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An iterative method involving a class of quasi-phi-nonexpansive mappings for solving split equality fixed point problems

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ABSTRACT. A new inertial iterative algorithm for approximating solution of split equality fixed point problem (SEFPP) for quasi-ϕ- nonexpansive mappings is introduced and studied in p-uniformly convex and uniformly smooth real Banach spaces, $p > 1$. A strong convergence theorem is proved without imposing any compactness-type condition on the mappings. Our theorems complement several important recent results that have been proved in 2-uniformly convex and uniformly smooth real Banach spaces. It is well known that these spaces do not include L_p, l_p and the Sobolev spaces $W^m{}_p(\Omega)$, for $2 < p < \infty$. Our theorems, in particular, are applicable in these spaces. Furthermore, application of our theorem to split equality variational inclusion problem is presented. Finally, numerical examples are presented to illustrate the convergence of our algorithms.

1. INTRODUCTION

Let D be a nonempty closed and convex subset of a real Banach space E and $K : D \to D$ be any mapping. A point $x \in D$ is called a fixed point of K if $Kx = x$. We shall denote the set of fixed points of a mapping K, by $F(K)$. Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear maps. The *split equality problem* (SEP) introduced by Moudafi [14] is the following problem:

$$\text{find } \ x^* \in C, \ y^* \in Q, \ \text{ such that } \ Ax^* = By^*. \tag{1.1}$$

We shall denote the solution set of problem (1.1) by

$$\Gamma = \{x \in C, \ y \in Q : \ Ax = By\}.$$

Observe that if $B = I$ (the identity map on H_2) and $H_3 = H_2$, then problem (1.1) reduces to the following *split feasibility problem* (SFP):

$$\text{find } \ x^* \in C, \ \text{ such that } \ Ax^* \in Q. \tag{1.2}$$

Numerous research efforts have been devoted to the study of the SEP due to its usefulness in applications. For example, the SEP has applications in game theory, in intensity-modulated radiation therapy preparation, in decomposition methods for partial differential equations, in fully discretized models of inverse problems which arise from phase retrievals and in medical image reconstruction (see, for example [5] [6], [4], and the references therein).

Remark 1.1. Let $T : C \to C$ and $S : Q \to Q$ be any two mappings, setting $C = F(T)$ and $Q = F(S)$, problem (1.1) reduces to the following *split equality fixed point problem* (SEFPP):

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$$\text{find} \quad x^* \in F(T), \ y^* \in F(S) \quad \text{such that} \quad Ax^* = By^*. \tag{1.3}$$

If $H_2 = H_3$ and $B = I$, the identity mapping on H_2, then the SEFPP (1.3) reduces to the following *split common fixed point problem* (SCFPP) introduced by Censor and Segal in [6]:

$$\text{find } x \in F(T) \text{ such that } Ax \in F(S).$$

In 2014, Zhao [17] proposed and studied an iterative algorithm for approximating a solution of SEFPP in real Hilbert spaces, and proved weak convergence.

Various algorithms for approximating solution of the SEFPP in a real Hilbert space have been proposed and studied by several authors (see, e.g, [15], [18], [17]). However, it is well known that most of the mathematical problems that arise in real life lie in Banach spaces more general than Hilbert spaces. This fact was rightly captured by Hazewinkel who wrote: "*···many, and probably most, mathematical objects and models do not naturally live in a Hilbert space*" [13], pg. viii.

The Theorems of Chidume *et al.* [11] extend the results of Zhao [17] from real Hilbert space to 2-uniformly convex and uniformly smooth real Banach spaces that have weakly sequentially continuous duality maps. Such spaces include the sequence spaces l_p , for $1 < p \leq 2$. They do not include the important real Banach spaces $l_p, 2 < p < \infty$; $L_p(1 < p < \infty, p \neq 2)$ and the Sobolev spaces $\mathbf{W}_p^m(\Omega)$, $1 < p < \infty, p \neq 2$. Chidume *et al.* [11] proved weak convergence of the sequence of their algorithm to a solution of the SEFFP. Under the assumption that the operators are semi-compact, they proved strong convergence.

In 2018, Chidume *et al.* [12] introduced a new iterative algorithm involving generalized projections in 2-uniformly convex and uniformly smooth real Banach spaces without requiring that the spaces admit weak sequential continuous duality mappings. These spaces include L_p, l_p and the Sobolev spaces $\mathbf{W}_p^m(\Omega)$ for $2 < p < \infty$. They proved that the sequence generated by their algorithm converges strongly to a solution of the SEFFP without requiring that the operators be semi-compact.

Motivated by the research on inertial algorithms to speed up convergence, Adamu and Adam [1] incorporated the inertial extrapolation term in an algorithm for approximating solution(s) of the SEFPP so as to obtain a method which accelerates the approximation of solution of the SEFPP in the setting of 2-uniformly convex and smooth real Banach spaces. They proved strong convergence of the sequence generated by their algorithm to a solution of the SEFPP.

Remark 1.2. It is well known that 2-uniformly convex and uniformly smooth real Banach spaces are more general than real Hilbert spaces, (they include $L_p, l_p, \mathbf{W}_p^m(\Omega)$spaces, for $1 < p \leq 2$). However, they exclude some very important real Banach spaces. In particular, they exclude $L_p, l_p, \mathbf{W}_m^p(\Omega)$ spaces, for $2 < p < \infty$. Consequently, all theorems proved in the literature in 2-uniformly convex real Banach spaces are not applicable in the following very important Banach spaces: L_p, l_p and the Sobolev spaces $\mathbf{W}_m^p(\Omega)$ spaces, for $2 < p < \infty$, because these spaces are not 2-uniformly convex.

Recently, Chidume [8] established new geometric inequalities in real Banach spaces which will be useful tools in p-uniformly convex and uniformly smooth real Banach spaces. These spaces include in particular, L_p, l_p and the Sobolev spaces, for $2 < p < \infty$. As an application, he proposed a new iterative algorithm for approximating a solution of a split equality fixed point problem (SEFPP) for quasi-ϕ-nonexpansive semigroups. Using some of the new geometric inequalities, he proved that the sequence generated by the algorithm converges strongly to a solution of the SEFPP in p-uniformly convex and uniformly smooth real Banach spaces, $p > 2$. These new geometric inequalities established

by Chidume [8] are now generating considerable research interest in the study of iterative methods, (see, e.g., [9, 10]).
It is our purpose in this paper to introduce new inertial iterative algorithms for approximating solutions of the SEFPP in real Banach spaces that will include L_p, l_p, and the Sobolev spaces, $\mathbf{W}_p^m(\Omega)$, for $2 < p < \infty$. Consequently, our theorems will complement, in particular, the results of Zhao [17], Chidume *et al.* [12], Chidume *et al.* [11], Adamu and Adam [1], and a host of other results to provide iterative algorithms for approximating solutions of the SEFPP, assuming existence in L_p, l_p and the Sobolev spaces for $2 < p < \infty$.

2. PRELIMINARIES

Let E be a strictly convex and smooth real Banach space. For $p > 1$, the generalized duality mapping J_p from E to 2^{E^*} is defined by

$$J_p x := \{u^* \in E^* : \langle x, u^* \rangle = \|x\|\|u^*\|, \|u^*\| = \|x\|^{p-1}, \ \forall\, x \in E\}.$$

If $p = 2$, J_2 is called the normalized duality mapping and is denoted by J. It is easy to see from the definition that $J_p(x) = \|x\|^{p-2} Jx$ and $\langle x, J_p x \rangle = \|x\|^p; \forall x \in E$.

Remark 2.3. If E is smooth, then J_p is single-valued and if E is strictly convex, J_p is one-to-one, and if E is reflexive, then J_p is surjective. Furthermore, if E is uniformly smooth and uniformly convex, then the dual space E^* is also uniformly smooth and uniformly convex and the normalized duality map J_p and its inverse, J_p^{-1}, are both uniformly continuous on bounded sets.

Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^*. For $p > 1$, Chidume [8] defined the following functionals:
$\phi_p : E \times E \to \mathbb{R}$ by,

$$\phi_p(x, y) = \|x\|^p - p\langle x, J_p y \rangle + (p-1)\|y\|^p, \ \forall\, x, y \in E, \tag{2.4}$$

$V_p : E \times E^* \to \mathbb{R}$ by

$$V_p(x, x^*) = \|x\|^p - p\langle x, x^* \rangle + (p-1)\|x^*\|^{\frac{p}{p-1}}.$$

It is easy to see from the definition that

$$V_p(x, x^*) = \phi_p(x, J_p^{-1} x^*), \forall x \in E, x^* \in E^*.$$

If $p = 2$, we denote ϕ_2 by ϕ and V_2 by V. So,

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \ \forall\, x, y \in E.$$

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2.$$

Definition 2.1. Let E be a real normed space with dimension $E \geq 2$. The modulus of convexity of E is the function $\delta_E(\epsilon) : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \Big\{1 - \|\frac{u+v}{2}\| : \|u\| = \|v\| = 1; \epsilon = \|u - v\|\Big\}.$$

Let $p > 1$ be a real number and $\delta_E(\epsilon) : (0, 2] \to [0, 1]$ be the modulus of convexity of E. Then, a normed space E is said to be p-uniformly convex if there exists a constant $c > 0$ such that

$$\delta_E(\epsilon) \geq c\epsilon^p$$

.

Definition 2.2. Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty closed and convex subset of E. The map $\Pi_C : E \to C$ defined by $\tilde{x} = \Pi_C(x) \in C$ such that $\phi(\tilde{x}, x) = \inf_{y \in C} \phi(y, x)$ is called the *generalized projection* of E onto C. Clearly, in a real Hilbert space H, the generalized projection Π_C coincides with the metric projection P_C from H onto C.

Lemma 2.1 ([2]). *Let C be a nonempty closed and convex subset of a reflexive, strictly convex and smooth Banach space E. Then*

$$\phi(u, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(u, y), \ \forall \, u \in C, \ y \in E.$$

Definition 2.3. Let E_1 and E_2 be two reflexive, strictly convex and smooth real Banach spaces. The collection of mappings $A : E_1 \to E_2$ that are linear and continuous is a normed linear space with norm defined by $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$. The dual operator $A^* : E_2^* \longrightarrow E_1^*$ defined by $\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle, \forall \, x \in E_1, \, y^* \in E_2^*$ is called the adjoint operator of A. The adjoint operator A^* has the property $\|A^*\| = \|A\|$.

Definition 2.4. Let C be a nonempty closed and convex subset of a real Banach space E and let $T : C \to C$ be any mapping. Then: T is said to be quasi-ϕ-nonexpansive if $F(T) := \{x \in C : Tx = x\} \neq \emptyset$ and

$$\phi_p(x, Ty) \leq \phi_p(x, y) \, \forall \, x \in F(T), \ y \in C.$$

Lemma 2.2 ([8]). *Let E be a p-uniformly convex and smooth real Banach space, and let $\{x_n\}$,$\{y_n\}$ be sequences in E. If $\lim\limits_{n \to \infty} \phi_p(x_n, y_n) = 0$, then, $\lim\limits_{n \to \infty} ||x_n - y_n|| = 0$.*

Lemma 2.3 ([8]). *Let E be a reflexive, strictly convex and smooth real Banach space with dual E^*. Then, for $p > 1$,*

$$V_p(x, x^*) + p\langle J_p^{-1} x^* - x, y^* \rangle \leq V_p(x, x^* + y^*), \ \ \forall \, x \in E, \ x^*, y^* \in E^*.$$

Lemma 2.4 ([8]). *Let E be p-uniformly convex and smooth real Banach space with dual space E^*. For $p > 1$, let $J_p : E \to E^*$ be the generalized duality map. Then,*

$$\|J_p^{-1} u - J_p^{-1} v\| \leq \kappa_p \|u - v\|^{\frac{1}{p-1}}, \ \forall \, u, v \ \in \ E^*.$$

where $\kappa_p = (\frac{1}{c})^{\frac{1}{p-1}}, \ for \ some \ c > 0$.

Lemma 2.5 ([16]). *Let C be a nonempty closed and convex subset of a reflexive, strictly convex and smooth Banach space E, $A : E \to 2^{E^*}$ be a maximal monotone operator with $A^{-1}(0) \neq \emptyset$, then for any $x \in E, y \in A^{-1}(0)$ and $r > 0$ we have*

$$\phi(y, Q_r^A x) + \phi(Q_r^A x, x) \leq \phi(y, x),$$

where $Q_r^A : E \to E$ is defined by $Q_r^A x := (J + rA)^{-1} Jx$.

Remark 2.4. We observe that since E_1 and E_2 are p- uniformly convex, they are reflexive and strictly convex. By our hypothesis, they are smooth. So, Lemma 2.1 and Lemma 2.5 are applicable. Hence we can use the functional ϕ in these Lemmas instead of the functional ϕ_p.

3. Main results

In the sequel, we assume that $J_{p_{E_1}}$, $J_{p_{E_2}}$, $J_{p_{E_3}}$ are the generalized duality maps on E_1, E_2, E_3 respectively, and $J^{-1}_{p_{E_1}}$, $J^{-1}_{p_{E_2}}$, $J^{-1}_{p_{E_3}}$ are the generalized duality maps on E_1^*, E_2^*, E_3^* respectively.

3.1. **Strong convergence theorem.**

Theorem 3.1. *Let E_1 and E_2 be p-uniformly convex and uniformly smooth real Banach spaces, $p > 1$ and E_3 be a uniformly smooth real Banach space. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ (such that $A, B \neq 0$) be bounded linear operators with adjoints A^* and B^*, respectively. Let $T : E_1 \to E_1$ and $S : E_2 \to E_2$ be closed quasi-ϕ-nonexpansive mappings. Setting $\Gamma = \{(x, y) \in F(T) \times F(S) : Ax = By\}$ and assuming $\Gamma \neq \emptyset$. Let $\{(x_n, y_n)\}$ be a sequence generated by*

$$\begin{cases} x_0, x_1 \in E_1,\ y_0, y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2,\ e_n = J_{p_{E_3}}(Aw_n - Bt_n); \\ w_n = x_n + \alpha_n(x_n - x_{n-1}),\ u_n = J^{-1}_{p_{E_1}}(J_{p_{E_1}} w_n - \gamma A^* e_n); \\ r_n = J^{-1}_{p_{E_1}}(a_n J_{p_{E_1}} u_n + (1 - a_n) J_{p_{E_1}} T u_n); \\ t_n = y_n + \alpha_n(y_n - y_{n-1}),\ v_n = J^{-1}_{p_{E_2}}(J_{p_{E_2}} t_n + \gamma B^* e_n); \\ z_n = J^{-1}_{p_{E_2}}(a_n J_{p_{E_2}} v_n + (1 - a_n) J_{p_{E_2}} S v_n); \\ C_{n+1} = \{u \in C_n : \phi_p(u, r_n) \leq \phi_p(u, w_n)\}; \\ Q_{n+1} = \{v \in Q_n : \phi_p(v, z_n) \leq \phi_p(v, t_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ y_{n+1} = \Pi_{Q_{n+1}} y_1;\ n \geq 1. \end{cases} \tag{3.5}$$

where $0 < a_n < 1$, $\alpha_n \in (0, 1)$, $0 < \gamma < \left[\frac{1}{\kappa_p(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}})}\right]^{p-1}$, κ_p a positive constant as in Lemma 2.4 . Then $\{(x_n, y_n)\}$ converges to some point (x^*, y^*) in Γ.

Proof. We divide the proof into 4 steps.

Step 1. We show that C_n and Q_n are closed and convex for any $n \geq 1$.
Since $C_1 = E_1,\ Q_1 = E_2$, C_1 and Q_1 are closed and convex.
Assume C_n and Q_n are closed and convex for some $n \geq 1$. Since for any $(u, v) \in C_n \times Q_n$,

$$\phi_p(u, r_n) \leq \phi_p(u, w_n) \Leftrightarrow p\langle u, J_{p_{E_1}} w_n - J_{p_{E_1}} r_n\rangle \leq \|J_{p_{E_1}} w_n\|^p - \|J_{p_{E_1}} r_n\|^p$$

and

$$\phi_p(v, z_n) \leq \phi_p(v, t_n) \Leftrightarrow p\langle v, J_{p_{E_2}} t_n - J_{p_{E_2}} z_n\rangle \leq \|J_{p_{E_2}} t_n\|^p - \|J_{p_{E_2}} z_n\|^p,$$

its easy to deduce that C_{n+1} and Q_{n+1} are closed and convex. Therefore, C_n and Q_n are closed and convex for any $n \geq 1$.

Step 2. We prove that $\Gamma \subset C_n \times Q_n$, for any $n \geq 1$. Let $K_1 := C_1 \times Q_1$ and

$$K_{n+1} := \{(u, v) \in C_n \times Q_n : \phi_p(u, r_n) + \phi_p(v, z_n) \leq \phi_p(u, w_n) + \phi_p(v, t_n)\}.$$

Then, by construction $K_n \subset C_n \times Q_n$.

Claim. $\Gamma \subset K_n$, for any $n \geq 1$. Clearly, $\Gamma \subset C_1 \times Q_1$. Assume $\Gamma \subset K_n$ for some $n \geq 1$. Let $(x, y) \in \Gamma$, then

$$\begin{aligned} \phi_p(x, r_n) &= \phi_p(x, J^{-1}_{p_{E_1}}(a_n J_{p_{E_1}} u_n + (1 - a_n) J_{p_{E_1}} T u_n)) \\ &= V_p(x, a_n J_{p_{E_1}} u_n + (1 - a_n) J_{p_{E_1}} T u_n) \\ &\leq a_n V_p(x, J_{p_{E_1}} u_n) + (1 - a_n) V_p(x, J_{p_{E_1}} T u_n) \\ &= a_n \phi_p(x, u_n) + (1 - a_n)\phi_p(x, T u_n) \\ &\leq \phi_p(x, u_n). \end{aligned} \tag{3.6}$$

By Lemma 2.3 we get

$$\begin{aligned}\phi_p(x,u_n) &= \phi_p(x, J^{-1}_{p_{E_1}}(J_{p_{E_1}}w_n - \gamma A^* J_{p_{E_3}}(Aw_n - Btn)) \\ &= V_p(x, J_{p_{E_1}}w_n - \gamma A^* J_{p_{E_3}}(Aw_n - Btn)) \\ &\le V_p(x, J_{p_{E_1}}w_n) - p\gamma\langle J^{-1}_{p_{E_1}}(J_{p_{E_1}}w_n - \gamma A^* e_n) - x, A^* e_n\rangle \\ &= \phi_p(x,w_n) - p\gamma\langle Au_n - Ax, e_n\rangle.\end{aligned} \tag{3.7}$$

Thus,

$$\phi_p(x,r_n) \le \phi_p(x,w_n) - p\gamma\langle Au_n - Ax, e_n\rangle. \tag{3.8}$$

Using a similar argument, we obtain that

$$\phi_p(y,z_n) \le \phi_p(y,t_n) - p\gamma\langle By - Bv_n, e_n\rangle. \tag{3.9}$$

Adding inequalities (3.8) and (3.9) and using the fact that $Ax = By$, we get

$$\phi_p(x,r_n) + \phi_p(y,z_n) \le \phi_p(x,w_n) + \phi_p(y,t_n) - p\gamma\langle Au_n - Bv_n, e_n\rangle. \tag{3.10}$$

Using the fact that $e_n = J_{p_{E_3}}(Aw_n - Bt_n)$, we estimate as follows

$$\begin{aligned}&-p\gamma\langle Au_n - Bv_n, e_n\rangle \\ &= -p\gamma\|Aw_n - Bt_n\|^p - p\gamma\langle Au_n - Bv_n, e_n\rangle + p\gamma\langle Aw_n - Bt_n, e_n\rangle \\ &= -p\gamma\|Aw_n - Bt_n\|^p + p\gamma\langle A(w_n - u_n), e_n\rangle + \gamma\langle B(v_n - t_n), e_n\rangle \\ &= -p\gamma\|Aw_n - Bt_n\|^p + p\gamma\langle J^{-1}_{p_{E_1}} J_{p_{E_1}}w_n - J^{-1}_{p_{E_1}}(J_{p_{E_1}}w_n - \gamma A^* e_n), A^* e_n\rangle \\ &\quad + p\gamma\langle J^{-1}_{p_{E_2}}(J_{p_{E_2}}t_n + \gamma B^* e_n) - J^{-1}_{p_{E_2}} J_{p_{E_2}}t_n, B^* e_n\rangle \\ &\le -p\gamma\|Aw_n - Bt_n\|^p + p\gamma\|A\|\|e_n\|\|J^{-1}_{p_{E_1}} J_{p_{E_1}}w_n - J^{-1}_{p_{E_1}}(J_{p_{E_1}}w_n - \gamma A^* e_n)\| \\ &+ p\gamma\|B\|\|e_n\|\|J^{-1}_{p_{E_2}}(J_{p_{E_2}}t_n + \gamma B^* e_n) - J^{-1}_{p_{E_2}} J_{p_{E_2}}t_n\| \\ &\le -p\gamma\|Aw_n - Bt_n\|^p + p\kappa_p\gamma\|A\|\|e_n\|\|\gamma A^* e_n\|^{\frac{1}{p-1}} + p\kappa_p\gamma\|B\|\|e_n\|\|\gamma B^* e_n\|^{\frac{1}{p-1}} \\ &\le -p\gamma\|Aw_n - Bt_n\|^p + p\kappa_p\gamma^{\frac{p}{p-1}}\|e_n\|^p\|A\|^{\frac{p}{p-1}} + p\kappa_p\gamma^{\frac{p}{p-1}}\|e_n\|^p\|B\|^{\frac{p}{p-1}} \\ &= -p\gamma\Big(1 - \kappa_p\gamma^{\frac{1}{p-1}}(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}})\Big)\|(Aw_n - Bt_n)\|^p.\end{aligned} \tag{3.11}$$

Substituting inequality (3.11) in inequality (3.10) and using the fact that $0 < \gamma < \Big[\frac{1}{\kappa_p(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}})}\Big]^{p-1}$, we have that

$$\begin{aligned}\phi_p(x,r_n) + \phi_p(y,z_n) &\le \phi_p(x,w_n) + \phi_p(y,t_n) \\ &\qquad - p\gamma\Big(1 - \kappa_p\gamma^{\frac{1}{p-1}}(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}})\Big)\|(Aw_n - Bt_n)\|^p \\ &\le \phi_p(x,w_n) + \phi_p(y,t_n).\end{aligned} \tag{3.12}$$

Hence $\Gamma \subset K_n$, for any $n \ge 1$. Thus, $\Gamma \subset C_n \times Q_n$, for any $n \ge 1$.

Step 3. We shall show that $\lim_{n\to\infty}(x_n, y_n) = (x^*, y^*) \in E_1 \times E_2$.

Let $(u,v) \in \Gamma$. Since $\Gamma \subset C_{n+1} \times Q_{n+1} \subset C_n \times Q_n$ and $x_{n+1} = \Pi_{C_{n+1}}x_1 \subset C_n$, then by Lemma 2.1 we have that

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \le \phi(u, x_1) - \phi(u, x_n)$$

which implies $\{\phi(x_n, x_1)\}$ is bounded. Furthermore, $\phi(x_n, x_1) \le \phi(x_{n+1}, x_1)$. Hence $\{\phi(x_n, x_1)\}$ is nondecreasing. Thus, $\lim_{n\to\infty}\phi(x_n, x_1)$ exists. This implies that $\{x_n\}$ is bounded and consequently $\{w_n\}$ is bounded. Similarly, $\phi(y_n, y_1)$ is convergent implies that, $\{y_n\}$ is bounded and consequently $\{t_n\}$ is bounded.

By Lemma 2.1 we have that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \to 0, \text{ as } n, m \to \infty.$$

Hence, by Lemma 2.2 we obtain that $\|x_m - x_n\| \to 0$, as $m, n \to \infty$, which implies that $x_n \to x^* \in E_1$, as $n \to \infty$. Following similar argument, we also obtain that $y_n \to y^* \in E_2$, as $n \to \infty$.

Step 4. We show that $(x^*, y^*) \in \Gamma$.

Using the definition of w_n and t_n, we have that

$$\begin{aligned} \|x_n - w_n\| &= \|\alpha_n(x_{n-1} - x_n)\| \leq \|x_{n-1} - x_n\| \to 0, \text{ as } n \to \infty, \\ \|y_n - t_n\| &= \|\alpha_n(y_{n-1} - y_n)\| \leq \|y_{n-1} - y_n\| \to 0, \text{ as } n \to \infty. \end{aligned}$$

$$\text{So, } \lim_{n\to\infty} \phi_p(x_n, w_n) = 0 = \lim_{n\to\infty} \phi_p(y_n, t_n).$$

Since $(x_{n+1}, y_{n+1}) \in C_{n+1} \times Q_{n+1}$, we have that

$$\begin{aligned} \phi_p(x_{n+1}, r_n) &\leq \phi_p(x_{n+1}, w_n) \to 0, \text{ as } n \to \infty, \\ \phi_p(y_{n+1}, z_n) &\leq \phi_p(y_{n+1}, t_n) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \phi_p(x_{n+1}, r_n) = 0 = \lim_{n\to\infty} \phi_p(y_{n+1}, z_n)$. Hence, by Lemma 2.2, we have that

$$\lim_{n\to\infty} \|x_{n+1} - r_n\| = 0 = \lim_{n\to\infty} \|y_{n+1} - z_n\|.$$

Therefore, $r_n \to x^*$. and $z_n \to y^*$. as $n \to \infty$. Let $\varepsilon = p\gamma\Big(1 - \kappa_p \gamma^{\frac{1}{p-1}} (\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}})\Big)$, then from inequality (3.12) we have that

$$\varepsilon\|(Aw_n - Bt_n)\|^p \leq \phi_p(x, w_n) + \phi_p(y, t_n) - \phi_p(x, w_n) - \phi_p(y, t_n).$$

This implies that

$$\begin{aligned} \lim_{n\to\infty} \big(\varepsilon\|Aw_n - Bt_n\|^p\big) &\leq \lim_{n\to\infty} \big(\phi_p(x, w_n) + \phi_p(y, t_n) - \phi_p(x, w_n) - \phi_p(y, t_n)\big) \\ &= \phi_p(x, x^*) + \phi_p(y, y^*) - \phi_p(x, x^*) - \phi_p(y, y^*) = 0. \end{aligned}$$

Using the condition on γ we get

$$0 = \lim_{n\to\infty} \|Aw_n - Bt_n\| = \|Ax^* - By^*\| \tag{3.13}$$

which implies

$$Ax^* = By^*. \tag{3.14}$$

Next, we show that $(x^*, y^*) \in F(T) \times F(S)$. Using Lemma 2.4 we obtain that

$$\begin{aligned} \|u_n - x^*\| &= \|J_{p_{E_1}}^{-1}(J_{p_{E_1}} w_n - \gamma A^* e_n) - J_{p_{E_1}}^{-1} J_{p_{E_1}} x^*\| \\ &\leq \kappa_p \|J_{p_{E_1}} w_n - \gamma A^* e_n - J_{p_{E_1}} x^*\|^{\frac{1}{p-1}} \\ &\leq \kappa_p \big(\|J_{p_{E_1}} w_n - J_{p_{E_1}} x^*\| + \gamma\|A\|\|Aw_n - Bt_n\|\big)^{\frac{1}{p-1}}. \end{aligned}$$

Using equation (3.13) and the fact that the normalized duality mapping $J_{p_{E_1}}^{-1}$ is uniformly continuous on bounded subsets of E_1, this implies that $\lim_{n\to\infty} u_n = x^*$. Also,

$$\begin{aligned}\|J_{p_{E_1}} r_n - J_{p_{E_1}} x^*\| &= \|a_n J_{p_{E_1}} u_n + (1-a_n) J_{p_{E_1}} Tu_n - J_{p_{E_1}} x^*\| \\ &= \|(1-a_n)(J_{p_{E_1}} Tu_n - J_{p_{E_1}} x^*) - a_n(J_{p_{E_1}} x^* - J_{p_{E_1}} u_n)\| \\ &\geq (1-a_n)\|J_{p_{E_1}} Tu_n - J_{p_{E_1}} x^*\| - a_n\|J_{p_{E_1}} x^* - J_{p_{E_1}} u_n\|,\end{aligned}$$

implies that $\lim_{n\to\infty} \|J_{p_{E_1}} Tu_n - J_{p_{E_1}} x^*\| = 0$. By norm-to-weak continuity of $J_{p_{E_1}}^{-1}$, we have that $Tu_n \rightharpoonup x^*$ as $n \to \infty$. Furthermore,

$$|\|Tu_n\| - \|x^*\|| = |\|J_{p_{E_1}} Tu_n\| - \|J_{p_{E_1}} x^*\|| \leq \|J_{p_{E_1}} Tu_n - J_{p_{E_1}} x^*\| \to 0.$$

Thus, $\lim_{n\to\infty} \|Tu_n\| = \|x^*\|$. Hence, by Kadec-Klee property of E_1, we have that $\lim_{n\to\infty} Tu_n = x^*$. Using this, closedness of T and the fact that $\lim_{n\to\infty} u_n = x^*$, we have that $Tx^* = x^*$. Following the same argument, we also have that $Sy^* = y^*$. Thus, $(x^*, y^*) \in F(T) \times F(S)$. This together with (3.14) imply that $(x^*, y^*) \in \Gamma$. This completes the proof. □

Corollary 3.1. *Let E_1 and E_2 be l_p, $L^p(G)$, or the Sobolev spaces $W^p{}_m(G)$, $1 < p < \infty$ and E_3 be a uniformly smooth real Banach space. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ (such that $A, B \neq 0$) be bounded linear operators with adjoints A^* and B^*, respectively. Let $T : E_1 \to E_1$ and $S : E_2 \to E_2$ be closed quasi-ϕ-nonexpansive mappings. Setting $\Gamma = \{(x, y) \in F(T) \times F(S) : Ax = By\}$ and assuming $\Gamma \neq \emptyset$. Let $\{(x_n, y_n)\}$ be a sequence generated by (3.5). where $0 < a_n < 1$, $\alpha \in (0,1)$ $0 < \gamma < \left[\frac{1}{\kappa_p(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}})}\right]^{p-1}$. Then $\{(x_n, y_n)\}$ converges to some point (x^*, y^*) in Γ.*

4. APPLICATIONS

4.1. Split Equality Variational Inclusion Problem (SEVIP).

Theorem 4.2. *Let E_1 and E_2 be uniformly smooth and p-uniformly convex real Banach spaces and E_3 be a smooth real Banach space. Let $M : E_1 \to 2^{E_1^*}$ and $N : E_2 \to 2^{E_2^*}$ be maximal montone operators such that $M^{-1}(0)$ and $N^{-1}(0)$ are nonempty. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators with adjoints A^* and B^* such that $A, B \neq 0$. Assuming that $\Omega = \{(x, y) \in M^{-1}(0) \times N^{-1}(0) : Ax = By\} \neq \emptyset$. Let $\{(x_n, y_n)\}$ be a sequence generated by*

$$\begin{cases} x_0, x_1 \in E_1,\ y_0, y_1 \in E_2,\ e_n = J_{p_{E_3}}(Aw_n - Bt_n); \\ w_n = x_n + \alpha_n(x_n - x_{n-1}),\ u_n = J_{p_{E_1}}^{-1}(J_{p_{E_1}} w_n - \gamma A^* e_n); \\ r_n = J_{p_{E_1}}^{-1}(a_n J_{p_{E_1}} u_n + (1-a_n) J_{p_{E_1}} Q_r^M u_n); \\ t_n = y_n + \alpha_n(y_n - y_{n-1}),\ v_n = J_{p_{E_2}}^{-1}(J_{p_{E_2}} t_n + \gamma B^* e_n); \\ z_n = J_{p_{E_2}}^{-1}(a_n J_{p_{E_2}} v_n + (1-a_n) J_{p_{E_2}} Q_r^N v_n); \\ C_{n+1} = \{u \in C_n : \phi_p(u, r_n) \leq \phi_p(u, w_n)\}; \\ Q_{n+1} = \{v \in Q_n : \phi_p(v, z_n) \leq \phi_p(v, t_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ y_{n+1} = \Pi_{Q_{n+1}} y_1;\ n \geq 1, \end{cases} \tag{4.15}$$

where $Q_r^M = (J_{p_{E_1}} + rM)^{-1} J_{p_{E_1}}$, $Q_r^N = (J_{p_{E_2}} + rN)^{-1} J_{p_{E_2}}$, $0 < a_n < 1$, $\alpha \in (0,1)$, $0 < \gamma < \left[\frac{1}{\kappa_p(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}})}\right]^{p-1}$. Then $\{(x_n, y_n)\}$ converges to some point (x^*, y^*) in Ω.

Proof. Setting $T = Q_r^M$ and $S = Q_r^N$, from Lemma 2.5 we can see that Q_r^M and Q_r^N are quasi-ϕ-nonexpansive. And by Theorem 3.1 we get the desired result. □

4.2. **Split equality problem.** The split equality problem is to find

$$x \in C,\ y \in Q \ \text{ such that } \ Ax = By.$$

Theorem 4.3. *Let E_1 and E_2 be uniformly smooth and p-uniformly convex real Banach spaces and E_3 be a smooth real Banach space. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators with adjoints A^* and B^* such that $A, B \neq 0$. Assuming $\Gamma \neq \emptyset$. Let $\{(x_n, y_n)\}$ be a sequence generated by*

$$\begin{cases} x_0, x_1 \in E_1,\ y_0, y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2,\ e_n = J_{p_{E_3}}(Aw_n - Bt_n); \\ w_n = x_n + \alpha_n(x_n - x_{n-1}),\ u_n = J^{-1}_{p_{E_1}}(J_{p_{E_1}} w_n - \gamma A^* e_n); \\ r_n = J^{-1}_{p_{E_1}}(a_n J_{p_{E_1}} u_n + (1 - a_n) J_{p_{E_1}} \Pi_C u_n); \\ t_n = y_n + \alpha_n(y_n - y_{n-1}),\ v_n = J^{-1}_{p_{E_2}}(J_{p_{E_2}} t_n + \gamma B^* e_n); \\ z_n = J^{-1}_{p_{E_2}}(a_n J_{p_{E_2}} v_n + (1 - a_n) J_{p_{E_2}} \Pi_Q v_n); \\ C_{n+1} = \{u \in C_n : \phi_p(u, r_n) \leq \phi_(u, w_n)\}; \\ Q_{n+1} = \{v \in Q_n : \phi_p(v, z_n) \leq \phi_P(v, t_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ y_{n+1} = \Pi_{Q_{n+1}} y_1;\ n \geq 1, \end{cases} \tag{4.16}$$

where $0 < a_n < 1$, $\alpha \in (0, 1)$, $0 < \gamma < \left[\frac{1}{\kappa_p(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}})}\right]^{p-1}$. *Then $\{(x_n, y_n)\}$ converges to some point (x^*, y^*) in Γ.*

Proof. Setting $T = \Pi_C$ and $S = \Pi_Q$, from Lemma 2.1 we have that Π_C and Π_Q are quasi-ϕ-nonexpansive. And by Theorem 3.1 we get the desired result.

□

5. Iterative algorithm of Krasnoselskii-type

Replacing α_n in iterative algorithm 4.16 by a fixed constant, λ such that $0 < \lambda < 1$, the algorithm reduces to the following Krasnoselskii-type algorithm.

$$\begin{cases} x_0, x_1 \in E_1,\ y_0, y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2,\ e_n = J_{p_{E_3}}(Aw_n - Bt_n); \\ w_n = x_n + \lambda(x_n - x_{n-1}),\ u_n = J^{-1}_{p_{E_1}}(J_{p_{E_1}} w_n - \gamma A^* e_n); \\ r_n = J^{-1}_{p_{E_1}}(a_n J_{p_{E_1}} u_n + (1 - a_n) J_{p_{E_1}} \Pi_C u_n); \\ t_n = y_n + \lambda(y_n - y_{n-1}),\ v_n = J^{-1}_{p_{E_2}}(J_{p_{E_2}} t_n + \gamma B^* e_n); \\ z_n = J^{-1}_{p_{E_2}}(a_n J_{p_{E_2}} v_n + (1 - a_n) J_{p_{E_2}} \Pi_Q v_n); \\ C_{n+1} = \{u \in C_n : \phi_p(u, r_n) \leq \phi_(u, w_n)\}; \\ Q_{n+1} = \{v \in Q_n : \phi_p(v, z_n) \leq \phi_P(v, t_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ y_{n+1} = \Pi_{Q_{n+1}} y_1;\ n \geq 1, \end{cases} \tag{5.17}$$

6. Numerical Illustration

In this section, we give a numerical implementation of our proposed algorithm (3.5) and we show the effect of the inertial parameter in the performance of our algorithm.

Example 6.1. Consider the simplified version of algorithm 3.5 in $\mathbb{R}^n$.

$$\begin{cases} x_0, x_1 \in \mathbb{R},\ y_0, y_1 \in \mathbb{R},\ C_1 = \mathbb{R},\ Q_1 = \mathbb{R},\ e_n = Aw_n - Bt_n; \\ w_n = x_n + \alpha_n(x_n - x_{n-1}); \\ r_n = a_n u_n + (1 - a_n)Tu_n,\ u_n = w_n - \gamma A^* e_n; \\ t_n = y_n + \alpha_n(y_n - y_{n-1}); \\ z_n = a_n v_n + (1 - a_n)Sv_n,\ v_n = t_n + \gamma B^* e_n; \\ C_{n+1} = \{u \in C_n : |u - r_n| \leq |u - w_n|\}; \\ Q_{n+1} = \{v \in Q_n : |v - z_n| \leq |v - t_n|\}; \\ x_{n+1} = P_{C_{n+1}} x_1,\ y_{n+1} = P_{Q_{n+1}} y_1;\ n \geq 1. \end{cases} \tag{6.18}$$

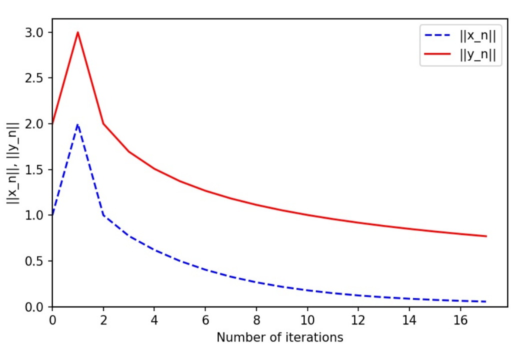
In algorithm (6.18) let $A : \mathbb{R} \to \mathbb{R}^2$, $B : \mathbb{R} \to \mathbb{R}^2$, $T : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ be defined by

$$Ax := \left(\frac{x}{2}, \frac{x}{3}\right),\ \text{then } A^*(u,v) = \frac{u}{2} + \frac{v}{3},\ Bx := \left(\frac{x}{4}, \frac{x}{5}\right),\ \text{then } B^*(u,v) = \frac{u}{4} + \frac{v}{5},$$

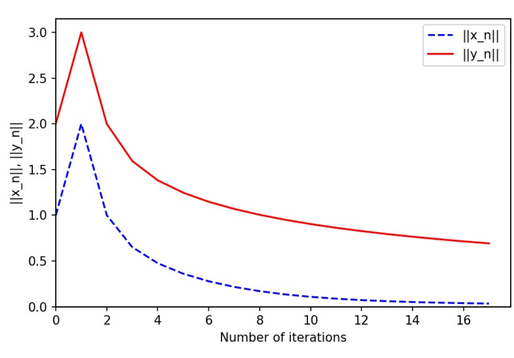
and set $Tx = \frac{x}{4}$, $Sx = \sin x$. In algorithm (6.18), we take $\gamma = 0.1$, $a_n = 0.5$ and we vary the inertial parameter α_n. Setting maximum number of iteration $n = 20$ and tolerance 10^{-4}.

TABLE 1. Numerical results of Example 6.1

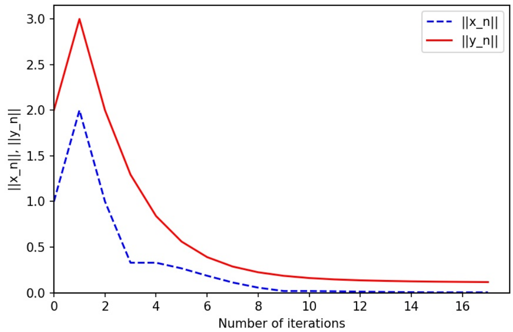
Table of values choosing $x_0 = 1$, $x_1 = 2$, $y_0 = 2$ and $y_1 = 3$		
20th Iterates of Algorithm 6.18		
	$\lvert x_{n+1} - 0\rvert$	$\lvert y_{n+1} - 0\rvert$
$\alpha_n = 0.05$	0.0553	0.7702
$\alpha_n = 0.2$	0.0341	0.692
$\alpha_n = 0.6$	0.0041	0.1164
$\alpha_n = \frac{1}{n}$	0.0248	0.6455
$\alpha_n = \frac{1}{n^2}$	0.0497	0.7582
$\alpha_n = \frac{4n}{4n+5}$	0.0099	0.2258



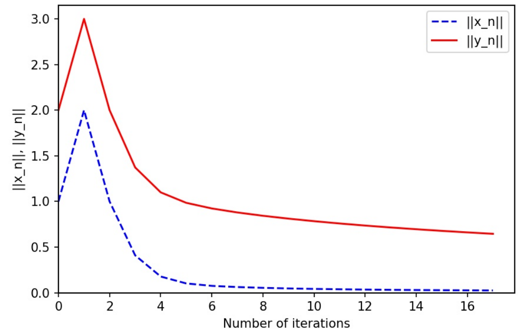
(A) $\alpha_n = 0.005$



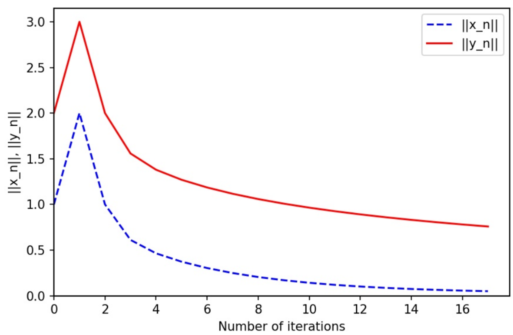
(B) $\alpha_n = 0.2$



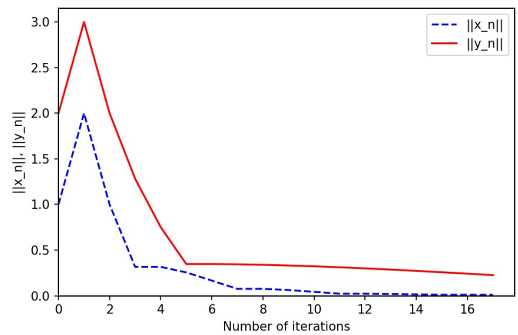
(C) $\alpha_n = 0.6$



(D) $\alpha_n = \frac{1}{n}$



(E) $\alpha_n = \frac{1}{n^2}$



(F) $\alpha_n = \frac{4n}{4n+5}$

FIGURE 1. Graph of the first 20 iterates of algorithm (6.18) for various α_n

Remark 6.5. In general, Krasnoselskii-type algorithms are known to converge as fast as a geometric progression. From Table 1 and Figure (A-F), for constant and variable choices of α_n as α_n approaches $\frac{1}{2}(\lambda = \frac{1}{2})$, we obtain better approximations.

7. DECLARATIONS

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7.2. **Conflict of interest.** The authors declare that they have no conflict of interest.

7.3. **Authors Contribution.** We will like to affirm that this manuscript was part of the findings of A.A. Adam in her course with C.E. Chidume with A. Adamu as his teaching assistant. The first author C.E. Chidume read and approved the final manuscript before his demise.

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