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Hausdorff series in semigroup rings of rectangular bands

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ABSTRACT. The Hausdorff series provides a solution to the equation $w = \log(e^u e^v)$ given by a recursive formula which can be expressed as nested commutators of u and v. Evolutions of the Haussdorff series in various algebras and rings has been considered in obtaining a closed form of this formula. We consider the rectangular band $L_m \times R_n$ determined by the left zero semigroup L_m and the right zero semigroup R_n of order m and n, respectively. Let $\mathbb{R}\langle L_m \times R_n\rangle$ be the semigroup ring spanned on $L_m \times R_n$ together with the identity element 1. We provide a closed form of the formula for solving the equation in $\mathbb{R}\langle L_m \times R_n\rangle$.

1. INTRODUCTION

In relation to the problem of construction a Lie group from a given Lie algebra, Campbell [3, 4] published two papers in 1897, without overcoming the convergence problem. These studies were followed by the Baker [2] in 1905 and Hausdorff [10] in 1906 solving the convergence problem, although they did not mention Lie groups. Thus, Campbell-Baker-Hausdorff (CBH) formula dedicated to the names of three influential mathematicians John Edward Campbell, Henry Frederick Baker, and Felix Hausdorff was revealed.

The Campbell-Baker–Hausdorff (CBH) series is of fundamental importance in the theory of Lie groups, their applications, physics, and physical chemistry. The CBH formula is a general result for the quantity $w = w(u, v) = \log(e^u e^v)$ or $e^w = e^u e^v$, where u and v are not necessarily commutative. Standard methods for the explicit construction of the CBH terms yield polynomial representations, which must be translated into the usually required commutator representation.

The expression of the CBH formula suggests that w can be presented as a series of nested commutators of u and v, which does not provide a general formula. Dynkin [6] in 1947 derived an explicit expression for the terms as a sum of iterated commutators over a certain set of sequences. The first few terms are shown below.

$$w = u + v + \frac{1}{2}[u, v] + \frac{1}{12}[u, [u, v]] + \frac{1}{12}[v, [v, u]] - \frac{1}{24}[u, [v, [u, v]]]$$
$$-\frac{1}{720}[v, [v, [v, [v, u]]]] - \frac{1}{720}[u, [u, [u, [u, v]]]] + \frac{1}{360}[u, [v, [v, [v, u]]]] + \cdots$$

Although the expression is known in the recursive form, evolutions of the Haussdorff series in various algebras and rings has been considered in obtaining a closed form of this formula. One may see the works by Gerritzen [9], Kurlin [12], and Drensky and Fındık [7].

Baker [1] gave evaluations of the Hausdorff series in some finite dimensional Lie algebras and computed the coefficients according to a fixed basis set.

In the theory of semigroups, a semigroup S is a rectangular band if $s_1 s_2 s_1 = s_1$ for each $s_1, s_2 \in S$. The initial equivalent definition of rectangular bands was first given by

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Clifford [5]. It is well known, see e.g. [11], that each rectangular band can be expressed as a cartesian product of a left zero semigroup and a right zero semigroup.

Given a finite semigroup S one may consider the semigroup ring $\mathbb{R}\langle S\rangle$ spanned on the elements of S together with the identity element 1, and compute the evolution of the CBH formula inside $\mathbb{R}\langle S\rangle$. Recently Fındık and the author [8] considered the semigroup rings $\mathbb{R}\langle L_m\rangle$ and $\mathbb{R}\langle R_n\rangle$ of the left and right semigroups of order m and n, respectively. They gave a solution to the equation $w = \log(e^u e^v)$, where u, v are in $\mathbb{R}\langle L_m\rangle$ or in $\mathbb{R}\langle R_n\rangle$ for which a closed formula was obtained in these semigroup rings.

In the present paper, as a generalization of the paper [8] on $\mathbb{R}\langle L_m\rangle$ and $\mathbb{R}\langle R_n\rangle$, we compute the evolution of CBH formula in the semigroup ring $\mathbb{R}\langle L_m \times R_n\rangle$ of the rectangular band $L_m \times R_n$ of order mn.

2. Main results

Let $L_m = \{l_1, \dots, l_m\}$ and $R_n = \{r_1, \dots, r_n\}$ be the left zero semigroup and right semigroup, respectively. We consider the rectangular band

$$A_{mn} = L_m \times R_n = \{a_{ij} = (l_i, r_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

with the multiplication rule $a_{ij}a_{kl} = a_{il}$. Note that A_{mn} is isomorphic to a left or a right zero semigroup when $n = 1$ or $m = 1$, respectively. Since these cases has been considered in [8], we assume that $m, n \geq 2$ throughout the paper.

Now let $\mathbb{R}\langle A_{mn}\rangle$ denote the semigroup ring spanned on the basis $\{1\} \cup A_{mn}$ over the field $\mathbb{R}$ of real numbers. Each element u of $\mathbb{R}\langle A_{mn}\rangle$ is of the form

$$u = x_{00} \cdot 1 + \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \cdot a_{ij} = x_{00} + \sum_{i,j} x_{ij} a_{ij}$$

for some $x_{00}, x_{ij} \in \mathbb{R}$. Now let us define the following notations.

$$x = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}, \quad r_{x_i} = \sum_{j=1}^{n} x_{ij}, \quad c_{x_j} = \sum_{i=1}^{m} x_{ij}$$

Clearly x, r_{x_i}, and c_{x_j} are the sum of all entries, of entries of i-th row, and of entries of j-th column of the matrix $[x_{ij}]_{m \times n}$, respectively.

The multiplication in $\mathbb{R}\langle A_{mn}\rangle$ is as follows. Let

$$u = x_{00} + \sum_{i,j} x_{ij} a_{ij}, \quad \text{and} \quad v = y_{00} + \sum_{i,j} y_{ij} a_{ij}$$

for some $x_{00}, x_{ij}, y_{00}, y_{ij} \in \mathbb{R}$. Then it is straightforward to see that

$$uv = x_{00}v + y_{00}u + \sum_{i=1}^{m} \sum_{j=1}^{n} (r_{x_i} c_{y_j}) a_{ij} = x_{00}v + y_{00}u + \sum_{i,j} r_{x_i} c_{y_j} a_{ij}.$$

Solving the equation $w = \log(e^u e^v) = z_{00} + \sum_{i,j} z_{ij} a_{ij}$, it is sufficient to consider the elements u, and v without constant terms by Lemma 2.1 of [8], which implies that $e^{w - z_{00}} = e^{u - x_{00}} e^{v - y_{00}}$. The next lemma supports our main result considering the exponent of a given element in the subsemigroup ring of A spanned on a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$.

Lemma 2.1. *Let $u = \sum_{i,j} x_{ij} a_{ij}$, $v = \sum_{i,j} y_{ij} a_{ij}$, $x = \sum_{i,j} x_{ij}$, and $y = \sum_{i,j} y_{ij}$. Then the following assertions hold.*

(*a*) $u^k = x^{k-2}u^2 = x^{k-2}\sum_{i,j} r_{x_i}c_{x_j}a_{ij}$ *for each* $k \geq 3$*, and*

$$e^u = 1 + u + \left(\frac{e^x - 1 - x}{x^2}\right)u^2.$$

(*b*)

$$u^2 = \sum_{i,j} r_{x_i}c_{x_j}a_{ij}, \quad v^2 = \sum_{i,j} r_{y_i}c_{y_j}a_{ij}, \quad uv = \sum_{i,j} r_{x_i}c_{y_j}a_{ij}$$
$$u^2v = x\sum_{i,j} r_{x_i}c_{y_j}a_{ij}, \quad uv^2 = y\sum_{i,j} r_{x_i}c_{y_j}a_{ij}, \quad u^2v^2 = xy\sum_{i,j} r_{x_i}c_{y_j}a_{ij}.$$

Proof. We shall prove only (a), since the proof of (b) is obtained by direct computations. One may easily obtain that $u^3 = xu^2$, which implies by induction that $u^k = x^{k-2}u^2$, $k \geq 3$. The following computations complete the proof.

$$e^u = \sum_{k\geq 0}\frac{u^k}{k!} = 1 + u + \sum_{m\geq 2}\frac{x^{k-2}u^2}{k!} = 1 + \left(\frac{e^x - 1 - x}{x^2}\right)u^2.$$

□

Now we are ready to state the main result of our paper.

Theorem 2.1. *Let* $u, v, w \in A$ *such that* $e^w = e^u e^v$*, where* $u = \sum_{i,j} x_{ij}a_{ij}$*,* $v = \sum_{i,j} y_{ij}a_{ij}$*, and* $w = \sum_{i,j} z_{ij}a_{ij}$*. Then for each* $1 \leq i \leq m$*,* $1 \leq j \leq n$*, we have the following.*

$$z_{ij} = t_{ij} - r_{z_i}c_{z_j}\frac{e^{x+y} - 1 - x - y}{(x+y)^2},$$

where

$$t_{ij} = x_{ij} + y_{ij} + r_{x_i}c_{y_j}B_1 + r_{x_i}c_{x_j}B_2 + r_{y_i}c_{y_j}B_3$$
$$r_{z_i} = \frac{x+y}{e^{x+y}-1}\left[r_{x_i}(1 + yB_1 + xB_2) + r_{y_i}(1 + yB_3)\right]$$
$$c_{z_j} = \frac{x+y}{e^{x+y}-1}\left[c_{x_j}(1 + xB_2) + c_{y_j}(1 + xB_1 + yB_3)\right]$$
$$B_1 = \frac{(e^x-1)(e^y-1)}{xy},\ B_2 = \frac{e^x - 1 - x}{x^2},\ B_3 = \frac{e^y - 1 - y}{y^2}.$$

Proof. Lemma 2.1 together with $e^u e^v = e^w$ give

$$(1 + u + \frac{e^x - 1 - x}{x^2}u^2)(1 + v + \frac{e^y - 1 - y}{y^2}v^2) = 1 + w + \frac{e^z - 1 - z}{z^2}w^2$$

and

$$1 + u + v + uv + \frac{e^x - 1 - x}{x^2}u^2 + \frac{e^y - 1 - y}{y^2}v^2 + \frac{e^y - 1 - y}{y^2}uv^2$$
$$+ \frac{e^x - 1 - x}{x^2}u^2v + \frac{e^x - 1 - x}{x^2}\frac{e^y - 1 - y}{y^2}u^2v^2 = 1 + w + \frac{e^z - 1 - z}{z^2}w^2$$

The coefficient of a_{ij} for each pair (i, j) from both sides, by Lemma 2.1 (b), is

$$x_{ij} + y_{ij} + r_{x_i}c_{y_j}\left[1 + \frac{e^x - 1 - x}{x} + \frac{e^y - 1 - y}{y} + \frac{e^x - 1 - x}{x}\frac{e^y - 1 - y}{y}\right]$$
$$+ r_{x_i}c_{x_j}\frac{e^x - 1 - x}{x^2} + r_{y_i}c_{y_j}\frac{e^y - 1 - y}{y^2} = z_{ij} + \frac{e^y - 1 - y}{y^2}r_{z_i}c_{z_j}$$

or

$$t_{ij} = x_{ij} + y_{ij} + r_{x_i} c_{y_j} B_1 + r_{x_i} c_{x_j} B_2 + r_{y_i} c_{y_j} B_3 = z_{ij} + \frac{e^z - 1 - z}{z^2} r_{z_i} c_{z_j}$$

where

$$B_1 = 1 + \frac{e^x - 1 - x}{x} + \frac{e^y - 1 - y}{y} + \frac{e^x - 1 - x}{x} \frac{e^y - 1 - y}{y} = \frac{(e^x - 1)(e^y - 1)}{xy}$$

$$B_2 = \frac{e^x - 1 - x}{x^2},\ B_3 = \frac{e^y - 1 - y}{y^2}.$$

Considering the sum of all mn equations we get that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} = x + y + xyB_1 + x^2 B_2 + y^2 B_3 = z + \frac{e^z - 1 - z}{z^2} z$$

or

$$e^{x+y} - 1 = e^z - 1$$

which gives that $z = x + y$. Now going back to the definition t_{ij}, we need r_{z_i} and c_{z_j} for obtaining the explicit expressions of z_{ij}. The following computations complete the proof by giving the desired expressions stated in the theorem.

$$\sum_{j=1}^{n} t_{ij} = r_{x_i} + r_{y_i} + r_{x_i} y B_1 + r_{x_i} x B_2 + r_{y_i} y B_3 = \frac{e^z - 1}{z} r_{z_i}$$

$$\sum_{i=1}^{m} t_{ij} = c_{x_j} + c_{y_j} + c_{y_j} x B_1 + c_{x_j} x B_2 + c_{y_j} y B_3 = \frac{e^z - 1}{z} c_{z_j}$$

□

Let us illustrate the result by an example as follows. Let $m = n = 2$, and

$$a_{11} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, a_{12} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, a_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, a_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then it is easy to check that $a_{ij} a_{kl} = a_{il}$ for $1 \leq i, j \leq 2$. Hence $A_{22} = \{a_{11}, a_{12}, a_{21}, a_{22}\}$ is a rectangular band of order 4. Now let

$$u = \sum_{i,j} x_{ij} a_{ij} = 2a_{11} - a_{12} + a_{21} + 3a_{22},$$

$$v = \sum_{i,j} y_{ij} a_{ij} = -a_{11} + a_{12} - 2a_{21} - a_{22},$$

$$w = \sum_{i,j} z_{ij} a_{ij} = \log(e^u e^v).$$

Then we have that

$$x = \sum_{i,j} x_{ij} = 2 - 1 + 1 + 3 = 5,$$

$$r_{x_1} = \sum_{j} x_{1j} = 2 - 1 = 1,$$

$$r_{x_2} = \sum_{j} x_{2j} = 1 + 3 = 4,$$

$$c_{x_1} = \sum_{i} x_{i1} = 2 + 1 = 3,$$

$$c_{x_2} = \sum_i x_{i2} = -1 + 3 = 2,$$

and

$$y = \sum_{i,j} y_{ij} = -1 + 1 - 2 - 1 = -3,$$
$$r_{y_1} = \sum_j y_{1j} = -1 + 1 = 0,$$
$$r_{y_2} = \sum_j y_{2j} = -2 - 1 = -3,$$
$$c_{y_1} = \sum_i y_{i1} = -1 - 2 = -3,$$
$$c_{y_2} = \sum_i y_{i2} = 1 - 1 = 0.$$

Also,

$$B_1 = \frac{(e^x - 1)(e^y - 1)}{xy} = -\frac{(e^5 - 1)(e^{-3} - 1)}{15},$$
$$B_2 = \frac{e^x - 1 - x}{x^2} = \frac{e^5 - 6}{25},$$
$$B_3 = \frac{e^y - 1 - y}{y^2} = \frac{e^{-3} + 2}{9}.$$

If the calculated values are substituted in the t_{ij}, r_{z_i} and c_{z_j}, then we find

$$t_{11} = 2 - 1 - 3B_1 + 3B_2 = 1 + \frac{(e^5 - 1)(e^{-3} - 1)}{5} + \frac{3(e^5 - 6)}{25},$$

$$t_{12} = -1 + 1 + 2B_2 = \frac{2(e^5 - 6)}{25},$$

$$t_{21} = 1 - 2 - 12B_1 + 12B_2 + 9B_3 = -1 + \frac{4(e^5 - 1)(e^{-3} - 1)}{5} + \frac{12(e^5 - 6)}{25} + (e^{-3} + 2)$$

$$t_{22} = 3 - 1 + 8B_2 = 2 + \frac{8(e^5 - 6)}{25},$$

$$r_{z_1} = \frac{2}{e^2 - 1}\left[1 + \frac{(e^5 - 1)(e^{-3} - 1)}{5} + \frac{(e^5 - 6)}{5}\right],$$

$$r_{z_2} = \frac{2}{e^2 - 1}\left[4(\frac{(e^5 - 1)(e^{-3} - 1)}{5} + \frac{(e^5 - 6)}{5}) - 3(1 - \frac{(e^{-3} + 2)}{3})\right],$$

$$c_{z_1} = \frac{2}{e^2 - 1}\left[3(1 + \frac{(e^5 - 6)}{5}) - 3(1 - \frac{(e^5 - 1)(e^{-3} - 1)}{3} - \frac{(e^{-3} + 2)}{3})\right],$$

$$c_{z_2} = \frac{2}{e^2 - 1}\left[2(1 + \frac{(e^5 - 6)}{5})\right].$$

Therefore

$$z_{11} = t_{11} - r_{z_1} c_{z_1} \frac{e^2 - 3}{4},$$
$$z_{12} = t_{12} - r_{z_1} c_{z_2} \frac{e^2 - 3}{4},$$

$$z_{21} = t_{21} - r_{z_2} c_{z_1} \frac{e^2 - 3}{4},$$
$$z_{22} = t_{22} - r_{z_2} c_{z_2} \frac{e^2 - 3}{4},$$

with above values.

3. Conclusions

Let S be a semigroup and $\mathbb{R}\langle S\rangle$ be the semigroup ring with basis $S \cup \{1\}$. Consider the equation

$$w = \log(e^u e^v)$$

where $u, v \in \mathbb{R}\langle S\rangle$. Clearly, $w = u + v$ when S is commutative. However it is not easy to compute w in the case of noncommutativity. The first step in this direction has been done for left zero semigroup L_m and right zero semigroup R_n in [8].

A natural generalization of the recent result is the rectangular band $L_m \times R_n$, and we do not obtain a larger semigroup structure from this point, since the $k - times$ product

$$(L_m \times R_n) \times \cdots \times (L_m \times R_n)$$

is isomorphic to $L_{km} \times R_{kn}$ which is already covered by the main result of the paper. Notice that our findings are not consequences of [8] when $km \neq 1$ or $kn \neq 1$.

Hence we provide the description of w in $\mathbb{R}\langle L_m \times R_n\rangle$ in the present paper. It would be interesting to held the problem for some other noncommutative semigroups.

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