CREAT. MATH. INFORM.
Volume **32** (2023), No. 1,
Pages 21 - 28

Online version at https://semnul.com/creative-mathematics/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2023.01.03

Generalized Functional Discriminating Measure For Finite Probability Distributions

PRAPHULL CHHABRA

ABSTRACT. Analysts like Renyi (1961), Csiszar (1963, 1966), Bregman (1967), Burba-Rao (1982), Jain and Saraswat (2012), etc., have introduced and analyzed the different functional discriminating measures for comparing two discrete probability distributions. But in this article, a new functional discriminating measure has been proposed that will compare finite (more than two) discrete probability distributions simultaneously. Further, some intra-relations among the measures at different values of the parameters have been discussed. Also, an interesting connection with Csiszar's generalized functional measure has been created. Some new inequalities compared to variational discrimination and Chi-square discrimination have been discussed as well.

1. INTRODUCTION

Discriminating measures are fundamental measures of distance between two likelihoods of dissemination or comparing two likelihoods of dissemination or comparing two probability distributions. The concept of discriminating measure is working productively to resolve distinctive issues related to the likelihood hypothesis. The real information is approximated by the measures of likelihood and measurements. This data leads to data misfortune. The essential reason is to evaluate how much data is contained within the information.

Now a days, these measures are being connected in a few disciplines such as: color picture division [17], estimation of likelihood dispersions [4, 8], design acknowledgment [9, 23], 3D picture division and word arrangement [20], choice making [16, 22, 24, 25], attractive reverberation picture investigation [27], fetched- touchy classification for therapeutic conclusion [19], turbulence stream [5], fuzzy divergence and applications[3, 10, 15, 21, 26], etc.

Let $\Theta_l = \{U = (u_1, u_2, u_3, ..., u_l) : u_i > 0, \sum_{i=1}^{l} u_i = 1\}$, $l \geq 2$ be the set of all complete finite discrete probability distributions, where u_i is a probability mass function.

Definition 1.1. Convex function: A function $g(x)$ is said to be convex over an interval (a, b) if it has for every $x_1, x_2 \in (a, b)$ and $0 \leq \mu \leq 1$, we have

$$g[\mu x_1 + (1-\mu) x_2] \leq \mu g(x_1) + (1-\mu) g(x_2),$$

and if uniformity does not hold, i.e., $\mu \neq 0$ or $\mu \neq 1$, then it is said to be entirely convex. In a broader sense, we can write

$$g\left[\sum_{i=1}^{m} \mu_i x_i\right] \leq \sum_{i=1}^{m} \mu_i g(x_i), \tag{1.1}$$

for all $x_i \in (a, b)$ and $\mu_i \geq 0$ with $\sum_{i=1}^{m} \mu_i = 1$.

Received: 27.01.2022. In revised form: 26.09.2022. Accepted: 04.10.2022
2000 *Mathematics Subject Classification.* 94A17, 26D15.
Key words and phrases. *New generalized functional discriminating measure, Csiszar's generalized functional discriminating measure, Discrete probability distributions, Convex functions, Variational discrimination, Chi- square discrimination.*

Convex functions have wide applications in immaculate and connected arithmetic. As of late numerous generalizations and expansions have been made for the convexity, like: strong convexity [28], s- convexity [2], GA- convexity [29], GG- convexity [12], preinvexity [11] and others.

2. New Generalized Functional Discriminating Measure

Let $U_1 = (u_{11},..., u_{l1}),..., U_m = (u_{1m},..., u_{lm})$ and $W_1 = (w_{11},..., w_{l1}),..., W_m = (w_{1m},..., w_{lm})$ be discrete probability distributions such that $U_j, W_j \in \Theta_l \; \forall \; j = 1, 2, ..., m$. Then, the following new functional discriminating measure is being introduced.

$$S_g^m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m)$$
$$= \sum_{i=1}^{l} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1} w_{i2} ... w_{im} g \left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + \frac{u_{i2}+w_{i2}}{2w_{i2}} + ... + \frac{u_{im}+w_{im}}{2w_{im}}}{m} \right), \tag{2.2}$$

where $g : (0, \infty) \to R$ (set of real no.) is real, continuous, and convex function. For $m = 1$, it will compare two probability distributions U_1 and W_1 at a time, for $m = 2$, it compares four probability distributions U_1, W_1, U_2, and W_2 at a time, and so on. This allows you to compare $2j$ probability distributions at the same time, where $j = 1, 2, ..., 2m$.
Jain Saraswat's discriminating measure [14] is a special case of (2.2) at $m = 1$, which is

$$S_g^1 (U_1, W_1) = \sum_{i=1}^{l} w_{i1} g \left(\frac{u_{i1} + w_{i1}}{2w_{i1}} \right) = \sum_{i=1}^{l} w_i g \left(\frac{u_i + w_i}{2w_i} \right). \tag{2.3}$$

3. Intra Relation Among Functional Discriminating Measures

In this segment, an imperative and productive connection among new useful discriminations has been discussed. These discriminations are essentially uncommon cases of (2.2) agreeing to the number of likelihood dispersions or the number of discrete probability distributions.

Theorem 3.1. *Let $g : (0, \infty) \to R$ be a differentiable, convex function, i.e., $g'' (x) \geq 0 \; \forall x > 0$. For $U_j, W_j \in \Theta_l \; \forall \; j = 1, 2..., m$, we have*

$$S_g^1 (U_1, W_1) \geq S_g^2 (U_1, U_2, W_1, W_2) \geq ... \geq S_g^m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m)$$
$$\geq S_g^{m+1} (U_1, U_2, ..., U_m, U_{m+1}, W_1, W_2, ..., W_m, W_{m+1}) \geq g(1), \tag{3.4}$$

where $S_g^m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m)$ is given by (2.2).

Proof. By utilizing the disparity (1.1) for different summations, we get

$$\sum_{i=1}^{l} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1} w_{i2} ... w_{im} w_{i(m+1)} g \left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + \frac{u_{i2}+w_{i2}}{2w_{i2}} + ... + \frac{u_{im}+w_{im}}{2w_{im}} + \frac{u_{i(m+1)}+w_{i(m+1)}}{2w_{i(m+1)}}}{m+1} \right)$$

$$\geq g \left[\sum_{i=1}^{l} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1} w_{i2} ... w_{im} w_{i(m+1)} \left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + \frac{u_{i2}+w_{i2}}{2w_{i2}} + ... + \frac{u_{im}+w_{im}}{2w_{im}} + \frac{u_{i(m+1)}+w_{i(m+1)}}{2w_{i(m+1)}}}{m+1} \right) \right]$$

$$= g \left[\frac{1}{m+1} \left(\sum_{i=1}^{l} \frac{u_{i1} + w_{i1}}{2} \sum_{i=1}^{l} w_{i2} .. \sum_{i=1}^{l} w_{im} \sum_{i=1}^{l} w_{i(m+1)} \right) \right]$$
$$+ .. + g \left[\frac{1}{m+1} \left(\sum_{i=1}^{l} \frac{u_{i(m+1)} + w_{i(m+1)}}{2} \sum_{i=1}^{l} w_{i1} .. \sum_{i=1}^{l} w_{im} \right) \right]$$

$$= g\left[\frac{1}{m+1}(1+1+...+1)\right] = g\left(\frac{m+1}{m+1}\right) = g(1)\text{, i.e.,}$$

$$S_g^{m+1}(U_1, U_2, ..., U_m, U_{m+1}, W_1, W_2, ..., W_m, W_{m+1}) \geq g(1). \tag{3.5}$$

Consequently, it demonstrated the last inequality of the relation (3.4). □

Now again apply the inequality (1.1) for $h_1, h_2, ..., h_{m+1}$, where $h_i \in (0, \infty)\ \forall\ i = 1, 2, ..., m+1$, we obtain

$$\frac{1}{m+1}[g(h_1) + g(h_2) + ... + g(h_m) + g(h_{m+1})] \geq g\left[\frac{h_1 + h_2 + ... + h_m + h_{m+1}}{m+1}\right]. \tag{3.6}$$

Let

$$h_1 = \frac{z_1 + z_2 + ... + z_m}{m}, h_2 = \frac{z_2 + z_3 + ... + z_m + z_{m+1}}{m}, ..., h_{m+1} = \frac{z_{m+1} + z_1 + ... + z_{m-1}}{m},$$

where $z_i \in (0, \infty)\ \forall\ i = 1, 2, ..., m+1$.
Then by using the inequality (3.6), we get

$$\frac{1}{m+1}\left[g\left(\frac{z_1 + z_2 + ... + z_m}{m}\right) + ... + g\left(\frac{z_{m+1} + z_1 + ... + z_{m-1}}{m}\right)\right]$$

$$\geq g\left[\frac{1}{m+1}\left(\frac{z_1+z_2+...+z_m}{m} + ... + \frac{z_{m+1}+z_1+...+z_{m-1}}{m}\right)\right] = g\left(\frac{m(z_1+z_2+...+z_m+z_{m+1})}{m(m+1)}\right)$$

$$= g\left(\frac{z_1 + z_2 + ... + z_m + z_{m+1}}{m+1}\right). \tag{3.7}$$

Now put $z_j = \frac{u_{ij}+w_{ij}}{2w_{ij}}$ in (3.7), multiply with $w_{ij}\ \forall\ j = 1, ..., m+1$ and for each $i = 1, ..., l$ and then summation $m+1$ times from $i = 1$ to $i = l$, we get

$$\frac{1}{m+1}\left[\sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{i(m+1)} g\left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{im}+w_{im}}{2w_{im}}}{m}\right)\right]$$

$$+...+\frac{1}{m+1}\left[\sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{i(m+1)} g\left(\frac{\frac{u_{i(m+1)}+w_{i(m+1)}}{2w_{i(m+1)}} + \frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{i(m-1)}+w_{i(m-1)}}{2w_{i(m-1)}}}{m}\right)\right]$$

$$\geq \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{i(m+1)} g\left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{i(m+1)}+w_{i(m+1)}}{2w_{i(m+1)}}}{m+1}\right)\text{, i.e.,}$$

$$S_g^{m}(U_1, U_2, ..., U_m, W_1, W_2, ..., W_m) \geq S_g^{m+1}(U_1, U_2, ..., U_{m+1}, W_1, W_2, ..., W_{m+1}). \tag{3.8}$$

As a result, the second last inequality of the relation (3.4) is established for all m, and the theorem is established.

Remark 3.1. If g is normalized, i.e., $g(1) = 0$, then we get the following set of relations

$$S_g^1(U_1, W_1) \geq S_g^2(U_1, U_2, W_1, W_2) \geq ... \geq S_g^m(U_1, U_2, ..., U_m, W_1, W_2, ..., W_m)$$

$$\geq S_g^{m+1}(U_1, U_2, ..., U_m, U_{m+1}, W_1, W_2, ..., W_m, W_{m+1}) \geq 0. \tag{3.9}$$

4. Relation With Csiszar's Generalized Functional Discrimination

In 2000, S.S. Dragomir [7] presented the following important discrimination measure:

$$C_g^m\left(U_1, U_2, ..., U_m, W_1, W_2, ..., W_m\right) = \sum_{i=1}^{l}\sum_{i=1}^{l}...\sum_{i=1}^{l} w_{i1}w_{i2}...w_{im}g\left(\frac{\frac{u_{i1}}{w_{i1}} + \frac{u_{i2}}{w_{i2}} + ... + \frac{u_{im}}{w_{im}}}{m}\right), \tag{4.10}$$

where $g : (0, \infty) \to R$ (set of real no.) is real, continuous, and convex function. Ciszar's discriminating measure [1, 6] is a special case of this measure, which is

$$C_g^1\left(U_1, W_1\right) = \sum_{i=1}^{l} w_{i1}g\left(\frac{u_{i1}}{w_{i1}}\right) = \sum_{i=1}^{l} w_i g\left(\frac{u_i}{w_i}\right). \tag{4.11}$$

Now, a vital connection among measures (2.2) and (4.10) is being evaluated by the following theorem.

Theorem 4.2. *Let $g : (0, \infty) \to R$ be a differentiable, convex and normalized function, i.e., $g''(x) \geq 0\ \forall x > 0$ and $g(1) = 0$ respectively. For $U_j, W_j \in \Theta_l\ \forall\ j = 1, 2..., m$, we have*

$$S_g^m\left(U_1, U_2, ..., U_m, W_1, W_2, ..., W_m\right) \leq \frac{1}{2}C_g^m\left(U_1, U_2, ..., U_m, W_1, W_2, ..., W_m\right). \tag{4.12}$$

Proof. Apply the inequality (1.1) for the domain $I \subset (0, \infty)$, by putting $\mu_1 = \mu_2 = \frac{1}{2}, \mu_3 = ... = \mu_m = 0$, we obtain

$$g\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}\left[g\left(x_1\right) + g\left(x_2\right)\right]. \tag{4.13}$$

Now put $x_1 = x$ and $x_2 = 1$ in above inequality, we obtain

$$g\left(\frac{x + 1}{2}\right) \leq \frac{1}{2}g\left(x\right). \tag{4.14}$$

Now in the inequality (4.14), take $x = \frac{\sum_{j=1}^{m}\frac{u_{ij}}{w_{ij}}}{m}$, multiply with $\prod_{j=1}^{m} w_{ij}$ for each i and then summation over m times from $i = 1$ to $i = l$, we obtain the required relation (4.12). □

5. Important Deductions

Now, some curious results are going to be determined by using the defined discrimination (2.2) in the inequalities (3.9).

Proposition 5.1. *Let $U_j, W_j \in \Theta_l\ \forall\ j = 1, 2, ..., m$, then we have*

$$V_m\left(U_1, U_2, ..., U_m, W_1, W_2, ..., W_m\right) \leq \sum_{j=1}^{m} V\left(U_j, W_j\right) \tag{5.15}$$

and

$$V_1\left(U_1, W_1\right) = V\left(U, W\right) \geq \frac{1}{2}V_2\left(U_1, U_2, W_1, W_2\right) \geq ... \geq \frac{1}{m}V_m\left(U_1, ..., U_m, W_1, ..., W_m\right)$$

$$\geq \frac{1}{m+1}V_{m+1}\left(U_1, ..., U_m, U_{m+1}, W_1, ..., W_m, W_{m+1}\right) \geq 0. \tag{5.16}$$

Proof. Let $g(x) = |x-1|, x > 0$. Here $g(x)$ is convex and normalized function because $g''(x) \geq 0 \; \forall \; x > 0$ and $g(1) = 0$ respectively, but $g(x)$ is not differentiable at $x = 1$.
Put $g(x)$ in (2.2), we have

$$S_g^m(U_1,...,U_m,W_1,...,W_m) = \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left| \frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{im}+w_{im}}{2w_{im}}}{m} - 1 \right|$$

$$= \frac{1}{m} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left| \left(\frac{u_{i1}+w_{i1}}{2w_{i1}} - 1 \right) + ... + \left(\frac{u_{im}+w_{im}}{2w_{im}} - 1 \right) \right|$$

$$= \frac{1}{2m} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left| \frac{u_{i1}-w_{i1}}{w_{i1}} + ... + \frac{u_{im}-w_{im}}{w_{im}} \right| = \frac{1}{2m} V_m(U_1,...,U_m,W_1,...,W_m), \tag{5.17}$$

and for $m = 1$

$$\frac{1}{2} \sum_{i=1}^{l} |u_i - w_i| = \frac{1}{2} V_1(U_1, W_1) = \frac{1}{2} V(U, W), \tag{5.18}$$

where $V(U,W)$ is the Variational distance [13], a special case of the $V_m(U_1,..,U_m,W_1,..,W_m)$ for comparing two probability distributions.
Now, equation (5.17) can be written as

$$\frac{1}{2m} V_m(U_1,...,U_m,W_1,...,W_m) \leq \frac{1}{2m} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left[\left| \frac{u_{i1}-w_{i1}}{w_{i1}} \right| + ... + \left| \frac{u_{im}-w_{im}}{w_{im}} \right| \right]$$

$$= \frac{1}{2m} \left[\sum_{i=1}^{l} |u_{i1} - w_{i1}| \sum_{i=1}^{l} w_{i2} ... \sum_{i=1}^{l} w_{im} + ... + \sum_{i=1}^{l} |u_{im} - w_{im}| \sum_{i=1}^{l} w_{i1} ... \sum_{i=1}^{l} w_{i(m-1)} \right]$$

$$= \frac{1}{2m} \left[\sum_{i=1}^{l} |u_{i1} - w_{i1}| + ... + \sum_{i=1}^{l} |u_{im} - w_{im}| \right] = \frac{1}{2m} \sum_{j=1}^{m} \sum_{i=1}^{l} |u_{ij} - w_{ij}| = \frac{1}{2m} \sum_{j=1}^{m} V(U_j, W_j).$$

$$\Rightarrow V_m(U_1,...,U_m,W_1,...,W_m) \leq \sum_{j=1}^{m} V(U_j, W_j).$$

Hence, the result (5.15) is obtained. Also, the sequence of inequalities (5.16) can be gotten by utilising (3.9). We are overlooking the points of interest. □

Proposition 5.2. *Let $U_j, W_j \in \Theta_l \; \forall \; j = 1,2,...,m$, then we have*

$$\chi_m^2(U_1,U_2,...,U_m,W_1,W_2,...,W_m) = \sum_{j=1}^{m} \chi^2(U_j, W_j) \tag{5.19}$$

and

$$\chi_1^2(U_1,W_1) = \chi^2(U,W) \geq \frac{1}{2^2} \chi_2^2(U_1,U_2,W_1,W_2) \geq ... \geq \frac{1}{m^2} \chi_m^2(U_1,...,U_m,W_1,...,W_m)$$
$$\geq \frac{1}{(m+1)^2} \chi_{m+1}^2(U_1,...,U_m,U_{m+1},W_1,...,W_m,W_{m+1}) \geq 0. \tag{5.20}$$

Proof. Let $g(x) = (x-1)^2, x > 0$. Here $g(x)$ is convex and normalized function because $g''(x) \geq 0 \; \forall \; x > 0$ and $g(1) = 0$ respectively.
Put $g(x)$ in (2.2), we obtain

$$S_g^m(U_1,...,U_m,W_1,...,W_m) = \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left[\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{im}+w_{im}}{2w_{im}}}{m} - 1 \right]^2$$

$$= \frac{1}{m^2}\sum_{i=1}^{l}...\sum_{i=1}^{l} w_{i1}...w_{im}\left[\left(\frac{u_{i1}+w_{i1}}{2w_{i1}}-1\right)+...+\left(\frac{u_{im}+w_{im}}{2w_{im}}-1\right)\right]^2$$

$$= \frac{1}{4m^2}\sum_{i=1}^{l}...\sum_{i=1}^{l} w_{i1}...w_{im}\left[\frac{u_{i1}-w_{i1}}{w_{i1}}+...+\frac{u_{im}-w_{im}}{w_{im}}\right]^2 = \frac{1}{4m^2}\chi_m^2\left(U_1,...,U_m,W_1,...,W_m\right), \tag{5.21}$$

and for $m = 1$

$$\frac{1}{4}\sum_{i=1}^{l}\frac{(u_i-w_i)^2}{w_i} = \frac{1}{4}\chi_1^2\left(U_1,W_1\right) = \frac{1}{4}\chi^2\left(U,W\right), \tag{5.22}$$

where $\chi^2(U,W)$ is the well known Chi-square discriminating measure [18], an uncommon case of the

$$\chi_m^2\left(U_1,...,U_m,W_1,...,W_m\right),$$

for comparing two probability distributions.
Now, the above equation (5.21) can be written as

$$\frac{1}{4m^2}\chi_m^2\left(U_1,...,U_m,W_1,...,W_m\right)$$

$$= \frac{1}{m^2}\sum_{i=1}^{l}...\sum_{i=1}^{l} w_{i1}..w_{im}\left[\sum_{j=1}^{m}\left(\frac{u_{ij}+w_{ij}}{2w_{ij}}-1\right)^2 + 2\sum_{1\le j<k\le m}\left(\frac{u_{ij}+w_{ij}}{2w_{ij}}-1\right)\left(\frac{u_{ik}+w_{ik}}{2w_{ik}}-1\right)\right]$$

$$= \frac{1}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{l}...\sum_{i=1}^{l} w_{i1}...w_{im}\left(\frac{u_{ij}+w_{ij}}{2w_{ij}}-1\right)^2$$

$$+\frac{2}{m^2}\sum_{1\le j<k\le m}\sum_{i=1}^{l}...\sum_{i=1}^{l} w_{i1}...w_{im}\left(\frac{u_{ij}+w_{ij}}{2w_{ij}}-1\right)\left(\frac{u_{ik}+w_{ik}}{2w_{ik}}-1\right)$$

$$= \frac{1}{m^2}\sum_{j=1}^{m}\left[\sum_{i=1}^{l} w_{i1}...\sum_{i=1}^{l} w_{ij}\left(\frac{u_{ij}+w_{ij}}{2w_{ij}}-1\right)^2...\sum_{i=1}^{l} w_{im}\right]$$

$$+\frac{2}{m^2}\sum_{1\le j<k\le m}\left[\sum_{i=1}^{l} w_{i1}...\sum_{i=1}^{l} w_{ij}\left(\frac{u_{ij}+w_{ij}}{2w_{ij}}-1\right)...\sum_{i=1}^{l} w_{ik}\left(\frac{u_{ik}+w_{ik}}{2w_{ik}}-1\right)...\sum_{i=1}^{l} w_{im}\right]$$

$$= \frac{1}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{l} w_{ij}\left(\frac{u_{ij}+w_{ij}}{2w_{ij}}-1\right)^2 + \frac{2}{m^2}\sum_{1\le j<k\le m}\sum_{i=1}^{l}\left(\frac{u_{ij}-w_{ij}}{2}\right)\sum_{i=1}^{l}\left(\frac{u11_{ik}-w_{ik}}{2}\right)$$

$$= \frac{1}{4m^2}\sum_{j=1}^{m}\chi^2\left(U_j,W_j\right).$$

$$\Rightarrow \chi_m^2\left(U_1,...,U_m,W_1,...,W_m\right) = \sum_{j=1}^{m}\chi^2\left(U_j,W_j\right).$$

Hence, prove the result (5.19). Also, the sequence of inequalities (5.20) can be obtained by using (3.9). We are omitting the details. □

6. Conclusion

The main conclusion of this article is to present a new composite functional discriminating measure for analysing more than two discrete probability distributions at a time. The obtained results (in sections 3, 4, and 5) are important in information theory and are original to the best of the author's knowledge. Also, the same is valid for continuous probability distributions, like Normal, Exponential, Uniform, Chi-square, etc. The application of obtained results in signal processing and pattern recognition is in the works, and these will be provided in the next article as soon as possible.

References

[1] Ali, S.M.; Silvey, S. D. A general class of coefficients of divergence of one distribution from another. *J. Roy. Statist. Soc. Ser. B* **28** (1966), 131–142.

[2] Adil Khan, M.; Chu, Y. M.; Khan, T. U.; Khan, J. Some new inequalities of Hermite- Hadamard type for s-convex functions with applications. *Open Math.* **15** (2017), no. 1, 1414–1430.

[3] Bajaj, R. K.; Hooda, D. S. Generalized measures of fuzzy directed divergence, total ambiguity and information improvement. *J Appl Math Stat Inform JAMSI* **6** (2010), no. 2, 31–44.

[4] Basseville, M. *Divergence measures for statistical data processing*. Publications Internes de l'IRISA, ISSN : 2102-6327, (2010).

[5] Carlos Granero-Belinchón, C.; Roux, S. G.; Garnier, N. B. Kullback-Leibler divergence measure of intermittency: Application to turbulence. *Physical Review E.* **APS 97** (2018), 013107.

[6] Csiszar, I. Eine informations theoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizitat von Markoffschen Ketten. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), Ser. A, 85–108.

[7] Dragomir, S. S. A generalized f-divergence for probability vectors and applications. *PanAmer. Math. J* **13** (2003), no. 4, 61–69.

[8] Gkelsinis, T.; Karagrigoriou, A. Theoretical aspects on measures of directed information with simulations. *Mathematics* **8** (2020), 587; doi:10.3390/math8040587.

[9] Kadian, R.; Kumar, S. Renyi's-Tsallis fuzzy divergence measure and its applications to pattern recognition and fault detection. *J. Intell. Fuzzy Syst.* **39** (2020), no. 1, 731–752.

[10] Khatod, N.; Saraswat, R. N. *New fuzzy divergence measures, series, its bounds and applications in strategic decision-making*. Lecture Notes in Electrical Engineering, (2020) — book, DOI: 10.1007/978-981-15-0214-9 67.

[11] Khurshid, Y.; Adil Khan, M.; Chu, Y. M.; Khan, Z. A. Hermite Hadamard Fejer inequalities for conformable fractional integrals via preinvex functions. *J. Funct. Spaces*, Article ID: 3146210 **(2019)**.

[12] Khurshid, Y.; Adil Khan, M.; Chu, Y. M. Conformable integral inequalities of the Hermite- Hadamard type in terms of GG and GA- convexties. *J. Funct. Spaces*, Article ID: 6926107 **(2019)**.

[13] Kolmogorov, A. N. On the approximation of distributions of sums of independent summands by infinitely divisible distributions. *Sankhya* **25** (1965), pp 159–174.

[14] Jain, K. C.; Saraswat, R. N. Some bounds of information divergence measure in term of Relative arithmetic-geometric divergence measure. *Int. J. Appl. Math. Stat.* **32** (2013), Number 2, pp. 48-58.

[15] Jha, P.; Mishra, V. K. Some new trigonometric, hyperbolic and exponential measures of fuzzy entropy and fuzzy directed divergence. *Int. J. Scientific Eng. Research* **3** (2012), no. 4, 1–5.

[16] Joshi, R.; Kumar, S. *An Exponential Jensen Fuzzy Divergence Measure with Applications in Multiple Attribute Decision-Making*, Math. Probl. Eng Article ID 4342098 **(2018)**, 9, pageshttps://doi.org/10.1155/2018/4342098.

[17] Nielsen, F.; Boltz, S. The Burbea-Rao and Bhattacharyya centroids. *IEEE Transactions on Information Theory* **57** (2011), no. 8, 5455–5466.

[18] Pearson, K. On the Criterion that a given system of deviations from the probable in the case of correlated system of variables is such that it can be reasonable supposed to have arisen from random sampling. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **50**, (1900), Series 5, Issue 302, 157–172.

[19] Santos-Rodriguez, R.; Garcia-Garcia, D.; Cid-Sueiro, J. Cost-sensitive classification based on Bregman divergences for medical diagnosis. *Proceedings of the 8th International Conference on Machine Learning and Applications (ICMLA'09)*, **9** (2009), Miami Beach, Fl., USA, 551–556.

[20] Taskar, B.; Lacoste-Julien, S.; Jordan, M. I. Structured prediction, dual extra gradient and Bregman projections. *J Mach Learn Res.* **7** (2006), 1627–1653.

[21] Tomar, V. P.; Ohlan, A. New parametric generalized exponential fuzzy divergence measure. *J. Uncertainty Anal. Appl.* **2** (2014), 24, DOI 10.1186/s40467-014-0024-2.

[22] Umar, A.; Saraswat, R. N., Novel Divergence Measure Under Neutrosophic Environment and Its Utility in Various Problems of Decision Making. *Int. J. Fuzzy Syst.* **9** (2020), no. 4, 82- 104.

[23] Umar, A.; Saraswat, R. N. New Generalized Intuitionistic Fuzzy Divergence Measure with Applications to Multi-attribute Decision Making and Pattern Recognition. *Recent Adv. Comput. Sci. Commun.* **14** (2021), 2247–2266.

[24] Umar, A.; Saraswat, R. N. Decision-making in machine learning using novel picture fuzzy divergence measure. *Neural Comput. Appl.* **34** (2022), 457–475.

[25] Umar, A.; Saraswat, R. N. Neoteric divergence measure for refined interval-valued neutrosophic sets and its application in decision makin. *Int. J. Math. Oper. Research* **20** (2021), no. 2, 182–206.

[26] Umar, A.; Saraswat, R. N. New Fuzzy Divergence Measure and Its Applications in Multi-criteria Decision-Making Using New Tool. *Springer Proceedings in Mathematics and Statistics*, 2020 conference-paper, **307** (2020), DOI: 10.1007/978-981-15-1157-8 17.

[27] Vemuri, B.; etc. all Total Bregman divergence and its applications to DTI analysis. *IEEE Transactions on Medical Imaging* **30** (2011), no. 2, 475 - 483.

[28] Zaheer, U. S.; Adil Khan, M.; Chu, Y. M. Majorization of theorems for strongly convex functions. *J. Inequal. Appl.* Article ID: 58,**(2019)**.

[29] Zhang, X. M.; Chu, Y. M.; Zhang, X. H. The Hermite Hadamard type inequality of GA- convex functions and its applications. *J. Inequal. Appl.* Article ID: 507560, **(2010)**.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF ENGINEERING AND MANAGEMENT
JAIPUR- 303807 (RAJASTHAN), INDIA
Email address: `praphull.chhabra@uem.edu.in`