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Fixed point results for P-contractions via w-distance

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ABSTRACT. In the present paper, we define the Pw-contraction by considering the inequality called P-contraction in metric space together with the w-distance. We then present fixed point theorems for both single-valued and multivalued Pw-contractions. We also support our results with suitable examples.

1. INTRODUCTION AND PRELIMINARIES

In 2017, Fulga and Proca [7, 8] presented some fixed point theorems for single-valued mappings via new type contractive inequalities inspired by E-contraction on metric space. Since this idea, called E-contraction, was first used by Popescu (see the references of [7, 8]), we prefer to use it as P-contraction in our papers to cite Popescu [2, 3, 5]. Popescu's original definition and related fixed point theorem are as follows: Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is called P-contraction if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq k \left[d(x, y) + |d(x, Tx) - d(y, Ty)|\right] \tag{1.1}$$

for all $x, y \in X$. It is easy to see that every contraction mapping on metric space is P-contraction, but the converse is not true as shown in some examples in [2, 8]. Thus, the following theorem addresses a more general class of mappings than the famous Banach fixed point theorem.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be P-contraction. Then T has a unique fixed point. Moreover, every Picard iteration converges to the fixed point.*

We recommend papers such as [1, 4, 9, 12] in addition to the above for fixed point results regarding the concept of P-contraction.

In this paper, we will present the fixed point theorems for both single-valued and multivalued mappings by considering P-contraction idea of Popescu together with the w-distance in metric space.

First of all, let's remember the basic definition and properties of w-distance which was introduced by Kada et al. [11] in metric space. See [17] for more information on w-distance.

Definition 1.1 ([11]). Let (X, d) be a metric space. A function $w : X \times X \to [0, \infty)$ is called w-distance in X if it satisfies the following:

- $w(x, z) \leq w(x, y) + w(y, z)$ for all $x, y, z \in X$,
- the mapping $w_x : X \to [0, \infty)$ is lower semicontinuous for each $x \in X$, where $w_x(\cdot) = w(x, \cdot)$, that is, if $\{y_n\}$ is a sequence in X with $y_n \to y \in X$, then

  $$w_x(y) \leq \liminf_{n\to\infty} w_x(y_n)$$

  for each $x \in X$,

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- for any $\varepsilon > 0$, there exists $\delta > 0$ such that $w(x, y) \le \delta$ and $w(x, z) \le \delta$ imply $d(y, z) \le \varepsilon$.

Now we present some examples, which more explanations of them, can be found in some papers in the literature such as [10, 11, 14, 16].

Example 1.1. Let (X, d) be a metric space.

(1) The metric d is a w-distance in X.
(2) Define $w_k(x, y) = k > 0$. Then w_k is a w-distance in X.
(3) Let A, which has at least two elements, be a closed and bounded subset of X, and let $c \ge diam(A) = \sup\{d(a, b) : a, b \in A\}$. Define

$$w_c(x, y) = \begin{cases} d(x, y), & x, y \in A \\ c, & x \notin A \text{ or } y \notin A \end{cases}.$$

Then w_c is a w-distance in X.
(4) Let $f : X \to X$ be a continuous function. Define

$$w_f(x, y) = \max\{d(fx, y), d(fx, fy)\}.$$

Then w_f is a w-distance in X.

Example 1.2. Let $(X, \|\cdot\|)$ be a normed linear space. Then the functions $w_1(x, y) = \|y\|$ and $w_2(x, y) = \|x\| + \|y\|$ are w-distances in X.

The following lemmas about w-distance play crucial role in the proofs of our theorems

Lemma 1.1 ([11]). *Let (X, d) be a metric space, w be a w-distance in X, $\{x_n\}$ and $\{y_n\}$ be two sequence in X, and $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, \infty)$ converging to 0. Then, for all $x, y, z \in X$, the following hold:*

(a) *If $w(x_n, y) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $w(x, y) = 0$ and $w(x, z) = 0$, then $y = z$.*
(b) *If $w(x_n, y_n) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for all $n \in \mathbb{N}$, then $y_n \to z$ as $n \to \infty$.*
(c) *If $w(x_n, x_m) \le \alpha_n$ for all $n, m \in \mathbb{N}$ with $n > m$, then $\{x_n\}$ is a Cauchy sequence in X.*
(d) *If $w(x, x_n) \le \alpha_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.*

Lemma 1.2 ([16]). *Let (X, d) be a metric space, K be a closed subset of X and w be a w-distance in X. Suppose that there exists $u \in X$ such that $w(u, u) = 0$. Then $u \in K$ if and only if*

$$w(u, K) = \inf\{w(u, z) : z \in K\} = 0.$$

In the literature, besides the fixed point theorems for single-valued mappings obtained with the help of the w-distance function, Latif and Albar [15] (resp. Latif and Abdou [14]) proved the following theorem for multivalued mappings inspired by Feng-Liu's [6] paper (resp. Klim and Wardowski's [13] paper).

Theorem 1.2 ([15, 14]). *Let (X, d) be a complete space and let $T : X \to P_C(X)$ be a weakly contractive (resp. generalized w-contractive) map. Suppose that a real-valued function f on X defined by $f(x) = w(x, Tx)$ is lower semicontinous. Then there exists $vz \in X$ such that $f(z) = 0$. Further, if $w(z, z) = 0$, then $z \in Tz$.*

2. Main results

First we introduce the following definition.

Definition 2.2. Let (X, d) be a metric space, w be a w-distance on X and $T : X \to X$ be a mapping. If there exists a nonnegative real number $c < 1$ satisfying satisfying

$$w(Tx, Ty) \le c[w(x, y) + |w(x, Tx) - w(y, Ty)|], \tag{2.2}$$

for all $x, y \in X$, then T is said to be P_w-contraction.

Now we present our first main result.

Theorem 2.3. *Let (X,d) be a complete metric space, w be a w-distance on X and $T : X \to X$ be a P_w-contraction. Assume that one following hold:*

(i) *T is continuous,*
(ii) *w is continuous,*
(iii) *for every $y \in X$ with $y \neq Ty$*

$$\inf\{w(x,y) + w(x,Tx) : x \in X\} > 0.$$

Then T has a unique fixed point $z \in X$. Moreover $w(z,z) = 0$.

Proof. Let $x_0 \in X$ be an arbitrary point. Consider the associated Picard sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for $n \geq 0$. Now, we will consider the following two cases:

(1) Assume there exists $n_0 \in \mathbb{N}$ such that $w(x_{n_0}, x_{n_0+1}) = 0$. In this case we claim that $w(x_{n_0+1}, x_{n_0+2}) = 0$. Indeed, by (2.2) we have

$$\begin{aligned} w(x_{n_0+1}, x_{n_0+2}) &= w(Tx_{n_0}, Tx_{n_0+1}) \\ &\leq c\left[w(x_{n_0}, x_{n_0+1}) + |w(x_{n_0}, Tx_{n_0}) - w(x_{n_0+1}, Tx_{n_0+1})|\right] \\ &= c\left[w(x_{n_0}, x_{n_0+1}) + |w(x_{n_0}, x_{n_0+1}) - w(x_{n_0+1}, x_{n_0+2})|\right] \\ &= cw(x_{n_0+1}, x_{n_0+2}), \end{aligned}$$

which is a contradiction unless $w(x_{n_0+1}, x_{n_0+2}) = 0$. Hence $w(x_{n_0+1}, x_{n_0+2}) = 0$ and so from the triangular inequality we have

$$w(x_{n_0}, x_{n_0+2}) \leq w(x_{n_0}, x_{n_0+1}) + w(x_{n_0+1}, x_{n_0+2}) = 0.$$

Now that we have $w(x_{n_0}, x_{n_0+1}) = 0$ and $w(x_{n_0}, x_{n_0+2}) = 0$, from Lemma 1.1 (a), we get $x_{n_0+1} = x_{n_0+2} = Tx_{n_0+1}$ hence x_{n_0+1} is a fixed point of T.

(2) Now suppose $w(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Then from (2.2) we have

$$\begin{aligned} w(x_{n+1}, x_{n+2}) &= w(Tx_n, Tx_{n+1}) \\ &\leq c\left[w(x_n, x_{n+1}) + |w(x_n, Tx_n) - w(x_{n+1}, Tx_{n+1})|\right] \\ &= c\left[w(x_n, x_{n+1}) + |w(x_n, x_{n+1}) - w(x_{n+1}, x_{n+2})|\right] \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$. In this case it must be $w(x_{n+1}, x_{n+2}) < w(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ (otherwise from (2.3) we get a contradiction as $0 < w(x_{n+1}, x_{n+2}) \leq cw(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N}$) and hence from (2.3) we have

$$w(x_{n+1}, x_{n+2}) \leq \frac{2c}{1+c} w(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. Therefore we have

$$w(x_{n+1}, x_{n+2}) \leq \lambda^{n+1} w(x_0, x_1),$$

for all $n \in \mathbb{N}$, where $\lambda = \frac{2c}{1+c} < 1$. Now for any $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} w(x_n, x_m) &\leq \sum_{i=n}^{m-1} w(x_i, x_{i+1}) \\ &= \sum_{i=n}^{m-1} \lambda^i w(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda} w(x_0, x_1) \end{aligned} \tag{2.4}$$

and so from Lemma 1.1 (c), $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since w is lower semicontinuous in the second variable and $x_m \to z$ as $m \to \infty$, from (2.4) we get

$$w(x_n, z) \leq \lim \inf_{m\to\infty} w(x_n, x_m) \leq \frac{\lambda^n}{1-\lambda} w(x_0, x_1). \tag{2.5}$$

Now, if T is continuous, then $x_{n+1} = Tx_n \to Tz$ and so by the uniqueness of the limit we get $z = Tz$. Moreover, we have $w(z, z) = 0$. Indeed, from (2.2)

$$w(z, z) = w(Tz, Tz) \leq cw(z, z), \tag{2.6}$$

which is a contradiction unless $w(z, z) = 0$.

If w is continuous, then from (2.4) and (2.5) we have

$$w(z, z) = \lim_{n,m\to\infty} w(x_n, x_m) = 0$$

and

$$\lim_{n\to\infty} w(x_n, z) = 0.$$

Now putting $x = x_n$ and $y = z$ in (2.2) we have

$$w(x_{n+1}, Tz) \leq c[w(x_n, z) + |w(x_n, x_{n+1}) - w(z, Tz)|]$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ and using the continuity of w, we have

$$w(z, Tz) \leq cw(z, Tz),$$

which is a contradiction unless $w(z, Tz) = 0$. Hence we have $w(z, z) = 0 = w(z, Tz)$ and so from Lemma 1.1 (a) we have $z = Tz$.

Finally, assume (iii) holds and $z \neq Tz$. Then from (2.4) and (2.5) we have

$$\begin{aligned} 0 &< \inf\{w(x, z) + w(x, Tx) : x \in X\} \\ &\leq \inf\{w(x_n, z) + w(x_n, Tx_n) : n \in \mathbb{N}\} \\ &= \inf\{w(x_n, z) + w(x_n, x_{n+1}) : n \in \mathbb{N}\} \to 0 \end{aligned}$$

as $n \to \infty$, which is a contradiction. Hence $z = Tz$ and as in (2.6), we have $w(z, z) = 0$.

To show the uniqueness of fixed point, suppose u be also a fixed point of T. Then from (2.2) we have $w(u, u) = 0$ and

$$w(z, u) = w(Tz, Tu) \leq cw(z, u)$$

which implies $w(z, u) = 0$. Hence from Lemma 1.1 (a) we have $u = z$. □

Now we provide a few illustrative examples along with a comparative example.

Example 2.3. Let $X = \left[0, \frac{1}{2}\right]$ with the usual metric d. Define $T : X \to X$ by $Tx = x^2$ and consider the w-distance in X as $w(x, y) = y$. Then we have

$$w(Tx, Ty) = y^2$$

and

$$w(x, y) + |w(x, Tx) - w(y, Ty)| = y + \left|x^2 - y^2\right|$$

for all $x, y \in X$. Hence we get

$$w(Tx, Ty) \leq \frac{1}{2}[w(x, y) + |w(x, Tx) - w(y, Ty)|]$$

for all $x, y \in X$, that is, T is P_w-contraction. Other conditions of Theorem 2.3 are clearly hold, and therefore T has a unique fixed point.

Note that T is not P-contraction with respect to usual metric d. Indeed, since

$$\lim_{x\to\frac{1}{2}^-}\frac{d(Tx,T\frac{1}{2})}{d(x,\frac{1}{2})+\left|d(x,Tx)-d(\frac{1}{2},T\frac{1}{2})\right|}=\lim_{x\to\frac{1}{2}^-}\frac{\frac{1}{4}-x^2}{\frac{1}{2}-x+\left|x-x^2-\frac{1}{4}\right|}=1$$

we can not find the constant $k\in[0,1)$ satisfying the inequality (1.1). Therefore Theorem 1.1 can not be applied to this example.

Example 2.4. Let $X=C[0,1]$ with the supremum norm. Define $T:X\to X$ by

$$Tf(t)=\int_0^t(t-s)f(s)ds$$

and consider the w-distance in X as $w(f,g)=\|f\|_\infty+\|g\|_\infty$. Then we have

$$\begin{aligned}|Tf(t)| &= \left|\int_0^t(t-s)f(s)ds\right|\\ &\le \|f\|_\infty\int_0^t(t-s)ds\\ &= \frac{t^2}{2}\|f\|_\infty\end{aligned}$$

and so

$$\|Tf\|_\infty\le\frac{1}{2}\|f\|_\infty$$

for all $f\in X$. Therefore we get

$$\begin{aligned}w(Tf,Tg) &= \|Tf\|_\infty+\|Tg\|_\infty\le\frac{1}{2}\|f\|_\infty+\frac{1}{2}\|g\|_\infty\\ &= \frac{1}{2}w(f,g)\le\frac{1}{2}\left[w(f,g)+|w(f,Tf)-w(g,Tg)|\right]\end{aligned}$$

for all $f,g\in X$, that is, T is P_w-contraction. Other conditions of Theorem 2.3 are clearly hold, and therefore T has a unique fixed point.

Example 2.5. Let $X=\left\{\frac{1}{2^n}:n\in\mathbb{N}\right\}\cup\{0\}$ with the usual metric. Define $T:X\to X$ by

$$T\frac{1}{2^n}=\frac{1}{2^{n+1}}\text{ for }n\in\mathbb{N}\text{ and }T0=0$$

and consider the w-distance in X as $w(x,y)=y$. Then we have (except for the obvious case)

$$\begin{aligned}w\left(T\frac{1}{2^n},T\frac{1}{2^m}\right) &= \frac{1}{2^{m+1}}\\ &= \frac{1}{2}w\left(\frac{1}{2^n},\frac{1}{2^m}\right)\\ &\le \frac{1}{2}\left[w\left(\frac{1}{2^n},\frac{1}{2^m}\right)+\left|w\left(\frac{1}{2^n},T\frac{1}{2^n}\right)-w\left(\frac{1}{2^m},T\frac{1}{2^m}\right)\right|\right]\end{aligned}$$

for all $m,n\in\mathbb{N}$, that is, T is P_w-contraction. Other conditions of Theorem 2.3 are clearly hold, and therefore T has a unique fixed point.

We need some notations to use in our theorem about multivalued mappings. Let (X,d) be a metric space, $P(X)$ be the family of all nonempty subsets of X, $P_C(X)$ be the family of all nonempty and closed subsets of X and w be a w-distance in X. Let $T:X\to P(X)$ be a multivalued mapping and $b\in(0,1)$. For $x\in X$, define the set $J_b^x\subseteq X$ as

$$J_b^x=\{y\in Tx:bw(x,y)\le w(x,Tx)\}.$$

Remark 2.1. 1. Let $T : X \to P(X)$ and $x \in X$ with $w(x, Tx) > 0$. Then J_b^x is nonempty for every $b \in (0,1)$. Indeed, if we choose $\varepsilon_b = (\frac{1}{b} - 1)w(x, Tx) > 0$, then by the definition of infimum, there exists $y_{\varepsilon_b} \in Tx$ such that

$$w(x, y_{\varepsilon_b}) \leq w(x, Tx) + \varepsilon_b.$$

Hence we have

$$bw(x, y_{\varepsilon_b}) \leq w(x, Tx)$$

and so $y_{\varepsilon_b} \in J_b^x$.

2. Let $w(x, Tx) = 0$ for $x \in X$. Then J_b^x may be empty even if $Tx \in P_C(X)$. For example, let $X = (0, 1]$ with the usual metric, $Tx = (0, x] \in P_C(X)$ for every $x \in X$ and $w(x, y) = y$. Then we have $w(x, Tx) = 0$ for every $x \in X$ and so we can not find $y \in Tx$ satisfying the inequality $bw(x, y) \leq w(x, Tx)$. Hence J_b^x is empty for every $b \in (0, 1)$.

3. Let (X, d) be complete and $Tx \in P_C(X)$. Then J_b^x is nonempty for every $b \in (0, 1)$. The situation $w(x, Tx) > 0$ was examined in Case 1. Now let $w(x, Tx) = 0$. Then there exists a sequence $\{y_n\}$ in Tx such that $\lim_{n\to\infty} w(x, y_n) = 0$. Hence by Lemma 1.1 (d), $\{y_n\}$ is a Cauchy sequence in X. Since X is complete there exists $y \in X$ such that $y_n \to y$ with respect to d. By the closedness of Tx, we get $y \in Tx$. On the other hand, since w is lower semicontinuous in the second variable we have

$$w(x, y) \leq \lim \inf_{n\to\infty} w(x, y_n) = 0.$$

This shows that there exists $y \in Tx$ such that

$$bw(x, y) = 0 = w(x, Tx)$$

for every $b \in (0, 1)$. Hence we have J_b^x is nonempty.

Now, taking into account both Popescu's and Feng-Liu's ideas we will present a fixed point theorem for multivalued mappings on metric space via w-distance.

Definition 2.3. Let (X, d) be a metric space, w be a w-distance on X, $T : X \to P(X)$ be a multivalued mapping and $b \in (0, 1)$. If for all $x \in X$, there exists $y \in J_b^x$ satisfying

$$w(y, Ty) \leq c[w(x, y) + |w(x, Tx) - w(y, Ty)|],$$

where c is a nonnegative real number c satisfying $\frac{2c}{b(1+c)} < 1$. Then T is said to be multivalued P_w-contraction.

Theorem 2.4. *Let (X, d) be a complete metric space, w be a w-distance on X and $T : X \to P_C(X)$ be a multivalued P_w-contraction. Assume that $f(x) = w(x, Tx)$ is lower semicontinuous. Then there exists $z \in X$ such that $f(z) = 0$. Further, if $w(z, z) = 0$ then z is a fixed point of T.*

Proof. First note that by Remark 2.1, J_b^x is nonempty for any $x \in X$. By the assumption, for arbitrary point $x_0 \in X$, there exists $x_1 \in J_b^{x_0}$ such that

$$w(x_1, Tx_1) \leq c[w(x_0, x_1) + |w(x_0, Tx_0) - w(x_1, Tx_1)|],$$

and, for $x_1 \in X$, there exists $x_2 \in J_b^{x_1}$ such that

$$w(x_2, Tx_2) \leq c[w(x_1, x_2) + |w(x_1, Tx_1) - w(x_2, Tx_2)|].$$

Continuing this process, we can construct an iterative sequence $\{x_n\}$ such that $x_{n+1} \in J_b^{x_n}$ and

$$w(x_{n+1}, Tx_{n+1}) \leq c[w(x_n, x_{n+1}) + |w(x_n, Tx_n) - w(x_{n+1}, Tx_{n+1})|], \tag{2.7}$$

for $n = 0, 1, 2, \cdots$. If there exists $n_0 \in \mathbb{N}$ such that $w(x_{n_0}, Tx_{n_0}) = 0$, then we have $f(x_{n_0}) = 0$. Let assume $w(x_n, Tx_n) > 0$ for all $n \in \mathbb{N}$. Now, if there exists $m \in \mathbb{N}$ such that

$$w(x_{m+1}, Tx_{m+1}) \geq w(x_m, Tx_m),$$

then from (2.7) we have (note that, $bw(x_m, x_{m+1}) \leq w(x_m, Tx_m)$ since $x_{m+1} \in J_b^{x_m}$)

$$\begin{aligned} w(x_{m+1}, Tx_{m+1}) &\leq c[w(x_m, x_{m+1}) + |w(x_m, Tx_m) - w(x_{m+1}, Tx_{m+1})|] \\ &= c[w(x_m, x_{m+1}) + w(x_{m+1}, Tx_{m+1}) - w(x_m, Tx_m)], \end{aligned}$$

and so

$$\begin{aligned} w(x_{m+1}, Tx_{m+1}) &\leq \frac{c}{1-c} w(x_m, x_{m+1}) - \frac{c}{1-c} w(x_m, Tx_m) \\ &= \frac{c}{1-c}[w(x_m, x_{m+1}) - w(x_m, Tx_m)] \\ &\leq \frac{c}{1-c}\left[\frac{1}{b} w(x_m, Tx_m) - w(x_m, Tx_m)\right] \\ &\leq \left(\frac{c}{1-c}\right)\left(\frac{1-b}{b}\right) w(x_m, Tx_m) \\ &< w(x_m, Tx_m) \\ &\leq w(x_{m+1}, Tx_{m+1}), \end{aligned}$$

which is a contradiction. Therefore $w(x_{n+1}, Tx_{n+1}) < w(x_n, Tx_n)$ for all $n \in \mathbb{N}$. Thus, we have from (2.7)

$$w(x_{n+1}, Tx_{n+1}) \leq c[w(x_n, x_{n+1}) + w(x_n, Tx_n) - w(x_{n+1}, Tx_{n+1})]$$

and so

$$\begin{aligned} w(x_{n+1}, Tx_{n+1}) &\leq \frac{c}{1+c}[w(x_n, x_{n+1}) + w(x_n, Tx_n)] \\ &\leq \frac{2c}{1+c} w(x_n, x_{n+1}). \end{aligned}$$

Now since $x_{n+2} \in J_b^{x_{n+1}}$ we have

$$\begin{aligned} bw(x_{n+1}, x_{n+2}) &\leq w(x_{n+1}, Tx_{n+1}) \\ &\leq \frac{2c}{1+c} w(x_n, x_{n+1}), \end{aligned}$$

and so

$$w(x_{n+1}, x_{n+2}) \leq \frac{2c}{b(1+c)} w(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. Therefore, we obtain

$$w(x_n, x_{n+1}) \leq \lambda^n w(x_0, x_1),$$

for all $n \in \mathbb{N}$, where $\lambda = \frac{2c}{b(1+c)} < 1$. Hence we have

$$\lim_{n\to\infty} w(x_n, x_{n+1}) = 0,$$

and also, for $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} w(x_n, x_m) &\leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + \cdots + w(x_{m-1}, x_m) \\ &\leq \lambda^n w(x_0, x_1) + \lambda^{n+1} w(x_0, x_1) + \cdots + \lambda^{m-1} w(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda} w(x_0, x_1). \end{aligned}$$

Since $\lambda < 1$, the last inequality shows that , by Lemma 1.1 (c), $\{x_n\}$ is a Cauchy sequence. According to the completeness of X, there exists $z \in X$ such that $\{x_n\}$ converges to z with respect to d. Since f is lower semicontinuous, we get that

$$\begin{aligned} 0 &\leq w(z,Tz) = f(z) \\ &\leq \liminf f(x_n) \\ &= \liminf w(x_n, Tx_n) \\ &\leq \liminf w(x_n, x_{n+1}) = 0, \end{aligned}$$

and so $f(z) = w(z,Tz) = 0$. Further, if $w(z,z) = 0$, it follows by Lemma 1.2, that $z \in Tz$. □

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References

[1] Altun, I.; Aslantas, M.; Sahin, H. Best proximity point results for p-proximal contractions. *Acta Math. Hungar.* **162** (2020), no. 2, 393–402.
[2] Altun, I.; Durmaz, G.; Olgun, M. P-contractive mappings on metric spaces. *J. Nonlinear Funct. Anal.*. **2018** (2018), Article ID 43, pp. 1–7.
[3] Altun, I.; Hançer, H. A.; Erduran, A. Nonself P-contractions on convex metric spaces and their fixed points. *The Journal of Analysis*. (2022), 1-12, https://doi.org/10.1007/s41478-022-00390-5.
[4] Altun, I.; Hancer, H. A.; Erduran, A. Fixed point results for single valued and set valued P-contractions and application to second order boundary value problems. *Carpathian J. Math.*. **36** (2020), no. 2, 205–214.
[5] Aslantas, M.; Sahin, H.; Altun, I. Best proximity point theorems for cyclic p-contractions with some consequences and applications. *Nonlinear Analysis: Modelling and Control*. **26** (2021), no. 1, 113–129.
[6] Feng, Y.; Liu, S. Fixed point theorems for multivalued contractive mappings and multivalued Caristi type mappings. it J. Math. Anal. Appl. **317** (2006), 103-112.
[7] Fulga, A.; Proca, A. A new generalization of Wardowski fixed point theorem in complete metric spaces. *Advances in the Theory of Nonlinear Analysis and its Applications*. **1** (2017), no. 1, 57-63.
[8] Fulga, A.; Proca, A. Fixed points for φ_E-Geraghty contractions. *J. Nonlinear Sci. Appl.* **10** (2017), 5125–513.
[9] Hancer, H.A. On multivalued P-contractive mappings that belongs to class of weakly Picard operators. *Fixed Point Theory*. **22** (2021), no. 2, 663-670.
[10] Iqbal, I.; Rizwan, M. Existence of the solution to second order differential equation through fixed point results for nonlinear F-contractions involving w_0-distance. *Filomat*. **34** (2020), no. 12, 4079–4094.
[11] Kada, O.; Suzuki, T.; Takahashi, W. Nonconvex minimization theorems and fixed point theorems in complete metric spaces. *Math. Japon.* **44** (1996), 381–391.
[12] Karapınar, E.; Fulga, A.; Aydi, H. Study on Pata E-contractions. *Advances in Difference Equations*. **1** (2020), 1-15.
[13] Klim, D.; Wardowski, D. Fixed point theorems for set-valued contractions in complete metric spaces. *J. Math. Anal. Appl.* **334** (2007), 132–139.
[14] Latif, A.; Abdou, A. A. N. Fixed points of generalized contractive maps. *Fixed Point Theory and Applications*. **2009** (2009), 1–9.
[15] Latif, A.; Albar, W. A. Fixed point results in complete metric spaces. *Demonstratio Mathematica*. **41** (2008), no. 1, 145–150.
[16] Lin, L. J.; Du, W. S. Some equivalent formulations of the generalized Ekeland's variational principle and their applications. *Nonlinear Analysis: Theory, Methods & Applications*. **67** (2007), no. 1, 187–199.
[17] Rakočević, V. *Fixed Point Results in W-Distance Spaces*, Chapman and Hall/CRC, New York, 2021.

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