, the edges in this copy of $H \subseteq G'$ will induce a copy of $L(H)$ in G. Hence every edge lie in a copy of $L(H)$. □

Lemma 3.3. *Any cycle, $C_m, m \geq 4$ is not a star-line graph, $K_{1,n}L(G), n \geq 3$, for any G.*

Proof. If $C_m, m \geq 4$ were a a star-line graph, $K_{1,n}L(G), n \geq 3$, then by Lemma 3.2, every edge of C_m would lie in a copy of $L(K_{1,n}) \cong K_n, n \geq 3$, which is not possible. □

For the proof of the next theorem we need the definition of a binary operation on graphs, called corona.

Definition 3.8. Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. The corona of G_1 and G_2, denoted by $G_2 \circ G_2$, is the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2.

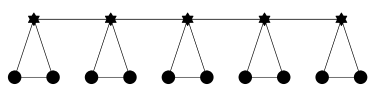

FIGURE 3. $P_4 \circ K_2$

Theorem 3.3. *Any cycle $C_m, m \geq 4$ can be embedded in a star-line graph, $K_{1,n}L(G), n \geq 3$, as an induced subgraph.*

Proof. Take G to be $C_m \circ \bar{K}_{n-2}$. Then $K_{1,n}L(G)$ will contain C_m as an induced subgraph. An illustration of the construction with C_5 is given in Figure 4. $K_{1,6}L(G)$ will have an induced C_5. □

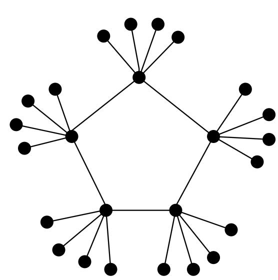

FIGURE 4. $G = C_5 \circ \bar{K}_4$

Theorem 3.4. *Star-line graphs, $K_{1,n}L(G), n \geq 3$ do not admit forbidden subgraph characterization.*

Proof. From Lemma 3.3 and Theorem 3.3, it is clear that $K_{1,n}L(G), n \geq 3$ do not have induced hereditary property. Hence, by Theorem 3.1, they do not admit forbidden subgraph characterization. □

Theorem 3.5. *H-line graphs admit forbidden subgraph characterization only when $H = P_3$*

Proof. It is clear from Theorem 3.2, Remark 3.4 and Theorem 3.4 that H-line graphs admit forbidden subgraph characterization only when $H = P_3$. □

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