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On the generalized trapezoid and midpoint type inequalities involving Euler's beta function

MEHMET ZEKI SARIKAYA and GIZEM KOZAN

ABSTRACT. The main object of this paper is to present some generalizations of fractional integral inequalities involving Euler's beta function of Hermite-Hadamard type which cover the previously published result such as Riemann integral, Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral.

1. INTRODUCTION

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. Some famous results for such estimations consist of Hermite-Hadamard, trapezoid, midpoint, Simpson or Jensen inequalities ect.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequalities [16]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions f: These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics. The absolute value of the difference of the second part of the (1.1) inequalities is known as the trapezoidal inequality in the literature and was given by Dragomir and Agarwal in 1998 [5]. Then, in 2004, the absolute value of the difference of the first part of the (1.1) inequalities, known as the midpoint inequality by Kirmanci, was given [11]. Thus, these two important inequalities have attracted the attention of many readers to date, and many studies have been carried out for different types of convex functions. For recent results and generalizations concerning Hermite-Hadamard's inequalities see [6], [9], [12]-[14], [20]-[23], [33] and the references given therein.

In [5], Dragomir and Agarwal proved the following results connected to the right part of (1.1).

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Corresponding author: Mehmet Zeki Sarıkaya; sarikayamz@gmail.com

Lemma 1.1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta + (1-t)b)dt. \tag{1.2}$$

Theorem 1.1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{8}\left(|f'(a)| + |f'(b)|\right). \tag{1.3}$$

In [11], Kirmaci proved the following results connected to the left part of (1.1). In [11] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

Lemma 1.2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L([a, b])$, then we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \qquad (1.4)\\ = \ &(b-a)\left[\int_0^{\frac{1}{2}} tf'(ta + (1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta + (1-t)b)dt\right]. \end{aligned}$$

Theorem 1.2. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)}{8}\left(|f'(a)| + |f'(b)|\right). \tag{1.5}$$

The subject of the fractional calculus (integrals and derivatives) has gained considerable popularity and importance during the past there decades or so, due mainly to its demostrated applications in nemerous seemingly diverse and widespread fields of science and engineering. The fractional integral does indeed provide several potentially useful tools for various problems involving special functions of mathematical science as well as their extensions and generalizations in one and more variables. This subject is still being studied extensively by many authors, see for instance ([1], [2], [10], [15], [17]-[32]). One of the important applications of fractional integrals is Hermite-Hadamard integral inequality, see [3], [7], [8], [18], [24]-[32]. First, let's recall the above concepts of the Riemann-Liouville fractional integral are defined as follows [10], [15] and [17]:

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ x > a,$$

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ x < b.$$

The k-Riemann-Liouville fractional integral are defined by follows:

$$J_{a+,k}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \ \ x > a,$$

$$J_{b-,k}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \ \ x < b$$

where

$$\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt,\quad \mathcal{R}(\alpha)>0$$

and

$$\Gamma_k(\alpha)=k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right),\quad \mathcal{R}(\alpha)>0;k>0$$

are given by Mubeen and Habibullah in [30].

Now, let's recall the basic expressions of Hermite-Hadamard inequality for fractional integrals as proved by Sarikaya et al. in [24] as follows:

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be a function with $a<b$ and $f\in L_1([a,b])$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\le\frac{f(a)+f(b)}{2}\tag{1.6}$$

with $\alpha>0$.

Meanwhile, in [24], Sarikaya et al. gave the following interesting trapeozid identity for Riemann-Liouville integral:

Lemma 1.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\\ =\;&\frac{b-a}{2}\int_0^1\left[(1-t)^\alpha-t^\alpha\right]f'(ta+(1-t)b)\,dt.\end{aligned}\tag{1.7}$$

Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\\ \le\;&\frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[f'(a)+f'(b)\right].\end{aligned}\tag{1.8}$$

On the other hand, in [28] Iqbal et al. gave the following results connected to the left part of Riemann-Liouville integral inequalities of Hermite-Hadamard type (1.6) by using the following Midpoint identity as follows.

Lemma 1.4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L^1[a,b]$, then the following identity for Riemann-Liouville fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]=\frac{b-a}{2}\sum_{k=1}^4 I_k,\tag{1.9}$$

where

$$I_1=\int_0^{\frac12}t^\alpha f'(tb+(1-t)a)\,dt,\qquad I_2=\int_0^{\frac12}(-t^\alpha)f'(ta+(1-t)b)\,dt,$$

$$I_3=\int_{\frac12}^1(t^\alpha-1)f'(tb+(1-t)a)\,dt,\quad I_4=\int_{\frac12}^1(1-t^\alpha)f'(ta+(1-t)b)\,dt.$$

Many papers study the Riemann-Liouville fractional integrals and give new and interesting generalizations of Hermite-Hadamard type inequalities using these kind of integrals, see for instance see ([1]-[3], [18], [24]-[32]).

The purpose of this paper is to establish new Hermite-Hadamard type inequalities involving Euler's beta functions. Using functions whose first derivatives absolute values are convex, we obtained new trapezoid and midpoint inequalities that are connected with the celebrated Hermite-Hadamard type which cover the previously published results.

2. Hermite-Hadamard inequalities involving Euler's beta function

In 1997, Chaudhry et al. [4] presented the following extension of Euler's beta function

$$\beta(m,n;\nu)=\int_0^1 t^{m-1}(1-t)^{n-1}e^{\frac{-\nu}{t(1-t)}}dt,\ \ \text{for } m,n,\nu>0$$

and they proved that this extension has connections with the Macdonald, error and Whittakers function. It clearly seems that $\beta(m,n;\nu)=\beta(m,n)$, where $\beta(m,n)$ is the classical beta function.

In this section, using Euler's beta function, we begin by the following theorem:

Theorem 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ with $a<b$, then the following inequalities hold:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\beta(m,n;\nu) \qquad (2.10)\\ \leq\ & \frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x)f(x)e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx\\ \leq\ & \beta(m,n;\nu)\frac{f(a)+f(b)}{2}\end{aligned}$$

where $\beta(m,n;\nu)$ is Euler's beta function and

$$\Omega(x)=(b-x)^{m-1}(x-a)^{n-1}+(b-x)^{n-1}(x-a)^{m-1}$$

for $m,n,\nu>0$.

Proof. For $t\in[0,1]$, let $x=ta+(1-t)b$, $y=(1-t)a+tb$. The convexity of f yields

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2} \qquad (2.11)$$

i.e.,

$$2f\left(\frac{a+b}{2}\right)\leq f(ta+(1-t)b)+f((1-t)a+tb). \qquad (2.12)$$

Multiplying both sides of (2.12) by $t^{m-1}(1-t)^{n-1}e^{\frac{-\nu}{t(1-t)}}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&2f\left(\frac{a+b}{2}\right)\int_0^1 t^{m-1}(1-t)^{n-1}e^{-\frac{\nu}{t(1-t)}}dt \\
\le\ &\int_0^1 t^{m-1}(1-t)^{n-1}f(ta+(1-t)b)\,e^{-\frac{\nu}{t(1-t)}}dt \\
&+\int_0^1 t^{m-1}(1-t)^{n-1}f((1-t)a+tb)\,e^{-\frac{\nu}{t(1-t)}}dt.
\end{aligned}$$

By using the change of the variable, we obtain

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\beta(m,n;\nu) \\
\le\ &\frac{1}{2(b-a)^{m+n-1}}\int_a^b\left[(b-x)^{m-1}(x-a)^{n-1}+(b-x)^{n-1}(x-a)^{m-1}\right]f(x)\,e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx
\end{aligned}$$

and the first inequality is proved.

To prove the other half of the inequality in (2.10), since f is convex, we have,

$$f(ta+(1-t)b)+f((1-t)a+tb)\le f(a)+f(b) \tag{2.13}$$

for every $t\in[0,1]$. Then multiplying both sides of (2.13) by $t^{m-1}(1-t)^{n-1}e^{\frac{-\nu}{t(1-t)}}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{2(b-a)^{m+n-1}}\int_a^b\left[(b-x)^{m-1}(x-a)^{n-1}+(b-x)^{n-1}(x-a)^{m-1}\right]f(x)\,e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx \\
\le\ &\frac{f(a)+f(b)}{2}\beta(m,n;\nu)
\end{aligned}$$

and the second inequality is proved. □

Corollary 2.1. *With the notations in Theorem 2.5, if we choose $n=m=1$, we have*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{(b-a)E(\nu)}\int_a^b f(x)\,e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx\le\frac{f(a)+f(b)}{2} \tag{2.14}$$

where $E(\nu)=\int_0^1 e^{\frac{-\nu}{t(1-t)}}dt,\ \nu>0.$

Remark 2.1. If in Corollary 2.1, we take $\nu\to 0$, then the inequalities (2.14) become the inequalities (1.1).

Corollary 2.2. *With the notations in Theorem 2.5, if we choose $m=1,\ n=\alpha$,(or $m=\alpha,\ n=1$) we have*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) && (2.15)\\ \leq\ & \frac{1}{2E(\alpha;\nu)(b-a)^{\alpha}}\int\limits_a^b\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]f(x)\,e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx \\ \leq\ & \frac{f(a)+f(b)}{2}\end{aligned}$$

where $E(\alpha;\nu)=\int\limits_0^1(1-t)^{\alpha-1}e^{\frac{-\nu}{t(1-t)}}dt,\ \alpha,\nu>0.$

Remark 2.2. If in Corollary 2.2, we take $\nu\to 0$, then the inequalities (2.15) become the inequalities (1.6).

In fact k-Riemann-Liouville fractional integrals of order α are generalization of Riemann-Liouville fractional integrals of order α. If we take $k\to 1$, the k-Riemann-Liouville fractional integrals of order α turn out to be Riemann-Liouville fractional integrals of order α. The following result is related to this;

Corollary 2.3. *With the notations in Theorem 2.5, if we choose $m=1,\ n=\frac{\alpha}{k}$,(or $m=\frac{\alpha}{k},\ n=1$) we have*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) && (2.16)\\ \leq\ & \frac{1}{2E_k(\alpha;\nu)(b-a)^{\frac{\alpha}{k}}}\int\limits_a^b\left[(x-a)^{\frac{\alpha}{k}-1}+(b-x)^{\frac{\alpha}{k}-1}\right]f(x)\,e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx \\ \leq\ & \frac{f(a)+f(b)}{2}\end{aligned}$$

where $E_k(\alpha;\nu)=\int\limits_0^1(1-t)^{\frac{\alpha}{k}-1}e^{\frac{-\nu}{t(1-t)}}dt,\ \alpha,\nu,k>0.$

Remark 2.3. *If in Corollary 2.3, we take $\nu\to 0$, then the inequalities (2.16) become the inequalities*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f(b)+\ J^{\alpha}_{b^-,k}f(a)\right]\leq\frac{f(a)+f(b)}{2}$$

which are proved by Hussain et. al. in [31].

3. Trapezoid inequalities involving Euler's beta function

In this section, we give an identity which use to assist us is proving our results as follows:

Lemma 3.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}\beta(m,n;\nu)-\frac{1}{2(b-a)^{m+n-1}}\int_a^b\Omega(x)f(x)e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx \tag{3.17}$$

$$=\frac{(b-a)}{2}\int_0^1\beta_t(m,n;\nu)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt$$

where $\beta_t(m,n;v)$ is incomplete Euler's beta function defiend by

$$\beta_t(m,n;\nu)=\int_0^t s^{m-1}(1-s)^{n-1}e^{\frac{-\nu}{s(1-s)}}ds,\quad 0<t\le 1$$

for $m,n,\nu>0$.

Proof. Here, we apply integration by parts in integrals of right part of (3.17), and by using the change of the variable $x=tb+(1-t)a$, then we have

$$\begin{aligned}F_1&=\int_0^1\beta_t(m,n;\nu)f'(tb+(1-t)a)dt\\&=\frac{f(b)}{b-a}\beta(m,n;\nu)-\frac{1}{b-a}\int_0^1 t^{m-1}(1-t)^{n-1}f(tb+(1-t)a)e^{\frac{-\nu}{t(1-t)}}dt\\&=\frac{f(b)}{b-a}\beta(m,n;\nu)-\frac{1}{(b-a)^{m+n}}\int_a^b(x-a)^{m-1}(b-x)^{n-1}f(x)e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx.\end{aligned}$$

And similarly, we obtain

$$F_2=\int_0^1\beta_t(m,n;\nu)f'(ta+(1-t)b)dt$$

$$\begin{aligned}&=-\frac{f(a)}{b-a}\beta(m,n;\nu)+\frac{1}{b-a}\int_0^1 t^{m-1}(1-t)^{n-1}f(ta+(1-t)b)e^{\frac{-\nu}{t(1-t)}}dt\\&=-\frac{f(a)}{b-a}(m,n;\nu)+\frac{1}{(b-a)^{m+n}}\int_a^b(x-a)^{n-1}(b-x)^{m-1}f(x)e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx.\end{aligned}$$

If we subtract F_1 from F_2 and multiply by $\frac{(b-a)}{2}$, we obtain proof of the (3.17). □

Remark 3.4. *If in Lemma 3.5, we get $m=n=1$ and $\nu\to 0$, then the identity (3.17) becomes the identity (1.2).*

Remark 3.5. If in Lemma 3.5, we get $m=1,\ n=\alpha,\ \nu\to 0$ (or $m=\alpha,\ n=1,\ \nu\to 0$), then the identity (3.17) becomes the identity (1.7).

Remark 3.6. If in Lemma 3.5, we get $m = 1,\ n = \frac{\alpha}{k}, \nu \to 0$ (or $m = \frac{\alpha}{k},\ n = 1\ \nu \to 0$), then the identity (3.17) reduces to

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}\, f(b) + J^{\alpha}_{b-,k}\, f(a)\right] \\ = \quad & \frac{b-a}{2}\int_0^1 \left[(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right] f'(ta+(1-t)b)\, dt \end{aligned}$$

which are proved by Hussain et. al. in [31].

Instead of the identity (3.17) in Lemma 3.5, the following identity will be used to prove the theorems.

Remark 3.7. By the change of variable in Lemma 3.5, the identity (3.17) reduces to

$$\begin{aligned} & \beta(m,n;\nu)\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x) f(x)\, e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}\, dx \qquad (3.18) \\ = \quad & \frac{(b-a)}{2}\int_0^1 \left[\beta_t(m,n;\nu) - \beta_{1-t}(m,n;\nu)\right] f'(tb+(1-t)a)\, dt. \end{aligned}$$

Now, we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for functions whose first derivatives absolute values are convex as follows:

Theorem 3.6. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f' \in L[a,b]$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} & \left|\beta(m,n;\nu)\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x) f(x)\, e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}\, dx\right| \qquad (3.19) \\ \leq \quad & (b-a)\left(\frac{|f'(a)|+|f'(b)|}{2}\right)\int_0^{\frac{1}{2}} \left[\beta_{1-t}(m,n;\nu) - \beta_t(m,n;\nu)\right] dt \end{aligned}$$

for $m, n, \nu > 0$.

Proof. Using identity (3.18) and the convexity of $|f'|$, we find that

$$\left|\beta\left(m,n;\nu\right)\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{2\left(b-a\right)^{m+n-1}}\int\limits_{a}^{b}\Omega\left(x\right)f\left(x\right)e^{\frac{-\nu(b-a)^{2}}{(b-x)(x-a)}}dx\right|$$

$$\begin{aligned}
&\leq \frac{(b-a)}{2}\int\limits_{0}^{1}\left|\beta_{t}\left(m,n;\nu\right)-\beta_{1-t}\left(m,n;\nu\right)\right|\left|f'\left((1-t)a+tb\right)\right|dt \\
&\leq \frac{(b-a)}{2}\int\limits_{0}^{\frac{1}{2}}\left[\beta_{1-t}\left(m,n;\nu\right)-\beta_{t}\left(m,n;\nu\right)\right]\left[(1-t)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dt \\
&\quad +\frac{(b-a)}{2}\int\limits_{\frac{1}{2}}^{1}\left[\beta_{t}\left(m,n;\nu\right)-\beta_{1-t}\left(m,n;\nu\right)\right]\left[(1-t)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dt \\
&= \frac{(b-a)}{2}\left(\left|f'\left(a\right)\right|+\left|f'\left(b\right)\right|\right)\int\limits_{0}^{\frac{1}{2}}\left[\beta_{1-t}\left(m,n;\nu\right)-\beta_{t}\left(m,n;\nu\right)\right]dt
\end{aligned}$$

which this completes the proof of the (3.19). □

Remark 3.8. If in Theorem 3.6, we get $n=m=1$ and $\nu\to 0$, then, the inequality (3.19) becomes the inequality (1.3).

Remark 3.9. If in Theorem 3.6, we get $m=1,\ n=\alpha,\ \nu\to 0$ (or $m=\alpha,\ n=1,\ \nu\to 0$), then the inequality (3.19) becomes the inequality (1.8).

Remark 3.10. If in Theorem 3.6, we get $m=1,\ n=\frac{\alpha}{k},\ \nu\to 0$ (or $m=\frac{\alpha}{k},\ n=1,\ \nu\to 0$), then the inequality (3.19) becomes

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_{k}\left(\alpha+k\right)}{2\left(b-a\right)^{\frac{\alpha}{k}}}\left[J_{a^{+},k}^{\alpha}\ f(b)+\ J_{b^{-},k}^{\alpha}\ f(a)\right]\right| \\
&\leq \frac{(b-a)}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\frac{\left|f'\left(a\right)\right|+\left|f'\left(b\right)\right|}{2}\right)
\end{aligned}$$

which are proved by Hussain et. al. in [31].

Theorem 3.7. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f'\in L\left[a,b\right]$. If $|f'|^{q}$ is convex on $[a,b]$ for some $q>1$, then the following inequality holds: for $m,n,\nu>0$*

$$\begin{aligned}
&\left|\beta\left(m,n;\nu\right)\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{2\left(b-a\right)^{m+n-1}}\int\limits_{a}^{b}\Omega\left(x\right)f\left(x\right)e^{\frac{-\nu(b-a)^{2}}{(b-x)(x-a)}}dx\right| \\
&\leq \frac{(b-a)}{2}\left(\int\limits_{0}^{1}\left|\beta_{t}\left(m,n;\nu\right)-\beta_{1-t}\left(m,n;\nu\right)\right|^{p}dt\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(a\right)\right|^{q}+\left|f'\left(b\right)\right|^{q}}{2}\right)^{\frac{1}{q}}
\end{aligned} \tag{3.20}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using identity (3.18), Hölder's inequality and the convexity of $|f'|^q$, we find that

$$\begin{aligned}
&\left|\beta(m,n;\nu)\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x) f(x) e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx\right| \\
\leq\;& \frac{(b-a)}{2}\left(\int_0^1 |\beta_t(m,n;\nu)-\beta_{1-t}(m,n;\nu)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'((1-t)a+tb)|^q dt\right)^{\frac{1}{q}} \\
\leq\;& \frac{(b-a)}{2}\left(\int_0^1 |\beta_t(m,n;\nu)-\beta_{1-t}(m,n;\nu)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[(1-t)|f'(a)|^q+t|f'(b)|^q\right] dt\right)^{\frac{1}{q}}
\end{aligned}$$

which this completes the proof of the (3.20). □

Remark 3.11. If in Theorem 3.7, we get $m=n=1$ and $\nu\to 0$, then, the inequality (3.20) becomes the inequality (2.4) of Theorem 2.3 in [5].

Remark 3.12. If in Theorem 3.7, we get $m=1,\ n=\alpha,\ \nu\to 0$ (or $m=\alpha,\ n=1,\ \nu\to 0$), then the inequality (3.20) becomes the inequality (2.7) of Theorem 8 in [26].

Remark 3.13. If in Theorem 3.7 we get $m=1,\ n=\frac{\alpha}{k},\ \nu\to 0$ (or $m=\frac{\alpha}{k},\ n=1,\ \nu\to 0$), then the inequality (3.20) becomes

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f(b)+J^{\alpha}_{b^-,k}f(a)\right]\right| \\
\leq\;& \frac{(b-a)}{2\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1,\ \frac{\alpha}{k}\in[0,1]$, which are proved Hussain et. al. in [31].

4. Midpoint inequalities involving Euler's beta function

Before starting and proving our next result, we need the following lemma.

Lemma 4.6. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$f\left(\frac{a+b}{2}\right)\beta(m,n;\nu)-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x) f(x) e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx=\frac{b-a}{2}\sum_{k=1}^{4}T_k \quad (4.21)$$

where

$$\begin{aligned}
T_1&=\int_0^{\frac{1}{2}}\beta_t(m,n;\nu)f'(tb+(1-t)a)\,dt, &\quad T_2&=\int_0^{\frac{1}{2}}(-\beta_t(m,n;\nu))f'(ta+(1-t)b)\,dt,\\
T_3&=\int_{\frac{1}{2}}^{1}(-\beta_{1-t}(m,n;\nu))f'(tb+(1-t)a)\,dt, &\quad T_4&=\int_{\frac{1}{2}}^{1}\beta_{1-t}(m,n;\nu)f'(ta+(1-t)b)\,dt
\end{aligned}$$

for $m,n,v>0$.

Proof. In the proof of (4.21), we apply integration by parts, then we have

$$\begin{aligned} T_1 &= \int_0^{\frac{1}{2}} \beta_t (m,n;\nu) f'(tb+(1-t)a)\,dt \\ &= \frac{1}{b-a}\beta_{\frac{1}{2}}(m,n;\nu) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_0^{\frac{1}{2}} t^{m-1}(1-t)^{n-1} f(tb+(1-t)a)\, e^{\frac{-\nu}{t(1-t)}}\,dt, \end{aligned}$$

$$\begin{aligned} T_2 &= \int_0^{\frac{1}{2}} \left(-\beta_t (m,n;\nu)\right) f'(ta+(1-t)b)\,dt \\ &= \frac{1}{b-a}\beta_{\frac{1}{2}}(m,n;\nu) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_0^{\frac{1}{2}} t^{m-1}(1-t)^{n-1} f(ta+(1-t)b)\, e^{\frac{-\nu}{t(1-t)}}\,dt, \end{aligned}$$

$$\begin{aligned} T_3 &= \int_{\frac{1}{2}}^{1} \left(-\beta_{1-t} (m,n;\nu)\right) f'(tb+(1-t)a)\,dt \\ &= \frac{1}{b-a}\beta_{\frac{1}{2}}(n,m) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_{\frac{1}{2}}^{1} (1-t)^{m-1} t^{n-1} f(tb+(1-t)a)\, e^{\frac{-\nu}{t(1-t)}}\,dt, \end{aligned}$$

$$\begin{aligned} T_4 &= \int_{\frac{1}{2}}^{1} \beta_{1-t} (m,n;\nu) f'(ta+(1-t)b)\,dt \\ &= \frac{1}{b-a}\beta_{\frac{1}{2}}(m,n;\nu) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_{\frac{1}{2}}^{1} (1-t)^{m-1} t^{n-1} f(ta+(1-t)b)\, e^{\frac{-\nu}{t(1-t)}}\,dt. \end{aligned}$$

Thus, by the above expressions, the desired identity (4.21) is obtained. □

Remark 4.14. If in Lemma 4.6, we take $m = n = 1$ and $\nu \to 0$ then, the identity (4.21) becomes the identity (1.4) of Lemma 1.2 by Kirmaci in [11].

Remark 4.15. If in Lemma 4.6, we take $m = 1,\ n = \alpha,\ \nu \to 0$ (or $m = \alpha,\ n = 1,\ \nu \to 0$), then the identity (4.21) reduces to (1.9) of Lemma 1.4 by Iqbal et al. in [28] .

Remark 4.16. If in Lemma 4.6, we take $m = 1,\ n = \frac{\alpha}{k},\ \nu \to 0$ (or $m = \frac{\alpha}{k},\ n = 1,\ \nu \to 0$), then the identity (4.21) reduces to the identity of Corollary 6 by Sarikaya and Ertugral in [20] .

Remark 4.17. By the change of variable in Lemma 4.6, the identity (4.21) reduces to

$$f\left(\frac{a+b}{2}\right)\beta(m,n;\nu)-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x) f(x) e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx \tag{4.22}$$

$$= \frac{(b-a)}{2}\int_0^{\frac{1}{2}}\left[\beta_t(m,n;\nu)+\beta_t(m,n;\nu)\right]\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt.$$

Finally, we extend some estimates of the left hand side of a Hermite-Hadamard type inequality for functions whose first derivatives absolute values are convex as follows:

Theorem 4.8. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f'\in L[a,b]$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)\beta(m,n;\nu)-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x) f(x) e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx\right| \tag{4.23}$$

$$\leq \frac{(b-a)}{2}\left[|f'(a)|+|f'(b)|\right]\left(\frac{1}{2}\beta(m,n;\nu)-\beta_{\frac{1}{2}}(m+1,n;\nu)-\beta_{\frac{1}{2}}(n+1,m;\nu)\right)$$

for $m,n,v>0$.

Proof. From (4.22) and using the convexity of $|f'|$, then we have

$$\left|f\left(\frac{a+b}{2}\right)\beta(m,n;\nu)-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x) f(x) e^{\frac{-\nu(b-a)^2}{(b-x)(x-a)}}dx\right| \tag{4.24}$$

$$\leq \frac{(b-a)}{2}\left(\int_0^{\frac{1}{2}}\left[\beta_t(m,n;\nu)+\beta_t(m,n;\nu)\right]dt\right)\left[|f'(a)|+|f'(b)|\right].$$

By changing the order of the integrals, we get

$$\int_0^{\frac{1}{2}}\left[\beta_t(m,n;\nu)+\beta_t(n,m;\nu)\right]dt \tag{4.25}$$

$$= \int_0^{\frac{1}{2}}\int_0^t s^{m-1}(1-s)^{n-1}e^{\frac{-\nu}{s(1-s)}}dsdt+\int_0^{\frac{1}{2}}\int_0^t (1-s)^{m-1}s^{n-1}e^{\frac{-\nu}{s(1-s)}}dsdt$$

$$= \frac{1}{2}\beta(m,n;\nu)-\beta_{\frac{1}{2}}(m+1,n;\nu)-\beta_{\frac{1}{2}}(n+1,m;\nu).$$

By writing (4.25) in the (4.24), we obtain the required inequality. Thus, the proof of (4.23) is completed . □

Remark 4.18. If in Theorem 4.8, we take $m=n=1$ and $\nu\to 0$ then, the inequality (4.23) reduces to the inequality (2.3) of Theorem 2.3 by Kirmaci in [11].

Remark 4.19. If in Theorem 4.8, we take $m=1,\ n=\alpha,\ \nu\to 0$ (or $m=\alpha,\ n=1,\ \nu\to 0$), then the inequality (4.23) reduces to the inequality (3) of Theorem 2 by Iqbal et al. in [28] .

Remark 4.20. If in Theorem 4.8, we take $m = 1,\ n = \frac{\alpha}{k},\ \nu \to 0$ (or $m = \frac{\alpha}{k},\ n = 1,\ \nu \to 0$), then the inequality (4.23) reduces to the inequality of Corollary 9 by Sarikaya and Ertugral in [20].

5. Conclusions

In this paper, we establish some new Hermite-Hadamard type inequalities involving Euler's beta functions. We get new inequalities of midpoint and trapezoidal type. In the future work of the authors, generalization or improvement of our results can be examined by using different kind of convex function classes or other type fractional integral operators.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND ARTS
DÜZCE UNIVERSITY
DÜZCE, TURKEY
Email address: `sarikayamz@gmail.com`
Email address: `gizemkozann@gmail.com`