CREAT. MATH. INFORM.
Volume **32** (2023), No. 1,
Pages 69 - 86

Online version at `https://semnul.com/creative-mathematics/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2023.01.08

Intuitionistic Fuzzy Prime Radical and Intuitionistic Fuzzy Primary Ideal Of Γ-Ring

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ABSTRACT. In this paper, we introduced the notion of intuitionistic fuzzy prime radical of an intuitionistic fuzzy ideal in Γ-rings. We also characterise intuitionistic fuzzy primary ideal of Γ-rings. We also analyse homomorphic behaviour of intuitionistic fuzzy primary ideal and intuitionistic fuzzy prime radical of Γ-rings.

1. INTRODUCTION

Motivational Ideas And Development. A ring of endomorphisms of an additive commutative group plays a very important role in many parts of mathematics. The property of a ring itself is also clarified when we consider it as a ring of endomorphism of an additive commutative group; but if we consider a set of homomorphism of an additive commutative group to another additive commutative group, the set is closed under addition and subtraction defined naturally but we cannot define natural multiplication of two homomorphisms in it. However, if we consider two additive commutative groups A and B and the additive commutative group M consisting of all homomorphisms from A to B then we can define the product of three elements f_1, g and f_2 where f_1, f_2 are the members of M and g is a homomorphism from B to A. In this case the product $f_1 g f_2$ is also an element of M. Thus we can define multiplication in M using Γ, where Γ is the additive commutative group of all homomorphisms from B to A. Similarly we can define a multiplication in Γ using M. Also we have

$$(f_1 g_1 f_2) g_2 f_3 = f_1 (g_1 f_2 g_2) f_3 = f_1 g_1 (f_2 g_2 f_3)$$

where f_1, f_2, f_3 are members of M and g_1, g_2 are members of Γ.

Again we know that the ring of all square matrices over a division ring plays a vital role in classical ring theory. However, if we consider the set M of all rectangular matrices of the type $m \times n (m \neq n)$ over a division ring then M is an additive commutative group but there appears to be no natural way of introducing a binary multiplication into it. Now if M is the additive commutative group of all rectangular matrices of the type $m \times n (m \neq n)$ over a division ring and Γ is that of all rectangular matrices of the type $n \times m$ over a division ring and also if $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ then $a\alpha b \in M$ and also

$$a\alpha(b\beta c) = a(\alpha b \beta)c = (a\alpha b)\beta c$$

Noting this fact N. Nobusawa [1] defined Γ-ring. Later on W.E Barnes [2] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. Any ring can be regarded as a Γ-ring by suitably choosing Γ. After that, J. Luh [3] and S. Kyuno [4], studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Since then, many researchers have investigated various

Received: 29.10.2021. In revised form: 23.05.2022. Accepted: 30.05.2022
2020 *Mathematics Subject Classification.* 03F55, 16Y80, 16D25, 03G25.
Key words and phrases. Γ*-rings, Intuitionistic fuzzy prime radical, Intuitionistic fuzzy primary ideal.*
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properties of this Γ-ring. Z. K. Warsi [5] studied the decomposition of primary ideals of Γ-ring. R. Paul [6] studied various types of ideals in Γ-ring and the corresponding operator rings.

Y. B. Jun [7], defined fuzzy prime ideal of a Γ-ring and obtained a number of characterization for a fuzzy ideal to be a fuzzy prime ideal. T. K. Dutta and T. Chanda [8] proved the same result in a different way and also proved a few more characterization of fuzzy prime ideals. B. A. Ersoy [9] defined fuzzy semiprime ideal and obtained some results. A. K. Aggarwal et al in [10] studied some theorems on fuzzy prime ideals of Γ-ring.

The idea of intuitionistic fuzzy sets was first published by Atanassaov [11, 12], as a generalization of the notion of fuzzy set given by Zadeh [13]. Kim et al in [14] considered the intuitionistic fuzzification of ideal of Γ-ring which were further studied by Palaniappan at al in [15, 16, 17]. The notion of intuitionistic fuzzy prime ideal and semiprime were studied by Palaniappan and Ramachandran in [18]. Authors in [19] studied the notion of intuitionistic fuzzy characteristic ideals of a Γ-ring and obtained a one to one correlation between the set of all intuitionistic fuzzy characteristic ideals of Γ-ring and that of its operator ring. Also in [20] they obtained an extension of intuitionistic fuzzy ideal which is used to characterise intuitionistic fuzzy prime and semiprime ideals of Γ-ring. The prime objective of studying the concepts of primary ideals and prime radical in the intuitionistic fuzzy environment is to lay down the foundation which leads to the study the decomposition property of intuitionistic fuzzy ideal in noetherian Γ-ring in terms of intuitionistic fuzzy primary ideals. The structuring of the paper is as follows.

In part 2 we recollect some groundwork for their use in the continuation of the development of the subject matter. In part 3 we set in motion of the notion of intuitionistic fuzzy prime radical of an intuitionistic fuzzy ideal of Γ-ring. In part 4 we utilized the work of part 3 to study the notion of intuitionistic fuzzy primary ideal in Γ-ring. In part 5, we study the homomorphic behaviour of intuitionistic fuzzy primary ideal and intuitionistic fuzzy prime radical of Γ-rings.

2. Preliminaries

Let us recall some definitions and results, which are necessary for the development of the paper,

Definition 2.1. ([1, 2]) If $(M, +)$ and $(\Gamma, +)$ are additive Abelian groups. Then M is called a Γ-ring (in the sense of Barnes [2]) if there exist mapping $M \times \Gamma \times M \to M$ [image of (x, α, y) is denoted by $x\alpha y$, $x, y \in M, \gamma \in \Gamma$] satisfying the following conditions:
(1) $x\alpha y \in M$.
(2) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$.
(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$. for all $x, y, z \in M$, and $\gamma \in \Gamma$.

A subset N of a Γ-ring M is a left (right) ideal of M if N is an additive subgroup of M and $M\Gamma N = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in N\}$, $(N\Gamma M)$ is contained in N. If N is both a left and a right ideal then N is a two-sided ideal, or simply an ideal of M. A mapping $f : M \to M^{'}$ of Γ-rings is called a Γ-homomorphism [2] if $f(x + y) = f(x) + f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$ for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.2. ([2]) Let M be a Γ-ring. A proper ideal L of M is called prime if for all pair of ideals S and T of M, $S\Gamma T \subseteq L$ implies that $S \subseteq L$ or $T \subseteq L$.

Theorem 2.1. ([6, 7]) *If L is an ideal of a Γ-ring M, the following conditions are equivalent:*

(i) L is a prime ideal of M;
(ii) If $a, b \in M$ and $a\Gamma M\Gamma b \subseteq L$ then $a \in L$ or $b \in L$.

Definition 2.3. ([5]) Let M be a Γ-ring. Then the radical of an ideal K of M is denoted by $\sqrt{K}$ and is defined as the set

$$\sqrt{K} = \{x \in M : (x\gamma)^{n-1}x \in K, \text{ for some } n \in \mathbf{N} \text{ and for all } \gamma \in \Gamma\ \}$$

where $(x\gamma)^{n-1}x = x$ for $n = 1$.

Definition 2.4. ([2]) An ideal K of a commutative Γ-ring M is said to be primary if for any two ideals I and J of M, $I\Gamma J \subseteq K$ implies either $I \subseteq K$ or $J \subseteq \sqrt{K}$, where $\sqrt{K}$ is the prime radical of K.

We now review some intuitionistic fuzzy logic concepts. We refer the reader to follow [11] and [18] for complete details.

Definition 2.5. ([11, 12]) An intuitionistic fuzzy set (IFS) A in X can be represented as an object of the form $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Remark 2.1. ([11, 12])
(i) When $\mu_A(x) + \nu_A(x) = 1$, i.e., $\nu_A(x) = 1 - \mu_A(x) = \mu_{A^c}(x), \forall x \in X$. Then A is called a fuzzy set.
(ii) An IFS $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ is shortly denoted by $A(x) = (\mu_A(x), \nu_A(x)), \forall x \in X$. We denote by $IFS(X)$ the set of all IFSs of X.

If $A, B \in IFS(X)$, then $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$ and $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. For any subset Y of X, the intuitionistic fuzzy characteristic function χ_Y is an intuitionistic fuzzy set of X, defined as $\chi_Y(x) = (1, 0), \forall x \in Y$ and $\chi_Y(x) = (0, 1), \forall x \in X\backslash Y$. Let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then the crisp set $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$ is called the (α, β)-level cut subset of A. Also the IFS $x_{(\alpha,\beta)}$ of X defined as $x_{(\alpha,\beta)}(y) = (\alpha, \beta)$, if $y = x$, otherwise $(0, 1)$ is called the intuitionistic fuzzy point (IFP) in X with support x. By $x_{(\alpha,\beta)} \in A$ we mean $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. Further if $f : X \to Y$ is a mapping and A, B be respectively IFS of X and Y. Then the image $f(A)$ is an IFS of Y is defined as $\mu_{f(A)}(y) = Sup\{\mu_A(x) : f(x) = y\}$, $\nu_{f(A)}(y) = Inf\{\nu_A(x) : f(x) = y\}$, for all $y \in Y$ and the inverse image $f^{-1}(B)$ is an IFS of X is defined as $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$, for all $x \in X$, i.e., $f^{-1}(B)(x) = B(f(x))$, for all $x \in X$. Also the IFS A of X is said to be f-invariant if for any $x, y \in X$, whenever $f(x) = f(y)$ implies $A(x) = A(y)$.

Definition 2.6. ([15]) Let A and B be two IFSs of a Γ-ring M and $\gamma \in \Gamma$. Then the product $A\Gamma B$ and the composition $A \circ B$ of A and B are defined by

$$(\mu_{A\Gamma B}(x), \nu_{A\Gamma B}(x)) = \begin{cases} (\vee_{x=y\gamma z}(\mu_A(y) \wedge \mu_B(z)), \wedge_{x=y\gamma z}(\nu_A(y) \vee \nu_B(z)), & \text{if } x = y\gamma z \\ (0, 1), & \text{otherwise} \end{cases}$$

and

$$(\mu_{A\circ B}(x), \nu_{A\circ B}(x))$$
$$= \begin{cases} (\vee_{x=\sum_{i=1}^{n} y_i\gamma z_i}(\mu_A(y_i) \wedge \mu_B(z_i)), \wedge_{x=\sum_{i=1}^{n} y_i\gamma z_i}(\nu_A(y_i) \vee \nu_B(z_i))), & \text{if } x = \sum_{i=1}^{n} y_i\gamma z_i \\ (0, 1), & \text{otherwise} \end{cases}$$

Remark 2.2. ([15]) If A and B be two IFSs of a Γ-ring M, then $A\Gamma B \subseteq A \circ B \subseteq A \cap B$

Definition 2.7. ([15]) Let A be an IFS of a Γ-ring M. Then A is called an intuitionistic fuzzy ideal (IFI) of M if for all $x, y \in M, \gamma \in \Gamma$, the following are satisfied
(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$;
(ii) $\mu_A(x\alpha y) \geq \mu_A(x) \vee \mu_A(y)$;
(iii) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$;
(iv)$\nu_A(x\alpha y) \leq \nu_A(x) \wedge \nu_A(y)$.

The set of all intuitionistic fuzzy ideals of Γ-ring M is denoted by $IFI(M)$. Note that if $A \in IFI(M)$, then $\mu_A(0_M) \geq \mu_A(x)$ and $\nu_A(0_M) \leq \nu_A(x), \forall x \in M$ (See [14]).

Remark 2.3. ([15, 17, 18]) If A, B and C be IFIs of a Γ-ring M, then $A\Gamma B$, $A \circ B$, $A \cap B$ are also IFI of M. Further, $A\Gamma B \subseteq C$ if and only if $A \circ B \subseteq C$.

Definition 2.8. ([18]) Let P be an intuitionistic fuzzy ideal (IFI) of a Γ-ring M. Then P is said to be prime if P is non-constant and for any IFIs A, B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

The set of all intuitionistic fuzzy prime ideal of a Γ-ring M is denoted by $IFPI(M)$.

Remark 2.4. ([18]) Let $x_{(p,q)}, y_{(t,s)} \in IFP(M)$. Then $x_{(p,q)}\Gamma y_{(t,s)} = (x\Gamma y)_{(p\wedge t, q\vee s)}$.

Definition 2.9. ([10]) A Γ-ring M is said to be commutative if $a\gamma b = b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Theorem 2.2. *([18]) Let M be a commutative Γ-ring and A be an IFI of M. Then following are equivalent*
(i) $x_{(p,q)}\Gamma y_{(t,s)} \subseteq A \Rightarrow x_{(p,q)} \subseteq A$ or $y_{(t,s)} \subseteq A$, where $x_{(p,q)}, y_{(t,s)} \in IFP(M)$.
(ii) A is an intuitionistic fuzzy prime ideal of M.

Theorem 2.3. *([18]) Let A be an IFI of Γ-ring M. Then each (p,q)-level cut set $A_{(p,q)}$ is either empty or an ideal of M, where $t \leq \mu_A(0_M)$ and $s \geq \nu_A(0_M)$. In particular $A_{(1,0)}$ which is denoted by A_*, i.e., the set $A_* = \{x \in M : \mu_A(x) = \mu_A(0_M)$ and $\nu_A(x) = \nu_A(0_M)\}$ is ideal of M. If $A \in IFPI(M)$, then A_* is a prime ideal of M.*

Theorem 2.4. *([18]) If P is an intuitionistic fuzzy prime ideal of a Γ-ring M, then the following conditions hold:*
(i) $P(0_M) = (1, 0)$,
(ii) P_* is a prime ideal of M,
(iii) $Img(P) = \{(1, 0), (t, s)\}$, where $t, s \in [0, 1)$ such that $t + s \leq 1$.

Definition 2.10. ([20]) A non-constant intuitionistic fuzzy ideal P of a Γ-ring M is called intuitionistic fuzzy semi-prime ideal if for any IFIs A of M, $A\Gamma A \subseteq P$ implies $A \subseteq P$.

Proposition 2.1. *([20]) Let P be a non-constant intuitionistic fuzzy ideal of a Γ-ring M, then the following conditions are equivalent:*
(i) P is an intuitionistic fuzzy semi-prime ideal of M
(ii) For any $a \in M$, $Inf_{m\in M,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 m\gamma_2 a)\} = \mu_P(a)$ and $Sup_{m\in M,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(a\gamma_1 m\gamma_2 a)\} = \nu_P(a)$.

3. Intuitionistic fuzzy prime radical of an IFI of a Γ-ring

Definition 3.11. Let A be a non-empty IFS of a Γ-ring M. Define a set $\wp(A)$ of all intuitionis tic fuzzy prime ideals of M that contains A, i.e., $\wp(A) = \{B : B \in IFPI(M), A \subseteq B\}$.

Proposition 3.2. *Let A_1, A_2 be two non-empty IFSs of a Γ-ring M. Then*

(i) $A_1 \subseteq A_2$ *implies that* $\wp(A_2) \subseteq \wp(A_1)$*;*
(ii) $\wp(A_1) \cup \wp(A_2) \subseteq \wp(A_1 \cap A_2)$*;*
(iii) $\wp(A_1) \cup \wp(A_2) = \wp(A_1 \Gamma A_2)$*, if* A_1, A_2 *are two IFIs of* M*;*
(iv) $\wp(A_1) \cup \wp(A_2) = \wp(A_1 \circ A_2)$*, if* A_1, A_2 *are two IFIs of* M*;*
(v) $\wp(\chi_I) \cup \wp(\chi_J) = \wp(\chi_{I\cap J})$ *if* I *and* J *are ideals of* M*.*

Proof.
(i) Let $B \in \wp(A_2)$. Then B is IFPI of M and $A_2 \subseteq B$. Since $A_1 \subseteq A_2$, $A_1 \subseteq B$. So $B \in \wp(A_1)$. Hence $\wp(A_2) \subseteq \wp(A_1)$.

(ii) Since $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$. Therefore by (i) we have $\wp(A_1) \subseteq \wp(A_1 \cap A_2)$ and $\wp(A_2) \subseteq \wp(A_1 \cap A_2)$. Thus $\wp(A_1) \cup \wp(A_2) \subseteq \wp(A_1 \cap A_2)$.

(iii) Since A_1 and A_2 are IFIs of the Γ-ring M, then $A_1 \Gamma A_2 \subseteq A_1 \cap A_2$ [by Remark (2.2)]. Therefore by (i) we have $\wp(A_1 \cap A_2) \subseteq \wp(A_1 \Gamma A_2)$. So by (ii) we have $\wp(A_1) \cup \wp(A_2) \subseteq \wp(A_1 \Gamma A_2)$.

Again, let $B \in \wp(A_1 \Gamma A_2)$. Then $A_1 \Gamma A_2 \subseteq B$ and $B \in IFPI(M)$, so either $A_1 \subseteq B$ or $A_2 \subseteq B$. Therefore $\wp(B) \subseteq \wp(A_1)$ or $\wp(B) \subseteq \wp(A_2)$.

Now $B \in IFPI(M)$ and $B \subseteq B$ so $B \in \wp(B)$. Thus we have $B \in \wp(A_1)$ or $B \in \wp(A_2)$. Therefore $B \in \wp(A_1) \cup \wp(A_2)$. So $\wp(A_1 \Gamma A_2) \subseteq \wp(A_1) \cup \wp(A_2)$. Hence $\wp(A_1) \cup \wp(A_2) = \wp(A_1 \Gamma A_2)$.

(iv) Since A_1 and A_2 are IFIs of the Γ-ring M, then $A_1 \Gamma A_2 \subseteq A_1 \circ A_2$ [by Remark (2.2)]. Then by (i) we have $\wp(A_1 \circ A_2) \subseteq \wp(A_1 \Gamma A_2)$.

Again, let $B \in \wp(A_1 \Gamma A_2)$. Then $A_1 \Gamma A_2 \subseteq B$ and $B \in IFPI(M)$. This implies that $A_1 \circ A_2 \subseteq B$, $B \in IFPI(M)$ (by Remark (2.3)). So $B \in \wp(A_1 \circ A_2)$. Thus $\wp(A_1 \Gamma A_2) \subseteq \wp(A_1 \circ A_2)$. Thus $\wp(A_1 \Gamma A_2) = \wp(A_1 \circ A_2)$. Hence from (iii) we get $\wp(A_1) \cup \wp(A_2) = \wp(A_1 \circ A_2)$.

(v) Assume that I and J are two ideals of the Γ-ring M. Clearly $\chi_I \cap \chi_J = \chi_{I\cap J}$. Thus $\wp(\chi_I) \cup \wp(\chi_J) \subseteq \wp(\chi_I \cap \chi_J) \subseteq \wp(\chi_{I\cap J})$.

Again, let $B \in \wp(\chi_{I\cap J})$. Then $\chi_{I\cap J} \subseteq B$. So $\chi_I \Gamma \chi_J \subseteq \chi_I \cap \chi_J = \chi_{I\cap J} \subseteq B$.

Since $B \in IFPI(M)$, we have $\chi_I \subseteq B$ or $\chi_J \subseteq B$. Thus $B \subseteq \wp(\chi_I)$ or $B \subseteq \wp(\chi_J)$. Therefore, $B \subseteq \wp(\chi_I) \cup \wp(\chi_J)$. Thus $\wp(\chi_{I\cap J}) \subseteq \wp(\chi_I) \cup \wp(\chi_J)$.
Hence $\wp(\chi_I) \cup \wp(\chi_J) = \wp(\chi_{I\cap J})$. □

Definition 3.12. Let A be an IFI of a Γ-ring M. Then the IFS $\sqrt{A}$ of M defined by

$$\sqrt{A} = \cap(\wp(A)) = \cap\{B : B \in IFPI(M); A \subseteq B\}$$

is said to be the intuitionistic fuzzy prime radical of A.

Proposition 3.3. *Let* A *be an IFI of a* Γ*-ring* M*. Then* $\sqrt{A}$ *is a non-constant IFI of* M *with* $\sqrt{A}(0_M) = (1, 0)$*.*

Proof. Let A be an IFI of a Γ-ring M. Then

$$\begin{aligned}\mu_{\sqrt{A}}(0_M) &= \mu_{\cap(\wp(A))}(0_M)\\ &= Inf\{\mu_B(0_M) : B \in IFPI(M); A \subseteq B\}\\ &= 1.\end{aligned}$$

Similarly, we can show $\nu_{\sqrt{A}}(0_M) = 0$. Thus $\sqrt{A}(0_M) = (1, 0)$.

Let $B \in IFPI(M)$. So there exists atleast one $x \in M$ such that $B(0_M) \neq (1, 0)$. Therefore $\sqrt{A}(0_M) \neq (1, 0)$. Thus $\sqrt{A}$ is non-constant IFS of M. Now for any $x, y \in M$, we have

$$\begin{aligned}\mu_{\sqrt{A}}(x-y) &= \mu_{\cap(\wp(A))}(x-y) = Inf\{\mu_B(x-y) : B \in IFPI(M); A \subseteq B\}\\ &\geq Inf\{\mu_B(x) \wedge \mu_B(y) : B \in IFPI(M); A \subseteq B\}\\ &= (Inf\{\mu_B(x) : B \in IFPI(M); A \subseteq B\}) \wedge (Inf\{\mu_B(y) : B \in IFPI(M); A \subseteq B\})\\ &= \mu_{\cap(\wp(A))}(x) \wedge \mu_{\cap(\wp(A))}(y)\\ &= \mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{A}}(y).\end{aligned}$$

Thus $\mu_{\sqrt{A}}(x-y) \geq \mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{A}}(y)$. Similarly, we can prove $\nu_{\sqrt{A}}(x-y) \leq \mu_{\sqrt{A}}(x) \vee \nu_{\sqrt{A}}(y)$.

Again for any $x, y \in M$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}\mu_{\sqrt{A}}(x\gamma y) &= \mu_{\cap(\wp(A))}(x\gamma y) = Inf\{\mu_B(x\gamma y) : B \in IFPI(M); A \subseteq B\}\\ &\geq Inf\{\mu_B(x) : B \in IFPI(M); A \subseteq B\}\\ &= \mu_{\cap(\wp(A))}(x)\\ &= \mu_{\sqrt{A}}(x).\end{aligned}$$

Similarly, we can show $\mu_{\sqrt{A}}(x\gamma y) \geq \mu_{\sqrt{A}}(y)$. Thus we have $\mu_{\sqrt{A}}(x\gamma y) \geq \mu_{\sqrt{A}}(x) \vee \mu_{\sqrt{A}}(y)$. Similarly, we can prove $\nu_{\sqrt{A}}(x\gamma y) \leq \nu_{\sqrt{A}}(x) \wedge \nu_{\sqrt{A}}(y)$. Hence $\sqrt{A}$ is a non-constant IFI of M. □

Proposition 3.4. *Let A be an IFI of a Γ-ring M. Then $\sqrt{A}$ is an intuitionistic fuzzy semi-prime ideal of M.*

Proof. We have already shown that $\sqrt{A}$ is a non-constant IFI of M. Now for any $m \in M$, we have

$$\begin{aligned}&Inf\{\mu_{\sqrt{A}}(m\gamma_1 x\gamma_2 m) : x \in M, \gamma_1, \gamma_2 \in \Gamma\}\\ &= Inf\{\mu_{\cap(\wp(A))}(m\gamma_1 x\gamma_2 m) : x \in M, \gamma_1, \gamma_2 \in \Gamma\}\\ &= Inf\{Inf\{\mu_B(m\gamma_1 x\gamma_2 m) : B \in IFPI(M); A \subseteq B\}, x \in M, \gamma_1, \gamma_2 \in \Gamma\}\\ &= Inf\{\mu_B(m) : B \in IFPI(M); A \subseteq B\}[\text{ As } B \in IFPI(M)]\\ &= \mu_{\cap(\wp(A))}(m)\\ &= \mu_{\sqrt{A}}(m).\end{aligned}$$

Similarly, we can prove $Sup\{\nu_{\sqrt{A}}(m\gamma_1 x\gamma_2 m) : x \in M, \gamma_1, \gamma_2 \in \Gamma\} = \nu_{\sqrt{A}}(m)$.

Hence $\sqrt{A}$ is an intuitionistic fuzzy semi-prime ideal of M (by Proposition (2.1)). □

Proposition 3.5. *Let A and B be two IFIs of a Γ-ring M. Then*
(i) $\sqrt{A}(x) = (1, 0)$ *if* $x \in (\sqrt{A})_*$
(ii) $A \subseteq \sqrt{A}$

(iii) If $A \subseteq B$ then $\sqrt{A} \subseteq \sqrt{B}$
(iv) $\sqrt{\sqrt{A}} = \sqrt{A}$
(v) $\sqrt{A \oplus B} = \sqrt{\sqrt{A} \oplus \sqrt{B}}$, where $A(0_M) = B(0_M) = (1, 0)$.

Proof.
(i) Let $x \in (\sqrt{A})_*$. Then

$$\begin{aligned} \mu_{\sqrt{A}}(x) &= \mu_{\sqrt{A}}(0_M) = \mu_{\cap(\wp(A))}(0_M) \\ &= Inf\{\mu_B(0_M) : B \in IFPI(M); A \subseteq B\} \\ &= 1. \end{aligned}$$

Similarly, we can prove $\nu_{\sqrt{A}}(x) = 0$. Thus $\sqrt{A}(x) = (1, 0)$.

(ii) For any $x \in M$

$$\begin{aligned} \mu_{\sqrt{A}}(x) &= \mu_{\cap(\wp(A))}(x) \\ &= Inf\{\mu_B(x) : B \in IFPI(M); A \subseteq B\} \\ &\geq \mu_A(x). \end{aligned}$$

Similarly, we can prove $\nu_{\sqrt{A}}(x) \leq \nu_A(x)$. Thus $A \subseteq \sqrt{A}$.

(iii) Let A and B be two IFIs of a Γ-ring M such that $A \subseteq B$. Then $\wp(B) \subseteq \wp(A)$. Thus $\cap(\wp(A)) \subseteq \cap(\wp(B))$, i.e., $\sqrt{A} \subseteq \sqrt{B}$.

(iv) Since $A \subseteq \sqrt{A}$, it follows that $\sqrt{A} \subseteq \sqrt{\sqrt{A}}$ and $\wp(A) \subseteq \wp(\sqrt{A})$. Thus $\cap(\wp(\sqrt{A})) \subseteq \cap(\wp(A))$, i.e., $\sqrt{\sqrt{A}} \subseteq \sqrt{A}$. Hence $\sqrt{\sqrt{A}} = \sqrt{A}$.

(v) Since $A \subseteq \sqrt{A}$ and $B \subseteq \sqrt{B}$, so $A \oplus B \subseteq \sqrt{A} \oplus \sqrt{B}$. Thus $\sqrt{A \oplus B} \subseteq \sqrt{\sqrt{A} \oplus \sqrt{B}}$.

Again $A \subseteq A \oplus B$ and $B \subseteq A \oplus B$ so $\sqrt{A} \subseteq \sqrt{A \oplus B}$ and $\sqrt{B} \subseteq \sqrt{A \oplus B}$ implies $\sqrt{A} \oplus \sqrt{B} \subseteq \sqrt{A \oplus B}$. Thus $\sqrt{\sqrt{A} \oplus \sqrt{B}} \subseteq \sqrt{\sqrt{A \oplus B}} = \sqrt{A \oplus B}$. Hence $\sqrt{A \oplus B} = \sqrt{\sqrt{A} \oplus \sqrt{B}}$. □

Proposition 3.6. *Let A be an $IFPI$ of a Γ-ring M. Then $\sqrt{A} = A$ and hence every $IFPI$ is intuitionistic fuzzy semi prime ideal.*

Proof. Assume that A is an $IFPI$ of Γ-ring M. Therefore $A \in IFPI(M)$.
$\sqrt{A} = \cap(\wp(A)) = \cap\{B : B \in IFPI(M); A \subseteq B\} \subseteq A$. Again $A \subseteq \sqrt{A}$. So $\sqrt{A} = A$.
The second assertion follows from Proposition (3.4). □

Lemma 3.1. *Let A be an IFI of M such that $A(0_M) = (1, 0)$, then $\sqrt{A_*} \subseteq (\sqrt{A})_*$, where $\sqrt{A_*} = \cap\{L : L$ is a prime ideal of M such that $A_* \subseteq L\}$.*

Proof. Let $x \in \sqrt{A_*}$. Then $x \in L$ for all prime ideal L of M such that $A_* \subseteq L$. Let B be an IF prime ideal of M such that $A \subseteq B$. Let $m \in A_*$. Then
$\mu_A(m) = \mu_A(0_M) = 1 = \mu_B(m)$ and $\nu_A(m) = \nu_A(0_M) = 0 = \nu_B(m)$. So $m \in B_*$. Hence $A_* \subseteq B_*$. As B is an IF prime ideal of M, B_* is a prime ideal of M (By Theorem (2.4)). Also $A_* \subseteq B_*$ so $x \in B_*$. Hence $B(x) = B(0_M) = (1, 0)$. Now

$$\begin{aligned} \mu_{\sqrt{A}}(x) &= \mu_{\cap(\wp(A))}(x) \\ &= Inf\{\mu_B(x) : B \in IFPI(M); A \subseteq B\} \\ &= 1 = \mu_{\sqrt{A}}(0_M). \end{aligned}$$

Similarly, we can prove that $\nu_{\sqrt{A}}(x) = \nu_{\sqrt{A}}(0_M)$. So $x \in (\sqrt{A})_*$. Thus $\sqrt{A_*} \subseteq (\sqrt{A})_*$. □

Lemma 3.2. *If A be an intuitionistic fuzzy ideal of M such that $Img(A) = \{(1,0),(t,s)\}$, where $t, s \in [0,1)$ such that $t + s \leq 1$. Then $(\sqrt{A})_* \subseteq \sqrt{A_*}$.*

Proof. Let $x \in (\sqrt{A})_*$. Then $\mu_{\sqrt{A}}(x) = \mu_{\sqrt{A}}(0_M) = 1$ and $\nu_{\sqrt{A}}(x) = \nu_{\sqrt{A}}(0_M) = 0$. Therefore, $\sqrt{A}(x) = (1,0)$. This implies that $P(x) = (1,0)$ for all intuitionistic fuzzy prime ideal P with the condition that $A \subseteq P$. Thus $x \in P_*$ whenever $P \in IFPI(M)$, $A \subseteq P$.

Let L be a prime ideal of M such that $A_* \subseteq L$. Now we define an IFS B of M as

$$\mu_B(m) = \begin{cases} 1, & \text{if } m \in L \\ t_1, & \text{if } m \in M \backslash L \end{cases}; \quad \nu_B(m) = \begin{cases} 0, & \text{if } m \in L \\ s_1, & \text{if } m \in M \backslash L. \end{cases}$$

where $t_1, s_1 \in (0,1)$ such that $t_1 > t$ and $s_1 < s$. Then B is an intuitionistic fuzzy prime ideal of M [by Theorem (2.4)] such that $A \subseteq B$. Hence $x \in B_* = L$. So $x \in \bigcap\{L : L \text{ is a prime ideal of } M \text{ such that } A_* \subseteq L\}$. Hence $x \in \sqrt{A_*}$. Thus we have $(\sqrt{A})_* \subseteq \sqrt{A_*}$. □

4. Intuitionistic fuzzy primary ideal of a Γ-ring

Definition 4.13. Let M be a Γ-ring. For any IFI A of M. The IFS $\sqrt{A}$ defined by

$$\mu_{\sqrt{A}}(x) = \vee\{\mu_A((x\gamma)^{n-1}x) : n \in \mathbf{N}\} \text{ and } \nu_{\sqrt{A}}(x) = \wedge\{\nu_A((x\gamma)^{n-1}) : n \in \mathbf{N})\}$$

is called the intuitionistic fuzzy prime radical of A, where $(x\gamma)^{n-1}x = x$, for $n = 1, \gamma \in \Gamma$.

Proposition 4.7. *For every IFIs A and B of Γ-ring M, we have*
(i) $A \subseteq \sqrt{A}$;
(ii) $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$;
(iii) $\sqrt{\sqrt{A}} = \sqrt{A}$.

Proof. Straightforward. □

Theorem 4.5. *For any IFI A of Γ-ring M, $\sqrt{A}$ is an IFI of M.*

Proof. Let $x, y \in M, \gamma \in \Gamma$.

$$\begin{aligned} \mu_{\sqrt{A}}(x+y) &= \vee_{k\geq 1}[\mu_A\{((x+y)\gamma)^k(x+y)\}] \\ &\geq \mu_A\{((x+y)\gamma)^{m+n}(x+y)\} \\ &= \mu_A\{(x\gamma)^{m+n}x\} \wedge \mu_A\{(y\gamma)^{m+n}y\} \wedge_{p+q=m+n} \mu_A\{(x\gamma)^p(y\gamma)^q x\} \\ &\qquad \wedge_{p+q=m+n}\mu_A\{(y\gamma)^p(x\gamma)^q y\} \\ &\geq \mu_A\{(x\gamma)^n x\} \wedge \mu_A\{(y\gamma)^n y\} \\ &= \mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{A}}(y). \end{aligned}$$

[As $((x+y)\gamma)^{m+n}(x+y)$ can be written as sum of the terms of the forms $(x\gamma)^{m+n}x$, $(y\gamma)^{m+n}y$, $(x\gamma)^p(y\gamma)^q x$ and $(y\gamma)^p(x\gamma)^q y$, for some $p, q \in \mathbf{N}$ such that $p + q = m + n$.]
Similarly, we can show that $\nu_{\sqrt{A}}(x+y) \leq \nu_{\sqrt{A}}(x) \vee \nu_{\sqrt{A}}(y)$. Further, since

$$\begin{aligned} \mu_A\{(x\gamma)^n x\} \vee \mu_A\{(y\gamma)^n y\} &\leq \mu_A\{(x\gamma)^n x\gamma(y\gamma)^n y\} \\ &\leq \vee_{k\geq 1}[\mu_A\{(x\gamma y)^k x\gamma y\}] \\ &= \mu_{\sqrt{A}}(x\gamma y). \end{aligned}$$

Thus $\mu_{\sqrt{A}}(x\gamma y) \geq \mu_A\{(x\gamma)^n x\} \vee \mu_A\{(y\gamma)^n y\}$. Similarly, we can show that $\nu_{\sqrt{A}}(x\gamma y) \leq \nu_A\{(x\gamma)^n x\} \wedge \nu_A\{(y\gamma)^n y\}$. Hence $\sqrt{A}$ is an IFI of M. □

Proposition 4.8. *Let A, B be two IFIs of a Γ-ring M. Then*
$$\sqrt{A\Gamma B} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}.$$

Proof. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ implies $\sqrt{A \cap B} \subseteq \sqrt{A}$ and $\sqrt{A \cap B} \subseteq \sqrt{B}$ and so $\sqrt{A \cap B} \subseteq \sqrt{A} \cap \sqrt{B}$.
For the reverse inclusion, let $x \in M, \gamma \in \Gamma$ be any element. Now

$$\begin{aligned}
\mu_{\sqrt{A}\cap\sqrt{B}}(x) &= \mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{B}}(x) \\
&= [\vee\{\mu_A((x\gamma)^m x) : m > 0\}] \wedge [\vee\{\mu_A((x\gamma)^n x) : n > 0\}] \\
&= \vee\{\mu_A((x\gamma)^m x) \wedge \mu_A((x\gamma)^n) : m, n > 0\} \\
&\leq \vee\{\mu_A((x\gamma)^{m+n} x) \wedge \mu_A((x\gamma)^{m+n} x) : m + n > 0\} \\
&= \vee\{\mu_{A\cap B}((x\gamma)^{m+n} x) : m + n > 0\} \\
&= \mu_{\sqrt{A\cap B}}(x).
\end{aligned}$$

Similarly, we van show that $\nu_{\sqrt{A}\cap\sqrt{B}}(x) \geq \nu_{\sqrt{A\cap B}}(x)$. Thus $\sqrt{A} \cap \sqrt{B} \subseteq \sqrt{A \cap B}$.
Hence $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.

Further, as $A\Gamma B \subseteq A \cap B$ implies $\sqrt{A\Gamma B} \subseteq \sqrt{A \cap B}$. For the other inclusion, let $x \in M, \gamma \in \Gamma$ be any element. Now

$$\begin{aligned}
\mu_{\sqrt{A\cap B}}(x) &= \mu_{\sqrt{A}\cap\sqrt{B}}(x) = \mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{B}}(x) \\
&= [\vee\{\mu_A((x\gamma)^m x) : m > 0\}] \wedge [\vee\{\mu_A((x\gamma)^n x) : n > 0\}] \\
&= \vee\{\mu_A((x\gamma)^m x) \wedge \mu_A((x\gamma)^n x) : m, n > 0\} \\
&\leq \vee\{\mu_{A\Gamma B}(z) : z = (x\gamma)^{m+n} x : m + n > 0\} \\
&= \mu_{\sqrt{A\Gamma B}}(x).
\end{aligned}$$

Similarly, we can show that $\nu_{\sqrt{A\cap B}}(x) \geq \nu_{\sqrt{A\Gamma B}}(x)$. Thus $\sqrt{A \cap B} \subseteq \sqrt{A\Gamma B}$.
Thus $\sqrt{A \cap B} = \sqrt{A\Gamma B}$. Hence $\sqrt{A\Gamma B} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$. □

Corollary 4.1. *If $\{A_i : 1 \leq i \leq n\}$ be a finite number of IFIs of a Γ-ring M, then*
$$\sqrt{A_1\Gamma A_2\Gamma A_3......\Gamma A_n} = \sqrt{A_1 \cap A_2 \cap A_3 \cap ..... \cap A_n} = \sqrt{A_1} \cap \sqrt{A_2} \cap \sqrt{A_3} \cap ..... \cap \sqrt{A_n}.$$

Definition 4.14. Let Q be a non-constant IFI of a Γ-ring M. Then Q is said to be an intuitionistic fuzzy primary ideal of M if for any two IFIs A, B of M such that $A\Gamma B \subseteq Q$ implies that either $A \subseteq Q$ or $B \subseteq \sqrt{Q}$.

Theorem 4.6. *Let $Q \in IFI(M)$. Then Q is an intuitionistic fuzzy primary ideal of M if and only if Q is non-constant and $A \circ B \subseteq Q$ implies that either $A \subseteq Q$ or $B \subseteq \sqrt{Q}$, where $A, B \in IFI(M)$.*

Proof. The proof follows from Remark (2.3), since $A \circ B \subseteq Q$ if and only if $A\Gamma B \subseteq Q$, where $A, B \in IFI(M)$. □

Theorem 4.7. *Let M be a commutative Γ-ring and Q be an IFI of M. Then for any two IFPs $x_{(p,q)}, y_{(t,s)} \in IFP(M)$ the following are equivalent:*
(i) Q is an intuitionistic fuzzy primary ideal of M
(ii) $x_{(p,q)}\Gamma y_{(t,s)} \subseteq Q$ implies $x_{(p,q)} \subseteq Q$ or $y_{(t,s)} \subseteq \sqrt{Q}$.

Proof. $(i) \Rightarrow (ii)$ Suppose that Q is an intuitionistic fuzzy primary ideal of M.
Let $x_{(p,q)}, y_{(t,s)} \in IFP(M)$ such that $x_{(p,q)}\Gamma y_{(t,s)} \subseteq Q$. This implies $(x\Gamma y)_{(p\wedge t, q\vee s)} \subseteq Q$,

i.e., $\mu_Q(x\gamma y) \geq p \wedge t$ and $\nu_Q(x\gamma y) \leq q \wedge s$, for every $\gamma \in \Gamma$.
Define two IFSs A, B of M as follows

$$A(z) = \begin{cases} (p,q), & \text{if } z \in <x> \\ (0,1), & \text{otherwise} \end{cases} ; \quad B(z) = \begin{cases} (t,s), & \text{if } z \in <y> \\ (0,1), & \text{otherwise} \end{cases}$$

Clearly, A, B are IFIs of M and $x_{(p,q)} \subseteq A$ and $y_{(t,s)} \subseteq B$. Now
$\mu_{A\Gamma B}(z) = \vee_{z=u\gamma v}[\mu_A(u) \wedge \mu_B(v)] = p \wedge q$ and $\nu_{A\Gamma B}(z) = \wedge_{z=u\gamma v}[\nu_A(u) \vee \nu_B(v)] = q \vee s$, where $u \in <x>$, $v \in <y>$. Thus $\mu_{A\Gamma B}(z) = p \wedge q \leq \mu_Q(x\gamma y)$ and $\nu_{A\Gamma B}(z) = q \vee a \geq \nu_Q(x\gamma y)$.
Thus when $z = u\gamma v$, where $u \in <x>$, $v \in <y>$. $(A\Gamma B)(z) \subseteq Q(z)$ otherwise $(A\Gamma B)(z) = (0,1)$. Thus get $A\Gamma B \subseteq Q$. As Q is intuitionistic fuzzy primary ideal of M, so either $A \subseteq Q$ or $B \subseteq \sqrt{Q}$. Thus we have $x_{(p,q)} \subseteq A \subseteq Q$ or $y_{(t,s)} \subseteq B \subseteq \sqrt{Q}$, i.e., $x_{(p,q)} \subseteq Q$ or $y_{(t,s)} \subseteq \sqrt{Q}$.

$(ii) \Rightarrow (i)$ Let A and B be two IFIs of Γ-ring M such that $A\Gamma B \subseteq Q$. Suppose that $A \nsubseteq Q$. Then $\exists x \in M$ such that $\mu_A(x) > \mu_Q(x)$ and $\nu_A(x) < \nu_Q(x)$. Let $\mu_A(x) = a, \nu_A(x) = b$. Let $y \in M$ and $\mu_B(y) = c, \nu_B(x) = d$.
If $z = x\gamma y$ for some $\gamma \in \Gamma$, then $(x_{(a,b)}\Gamma y_{(c,d)})(z) = (a \wedge c, b \vee d)$. Hence
$\mu_Q(z) = \mu_Q(x\gamma y) \geq \mu_{A\Gamma B}(x\gamma y) \geq [\mu_A(x) \wedge \mu_B(y)] = a \wedge c = \mu_{x_{(a,b)}\Gamma y_{(c,d)}}(x\gamma y) = \mu_{x_{(a,b)}\Gamma y_{(c,d)}}(z)$
$\nu_Q(z) = \nu_Q(x\gamma y) \leq \nu_{A\Gamma B}(x\gamma y) \leq [\nu_A(x) \vee \nu_B(y)] = b \vee d = \nu_{x_{(a,b)}\Gamma y_{(c,d)}}(x\gamma y) = \nu_{x_{(a,b)}\Gamma y_{(c,d)}}(z)$.
If $\mu_{x_{(a,b)}\Gamma y_{(c,d)}}(z) = 0, \nu_{x_{(a,b)}\Gamma y_{(c,d)}}(z) = 1$, then $\mu_Q(z) \geq \mu_{x_{(a,b)}\Gamma y_{(c,d)}}(z), \nu_Q(z) \leq \nu_{x_{(a,b)}\Gamma y_{(c,d)}}(z)$.
Hence $x_{(a,b)}\Gamma y_{(c,d)} \subseteq Q$. By (i) either $x_{(a,b)} \subseteq Q$ or $y_{(c,d)} \subseteq \sqrt{Q}$.
i.e., either $\mu_Q(x) \geq a, \nu_Q(x) \leq b$ or $\mu_{\sqrt{Q}}(y) \geq c, \nu_{\sqrt{Q}}(y) \leq d$.
Since $a \nleq \mu_Q(x), b \ngeq \nu_Q(x)$ implies that $x_{(a,b)} \nsubseteq Q$ and so $y_{(c,d)} \subseteq \sqrt{Q}$. This implies that $\mu_{\sqrt{Q}}(y) \geq c = \mu_B(y)$ and $\nu_{\sqrt{Q}}(y) \leq d = \nu_B(y), \forall y \in M$. Which implies that $B \subseteq \sqrt{Q}$. Hence Q is an intuitionistic fuzzy primary ideal of M. □

Proposition 4.9. *Let Q be an IFI of a Γ-ring M. If Q is an intuitionistic fuzzy primary ideal of M, then for all $x, y \in M, \gamma \in \Gamma$ such that $\mu_Q(x\gamma y) > \mu_Q(x), \nu_Q(x\gamma y) < \nu_Q(x)$ implies that $\mu_Q(x\gamma y) < \mu_{\sqrt{Q}}(y), \nu_Q(x\gamma y) > \nu_{\sqrt{Q}}(y)$.*

Proof. $\mu_Q(x\gamma y) = r > \mu_Q(x), \nu_Q(x\gamma y) = s < \nu_Q(x)$. Then $(x\gamma y)_{(r,s)} \in Q$ and $x_{(r,s)} \notin Q$. Since Q is an intuitionistic fuzzy primary ideal of M then $y_{(r,s)} \in \sqrt{Q}$. Thus $\mu_{\sqrt{Q}}(y) \geq r = \mu_Q(x\gamma y)$ and $\nu_{\sqrt{Q}}(y) \leq s = \nu_Q(x\gamma y)$. This complete the proof. □

Theorem 4.8. *Let Q be an intuitionistic fuzzy primary ideal of Γ-ring M. Then $Q_* = \{x \in M : \mu_Q(x) = \mu_Q(0_M)$ and $\nu_Q(x) = \nu_Q(0_M)\}$ is a primary ideal of M.*

Proof. Let $x, y \in Q_*$. Then $\mu_Q(x) = \mu_Q(y) = \mu_Q(0_M)$ and $\nu_Q(x) = \nu_Q(y) = \nu_Q(0_M)$. Now $\mu_Q(x-y) \geq \mu_Q(x) \wedge \mu_Q(y) = \mu_Q(0_M)$ and $\nu_Q(x-y) \leq \nu_Q(x) \vee \nu_Q(y) = \nu_Q(0_M)$ implies that $\mu_Q(x-y) = \mu_Q(0_M)$ and $\nu_Q(x-y) = \nu_Q(0_M)$. So $x - y \in Q_*$.

Further, let $x \in M$ and $y \in Q_*$, then $\mu_Q(y) = \mu_Q(0_M)$ and $\nu_Q(y) = \nu_Q(0_M)$.
Let $\gamma \in \Gamma$ be any element, then $\mu_Q(x\gamma y) \geq \mu_Q(x) \vee \mu_Q(y) = \mu_Q(x) \vee \mu_Q(0_M) = \mu_Q(0_M)$.
But $\mu_Q(0_M) \geq \mu_Q(x\gamma y)$ always implies $\mu_Q(x\gamma y) = \mu_Q(0_M)$. Similarly, $\nu_Q(x\gamma y) = \nu_Q(0_M)$.
Thus $x\gamma y \in Q_*$. This shows that Q_* is a right ideal of Γ-ring M. Similarly, we can show Q_* is a left ideal of Γ-ring M. Thus Q_* is a ideal of Γ-ring M.
Further, let $x, y \in M, \gamma \in \Gamma$ such that $x\gamma y \in Q_*$, i.e., $\mu_Q(x\gamma y) = \mu_Q(0_M)$ and $\nu_Q(x\gamma y) = \nu_Q(0_M)$. Suppose that $x \notin Q_*$, then we claim that $y \in \sqrt{Q_*}$, i.e., there exist some $m \in \mathbf{N}$

and $\gamma \in \Gamma$ such that $(y\gamma)^m y \in Q_*$.

As $x \notin Q_* \Rightarrow \mu_Q(x) < \mu_Q(0_M)$ and $\nu_Q(x) > \mu_Q(0_M)$. Thus we have $\mu_Q(x\gamma y) > \mu_Q(x), \nu_Q(x\gamma y) < \nu_Q(x)$. Then by above proposition (4.9) we have $\mu_Q(x\gamma y) < \mu_{\sqrt{Q}}(y), \nu_Q(x\gamma y) > \nu_{\sqrt{Q}}(y)$, i.e., $\mu_{\sqrt{Q}}(y) > \mu_Q(0_M), \nu_{\sqrt{Q}}(y) < \nu_Q(0_M)$ implies that $\vee\{\mu_Q((y\gamma)^m y) : m > 0\} > \mu_Q(0_M), \wedge\{\nu_Q((y\gamma)^m y) : m > 0\} < \nu_Q(0_M)$. Thus there exists some $m \in \mathbf{N}, \gamma \in \Gamma$ such that $\mu_Q((y\gamma)^m y) > \mu_Q(0_M)$ and $\nu_Q((y\gamma)^m y) < \nu_Q(0_M)$, i.e., $\mu_Q((y\gamma)^m y) = \mu_Q(0_M)$ and $\nu_Q((y\gamma)^m y) = \nu_Q(0_M)$ and so $(y\gamma)^m y \in Q_*$. Thus $y \in \sqrt{Q_*}$. This complete the proof. □

Theorem 4.9. *Let Q be an IFS of a Γ-ring M. If $Q(0_M) = (1, 0)$, Q_* is a primary ideal of M and $Img(Q) = \{(1, 0), (t, s)\}$, where $t, s \in [0, 1)$ such that $t + s \leq 1$. Then Q is an intuitionistic fuzzy primary ideal of M.*

Proof. Clearly Q is non-constant IFI of M as Q_* is a ideal of M. Let $A, B \in IFI(M)$ such that $A\Gamma B \subseteq Q$. Suppose $A \nsubseteq Q$ and $B \nsubseteq \sqrt{Q}$. Then $\exists, x, y \in M$ such that $\mu_A(x) > \mu_Q(x)$, $\nu_A(x) < \nu_Q(x)$ and $\mu_B(y) > \mu_{\sqrt{Q}}(y)$, $\nu_B(y) < \nu_{\sqrt{Q}}(y)$. Since $Q(0_M) = (1, 0) = \sqrt{Q}(0_M)$ gives that $x \notin Q_*$ and $y \notin (\sqrt{Q})_*$. Again since $\sqrt{Q_*} \subseteq (\sqrt{Q})_*$, so $y \notin \sqrt{Q_*}$.
Hence $x\Gamma M\Gamma y \nsubseteq Q_*$ (by Theorem 9, [2]) as Q_* is a primary ideal of M.
Hence $\mu_Q(x\gamma_1 m\gamma_2 y) = t \neq 1$, $\nu_Q(x\gamma_1 m\gamma_2 y) = s \neq 0$, for some $\gamma_1, \gamma_2 \in \Gamma$, $m \in M$.

Since $x \notin Q_*$, $\mu_Q(x) \neq \mu_Q(0_M) = 1$, $\nu_Q(x) \neq \nu_Q(0_M) = 0$. So $\mu_Q(x) = t, \nu_Q(x) = s$. Thus $\mu_A(x) > \mu_Q(x) = t$, $\nu_A(x) < \nu_Q(x) = s$.

Again since $\mu_Q(y) \leq \mu_{\sqrt{Q}}(y) < \mu_B(y)$ and $\nu_Q(y) \geq \nu_{\sqrt{Q}}(y) > \nu_B(y)$, $Q(y) \neq (1, 0)$. So $t = \mu_Q(y) < \mu_B(y)$ and $s = \nu_Q(y) > \nu_B(y)$. Now

$$\begin{aligned} t &= \mu_Q(x\gamma_1 m\gamma_2 y) \\ &\geq \mu_{A\Gamma B}(x\gamma_1 m\gamma_2 y) \\ &\geq \mu_A(x) \wedge \mu_B(y) \\ &> t. \end{aligned}$$

which is a contradiction. Hence Q is an intuitionistic fuzzy primary ideal of M. □

Example 4.1. If K is a primary ideal of M, then the intuitionistic fuzzy characteristic function χ_K is an intuitionistic fuzzy primary ideal of M.

Proof. Here we have

$$\mu_{\chi_K}(x) = \begin{cases} 1, & \text{if } x \in K \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{\chi_K}(x) = \begin{cases} 0, & \text{if } x \in K \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, $\mu_{\chi_K}(0_M) = 1, \nu_{\chi_K}(0_M) = 0$ and $(\chi_K)_* = K$ is a primary ideal of M. Hence χ_K is an intuitionistic fuzzy primary ideal of M. □

Proposition 4.10. *Let M be a Γ-ring and Q be a non-constant intuitionistic fuzzy primary ideal of M. Then there exists an intuitionistic fuzzy prime ideal P of M such that $P \in \wp(Q)$.*

Proof. Since Q is non-constant, then there exists $m \in M$ such that $\mu_Q(m) \neq \mu_Q(0_M)$ and $\nu_Q(m) \neq \nu_Q(0_M)$. Let $\mu_Q(m) < t < \mu_Q(0_M)$ and $\nu_Q(m) > s > \nu_Q(0_M)$. Then $Q_{(t,s)} \neq M$ and $Q_{(t,s)}$ is an ideal of M. So there exists a prime L of M such that $Q_{(t,s)} \subset L \subset M$.
Let P be an IFS on M defined by

$$\mu_P(x) = \begin{cases} 1, & \text{if } x \in L \\ t, & \text{otherwise} \end{cases}; \quad \nu_P(x) = \begin{cases} 0, & \text{if } x \in L \\ s, & \text{otherwise.} \end{cases}$$

Then P is an intuitionistic fuzzy prime ideal of M (by Theorem (2.4))

Let $x \in M$. Then either $\mu_Q(x) \geq t, \nu_Q(x) \leq s$ or $\mu_Q(x) > t, \nu_Q(x) < s$.
In the second case we get $\mu_Q(x) \leq \mu_P(x)$, $\mu_Q(x) \geq \mu_P(x)$.
In the first case we get $x \in Q_{(t,s)} \subset L$, so $\mu_P(x) = 1, \nu_P(x) = 0$. Hence in both the cases we get $\mu_Q(x) \leq \mu_P(x)$, $\mu_Q(x) \geq \mu_P(x)$. Thus $Q \subseteq P$. Hence $P \in \wp(Q)$. □

Proposition 4.11. *Let M be a Γ-ring and $\sum_i^n [e_i, \delta_i], e_i \in M, \delta_i \in \Gamma$, for $i = 1, 2, 3....., n$ be the left unity of M and A be a non-constant intuitionistic fuzzy ideal of M. Let $m \in M$ be such that $min\{\mu_A(e_i) : i = 1, 2, ...n\} < \mu_A(m)$ and $max\{\nu_A(e_i) : i = 1, 2, ...n\} > \nu_A(m)$. Then there exists $e \in \{e_i : i = 1, 2, ...n\}$ such that $\mu_{\sqrt{A}}(e) < \mu_A(m)$ and $\nu_{\sqrt{A}}(e) > \nu_A(m)$.*

Proof. Let $\mu_A(m) = s_1, \nu_A(m) = s_2$ and $min\{\mu_A(e_i) : i = 1, 2, ...n\} = t_1 = \mu_A(e^{'})$, $max\{\nu_A(e_i) : i = 1, 2, ...n\} = t_2 = \nu_A(e^{'})$, where $e^{'} \in \{e_i : i = 1, 2, ...n\}$.

Suppose that $r_1, r_2 \in [0, 1)$ such that $t_1 < r_1 < s_1$ and $t_2 > r_2 > s_2$. Then (r_1, r_2)-cut set $A_{(r_1,r_2)}$ is an ideal of M. Since $e^{'} \notin A_{(r_1,r_2)}$. Let L be a prime ideal of M such that $A_{(r_1,r_2)} \subseteq L$ and $L \neq M$.

Let B be an IFS of M defined by

$$\mu_B(m) = \begin{cases} 1, & \text{if } m \in L \\ r_1, & \text{if } m \notin L \end{cases}; \quad \nu_B(m) = \begin{cases} 0, & \text{if } m \in L \\ r_2, & \text{if } m \notin L. \end{cases}$$

Then by proposition (4.10) we can prove that $B \in \wp(A)$. Now as L is a proper ideal of M, there exists atleast one $e \in \{e_i : i = 1, 2, ...n\}$ such that $e \notin L$, for if $e_i \in L$ for all $i = 1, 2, 3, ..., n$, then $x = \sum_i e_i \delta_i x$ for all $x \in M$ that is $L = M$, a contradiction.
Hence $\mu_B(e) = r_1$ and $\nu_B(e) = r_2$. As $B \in \wp(A)$, $\sqrt{A} \subseteq B$,
Now $\mu_{\sqrt{A}}(e) \leq \mu_B(e) = r_1 < \mu_A(m)$ and $\nu_{\sqrt{A}}(e) \geq \nu_B(e) = r_2 > \nu_A(m)$.
This complete the result. □

Now we have the converse of Theorem (4.9)

Theorem 4.10. *Let M be a Γ-ring and Q be an intuitionistic fuzzy primary ideal of M. Then $Q(0_M) = (1, 0)$, $|Img(Q)| = 2$ and Q_* is a primary ideal of M.*

Proof. Suppose that $\mu_Q(0_M) = t < 1$ and $\nu_Q(0_M) = s > 0$.
Let $min_i\{\mu_Q(e_i)\} = \alpha < \mu_Q(0_M)$ and $max_i\{\nu_Q(e_i)\} = \beta > \nu_Q(0_M)$. Then there exists $e \in \{e_i : i = 1, 2, ...n\}$ such that $\mu_{\sqrt{Q}}(e) = t_1 < t$ and $\nu_{\sqrt{Q}}(e) = s_1 < s$ (by Proposition (4.11)).

Let $t < p \leq 1$ and $s > q \geq 0$. Then $\alpha < t_1 < p \leq 1$ and $\beta > s_1 > s \geq 0$.
Let A, B be two IFSs on M defined by

$$\mu_A(x) = \begin{cases} p, & \text{if } x \in Q_* \\ \alpha, & \text{if } x \notin Q_* \end{cases}; \quad \nu_A(x) = \begin{cases} q, & \text{if } x \in Q_* \\ \beta, & \text{if } x \notin Q_*. \end{cases}$$

and $B(x) = (t, s)$, for all $x \in M$. Then A and B are IFIs of M. Let $x \in M$ be any element.

If $x \in Q_*$, then $Q(x) = B(x) = (t, s)$ and so $\mu_{A\Gamma B}(x) = \vee_{x=y\gamma z}[\mu_A(y) \wedge \mu_B(z)] \leq t = \mu_Q(x)$ and $\nu_{A\Gamma B}(x) = \wedge_{x=y\gamma z}[\nu_A(y) \vee \nu_B(z)] \geq s = \nu_Q(x)$.

If $x \notin Q_*$, then $A(x) = (\alpha, \beta)$, then $\mu_{A\Gamma B}(x) = \alpha = min_i\{\mu_Q(e_i)\} \leq \mu_Q(x)$ and
$\nu_{A\Gamma B}(x) = \beta = max_i\{\nu_Q(e_i)\} \geq \nu_Q(x)$. So $A\Gamma B \subseteq Q$.

Also $\mu_A(0_M) = p > t = \mu_Q(0_M)$ and $\nu_A(0_M) = q < s = \nu_Q(0_M)$. So $A \nsubseteq Q$.

Again for some $e \in \{e_i : i = 1, 2, ..., n\}$, $\mu_B(e) = t > t_1 = \mu_{\sqrt{Q}}(e)$ and $\nu_B(e) = s < s_1 = \nu_{\sqrt{Q}}(e)$ implies that $B \nsubseteq \sqrt{Q}$. This is a contradiction since Q is an intuitionistic fuzzy primary ideal of M. Hence $\mu_Q(0_M) = 1$ and $\nu_Q(0_M) = 0$, i.e., $Q(0_M) = (1, 0)$.

Since Q is non-constant, so $|Img(Q)| \geq 2$. Suppose that $|Img(Q)| \geq 3$. Let $min_i\{\mu_Q(e_i)\} = \alpha$ and $max_i\{\nu_Q(e_i)\} = \beta$. Then there exists $(t, s) \in Img(Q)$ such that $\alpha < t < 1$ and $\beta > s > 0$. Let $m \in M$ be such that $\mu_Q(m) = t, \nu_Q(m) = s$. Then there exists $e \in \{e_i : i = 1, 2, ...n\}$ such that $\mu_{\sqrt{Q}}(e) < \mu_Q(m), \nu_{\sqrt{Q}}(e) > \nu_Q(m)$.

Let A, B be two IFSs of M such that

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in Q_{(t,s)} \\ \alpha, & \text{if } x \notin Q_{(t,s)} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in Q_{(t,s)} \\ \beta, & \text{if } x \notin Q_{(t,s)}. \end{cases}$$

and $B(x) = (t, s)$, for all $x \in M$. Then A and B are IFIs of M and $A\Gamma B \subseteq Q$.

Now $\mu_A(m) = 1 > t = \mu_Q(m)$ and $\nu_A(m) = 0 < s = \nu_Q(m)$. Thus $A \nsubseteq Q$. Also for some $e \in \{e_i : i = 1, 2, ..., n\}$ $\mu_B(e) = t = \mu_Q(m) > \mu_{\sqrt{Q}}(e)$ and $\nu_B(e) = s = \nu_Q(m) < \mu_{\sqrt{Q}}(e)$. Hence $B \nsubseteq \sqrt{Q}$. Thus we see that $A \nsubseteq Q$ and $B \nsubseteq \sqrt{Q}$, which is a contradiction. Hence $|Q(M)| = 2$.

Let $Img(Q) = \{(1, 0), (t, s)\}$, where $t, s \in [0, 1)$ such that $t + s \leq 1$. Let I, J be two ideals of M such that $I\Gamma J \subseteq Q_*$. Let $A = \chi_I$, $B = \chi_J$. Then $A\Gamma B \subseteq Q$. Since Q is intuitionistic fuzzy primary ideal, either $A \subseteq Q$ or $B \subseteq \sqrt{Q}$.

If $A \subseteq Q$, then $I \subseteq Q_*$ and if $B \subseteq \sqrt{Q}$, then $J \subseteq (\sqrt{Q})_* \subseteq \sqrt{Q_*}$ (by Lemma (3.2)). Hence Q_* is primary ideal of M. □

Corollary 4.2. *Let I be an ideal of the Γ-ring M such that χ_I is an intuitionistic fuzzy primary ideal of M. Then I is a primary ideal of M.*

Proof. Since χ_I is an intuitionistic fuzzy primary ideal of M, then $I = (\chi_I)_* = \chi_{I_*}$ is a primary ideal of M. □

From Theorem (4.9) and Theorem (4.10) we have

Theorem 4.11. *If Q is an intuitionistic fuzzy primary ideal of a Γ-ring M, then the following conditions hold:*
(i) $Q(0_M) = (1, 0)$,
(ii) Q_* is a primary ideal of M,
(iii) $Img(Q) = \{(1, 0), (t, s)\}$, where $t, s \in [0, 1)$ such that $t + s \leq 1$.

Example 4.2. Consider $M = \Gamma = Z$, the ring of integers. Then M is a Γ-ring. Consider the IFS Q on M defined by

$$\mu_Q(x) = \begin{cases} 1, & \text{if } x \in < p^n > \\ t, & \text{if } x \notin < p^n > \end{cases}; \quad \nu_Q(x) = \begin{cases} 0, & \text{if } x \in < p^n > \\ s, & \text{if } x \notin < p^n > . \end{cases}$$

where p is a prime number and $n > 1$ a positive integer, $t, s \in [0, 1)$ such that $t + s \leq 1$. Then it is easy to check that Q is an intuitionistic fuzzy primary ideal of M.

Remark 4.5. Every intuitionistic fuzzy prime ideal of a Γ-ring M is an intuitionistic fuzzy primary ideal but converse is not true.

Proof. It follows from definition (4.14) and Proposition (3.6). For the converse part, the IFS Q as defined in Example (4.2) is an intuitionistic fuzzy primary ideal but it is not an intuitionistic fuzzy prime ideal (as $Q_* =< p^n >$ is not a prime ideal of M). □

Theorem 4.12. *Let M be a Γ-ring and Q be an intuitionistic fuzzy primary ideal of M. Then $\sqrt{Q}$ is an intuitionistic fuzzy prime ideal of M.*

Proof. Since Q is an intuitionistic fuzzy primary ideal of M, $Q(0_M) = (1, 0)$, Q_* is a primary ideal of M and $Img(Q) = \{(1, 0), (t, s)\}$, where $t, s \in [0, 1)$ such that $t + s \leq 1$. (by Theorem (4.10)). Now $(\sqrt{Q})_* = \sqrt{Q_*}$ is a prime ideal of M and $\sqrt{Q}(x) = (1, 0)$ for $x \in Q_*$.

Let A be an IFS of M such that

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in (\sqrt{Q})_* \\ t, & \text{if } x \notin (\sqrt{Q})_* \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in (\sqrt{Q})_* \\ s, & \text{if } x \notin (\sqrt{Q})_*. \end{cases}$$

Then $A \in \wp(A)$ and $A_* = (\sqrt{Q})_* = \sqrt{Q_*}$.

Let $x \notin (\sqrt{Q})_*$. Then
$t = \mu_Q(x) \leq \mu_{\sqrt{Q}}(x) \leq \mu_A(x) = t$ and $s = \nu_Q(x) \geq \nu_{\sqrt{Q}}(x) \geq \nu_A(x) = s$.
Thus $Img(\sqrt{Q}) = \{(1, 0), (t, s)\}$, where $t, s \in [0, 1)$ such that $t + s \leq 1$, $\sqrt{Q}(0_M) = (1, 0)$ and $(\sqrt{Q})_*$ is a prime ideal of M. Hence $\sqrt{Q}$ is an intuitionistic fuzzy prime ideal of M (By Theorem (2.4)). □

5. Homomorphic behaviour of intuitionistic fuzzy primary ideals and intuitionistic fuzzy prime radical of Γ-ring

Lemma 5.3. *If f is a homomorphism of a Γ-ring M onto a Γ-ring $M^{'}$ and A is an f-invariant IFI of M, then $f(A_*) = (f(A))_*$.*

Proof. Clearly, $\mu_{f(A)}(0_{M'}) = Sup\{\mu_A(x) : f(x) = 0_{M'}\} = Sup\{\mu_A(x) : f(x) = f(0_M)\} = \mu_A(0_M)$. Similarly, we can show that $\nu_{f(A)}(0_{M'}) = \nu_A(0_M)$. Thus $f(A)(0_{M'}) = A(0_M)$.

Let $y \in f(A_*)$. Then $y = f(x)$ for some $x \in A_*$. Hence $A(x) = A(0_M) = f(A)(0_{M'})$.

$$\begin{aligned} \mu_{f(A)}(y) &= Sup\{\mu_A(z) : f(z) = y\} = Sup\{\mu_A(z) : f(z) = f(x)\} = \mu_A(x) \\ &= \mu_{f(A)}(0_{M'}). \end{aligned}$$

Similarly, we can show that $\nu_{f(A)}(y) = \nu_{f(A)}(0_{M'})$. So $y \in (f(A))_*$. Hence $f(A_*) \subseteq (f(A))_*$.

Again let $f(x) \in (f(A))_*$.
$\mu_{f(A)}(0_{M'}) = \mu_{f(A)}(f(x)) = Sup\{\mu_A(t) : f(t) = f(x)\} = \mu_A(x)$. Similarly, we can prove $\nu_{f(A)}(0_{M'}) = \nu_A(x)$. So $A(x) = (f(A))(0_{M'}) = A(0_M)$ implies that $x \in A_*$, i.e., $f(x) \in f(A_*)$. Thus $(f(A))_* \subseteq f(A_*)$. Hence the result prove. □

Lemma 5.4. ([14]) *Let f be a homomorphism of a Γ-ring M onto a Γ-ring $M^{'}$. If A is an f-invariant IFI of M, then $f(A)$ is an IFI of $M^{'}$.*

Theorem 5.13. *Let f be a homomorphism of a Γ-ring M onto a Γ-ring $M^{'}$. If A is an f-invariant intuitionistic fuzzy primary ideal of M, then $f(A)$ is an intuitionistic fuzzy primary ideal of $M^{'}$.*

Proof. Let A be an f-invariant intuitionistic fuzzy primary ideal of M. Then $f(A)$ is IFI of $M^{'}$ (by Lemma (5.4)). Since A is intuitionistic fuzzy primary ideal of M, then $A(0_M) = (1,0)$, A_* is a primary ideal of M and $A(M) = \{(1,0),(t,s)\}$, where $t,s \in [0,1)$ such that $t+s \leq 1$. From the proof of the Lemma (5.3) we have $f(A)(0_{M'}) = A(0_M) = (1,0)$. Also $(f(A))_* = f(A_*)$ is a primary ideal of $M^{'}$. Now we prove $f(A(M)) = \{(1,0),(t,s)\}$ where $t,s \in [0,1)$ such that $t+s \leq 1$.

Let $x \in M$ be such that $\mu_A(x) = t, \nu_A(x) = s$. Then $\mu_{f(A)}(f(x)) = Sup\{\mu_A(z) : f(z) = f(x)\} = \mu_A(x) = t$ and $\nu_{f(A)}(f(x)) = Inf\{\nu_A(z) : f(z) = f(x)\} = \nu_A(x) = s$. As A is f-invariant also $f(A)(0_{M'}) = (1,0)$. So $f(A(M)) = \{(1,0),(t,s)\}$. By Theorem (4.9) it follows that $f(A)$ is an intuitionistic fuzzy primary ideal of M □

Example 5.3. Let $M = \Gamma = Z$, the ring of integers and f be a Γ-homomorphism from M to M defined by $f(x) = 2x$, and let

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in 3Z \\ 0.2, & \text{if otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in 3Z \\ 0.7, & \text{otherwise.} \end{cases}$$

be an intuitionistic fuzzy primary ideal of M. Then
$f(A)(0) = (Sup\{\mu_A(x) : f(n) = 0\}, Inf\{\nu_A(x) : f(n) = 0\}) = (\mu_A(0), \nu_A(0)) = (1,0)$ and $f(A)(1) = (Sup\{\mu_A(x) : f(n) = 1\}, Inf\{\nu_A(x) : f(n) = 1\}) = (0,1)$ [As $f^{-1}(1) = \emptyset$]. Similarly, we can find that $f(A)(3) = f(A)(5) = (0,1)$ and $f(A)(2) = f(A)(4) = (0.2, 0.7)$ and so on we get

$$\mu_{f(A)}(x) = \begin{cases} 1, & \text{if } x \in 6Z \\ 0.2, & \text{if } x \in 2Z - 6Z \\ 0, & \text{if } x \in Z - 2Z \end{cases} ; \quad \nu_{f(A)}(x) = \begin{cases} 0, & \text{if } x \in 6Z \\ 0.7, & \text{if } x \in 2Z - 6Z \\ 1, & \text{if } x \in Z - 2Z, \end{cases}$$

is not an intuitionistic primary fuzzy ideal of M (As $|Img(A)| = 3 \neq 2$). This shows that the assumption that f be an epimorphism in Theorem (5.13) cannot be dropped.

Lemma 5.5. *Let f be a homomorphism of a Γ-ring M onto a Γ-ring $M^{'}$. If B is an IFI of $M^{'}$, then $(f^{-1}(B))_* = f^{-1}(B_*)$.*

Proof. Let $y \in (f^{-1}(B))_* \Leftrightarrow (f^{-1}(B))(y) = (f^{-1}(B))(0_M)$
$\Leftrightarrow B(f(y)) = B(f(0_M)) = B(0_{M'}) = (1,0)$
$\Leftrightarrow f(y) \in B_* \Leftrightarrow y \in f^{-1}(B_*)$.

Hence $(f^{-1}(B))_* = f^{-1}(B_*)$. □

Lemma 5.6. ([14, 15]) *Let f be a homomorphism of a Γ-ring M onto a Γ-ring $M^{'}$. If B is an IFI of $M^{'}$, then $f^{-1}(B)$ is an IFI of M.*

Theorem 5.14. *Let f be a homomorphism of a Γ-ring M onto a Γ-ring $M^{'}$. If B is an intuitionistic fuzzy primary ideal of $M^{'}$, then $f^{-1}(B)$ is an intuitionistic fuzzy primary ideal of M.*

Proof. By lemma (5.6) $f^{-1}(B)$ is an IFI of M. Also $(f^{-1}(B)(0_M) = B(f(0_M)) = B(0_{M'}) = (1,0)$. As B is an intuitionistic fuzzy primary ideal of N. Now $B(M^{'}) = \{(1,0),(t,s)\}$, where $t,s \in [0,1)$ such that $t+s \leq 1$. Let $y \in M^{'}$ be such that $\mu_B(y) = t, \nu_B(y) = s$, then there exists $x \in M$ such that $f(x) = y$. Now $f^{-1}(B)(x) = B(f(x)) = (t,s)$. Thus $f^{-1}(B(M)) = \{(1,0),(t,s)\}$, where $t,s \in [0,1)$ such that $t+s \leq 1$. Also by lemma (5.5) we have $(f^{-1}(B))_* = f^{-1}(B_*)$ is a primary ideal of M. Hence by Theorem (4.9) $f^{-1}(B)$ is an intuitionistic fuzzy primary ideal of M. □

Theorem 5.15. *Let f be a homomorphism of a Γ-ring M onto a Γ-ring $M^{'}$. If A is an IFI of M such that A is constant on $Kerf$, then $\sqrt{f(A)} = f(\sqrt{A})$.*

Proof. Clearly, $\sqrt{f(A)}$ and $f(\sqrt{A})$ are IFIs of $M^{'}$. Let $y \in M^{'}, \gamma \in \Gamma$ be any element, as f is onto so there exist some $x \in M$ such that $f(x) = y$. Now $f((x\gamma)^m x) = (y\gamma)^m y$.

$$\begin{aligned}
\mu_{f(\sqrt{A})}(y) &= Sup\{\mu_{\sqrt{A}}(x) : x \in f^{-1}(y)\} \\
&= Sup\{\vee\{\mu_A((x\gamma)^m x) : m > 0\} : x \in f^{-1}(y)\} \\
&= \vee\{Sup\{\mu_A((x\gamma)^m x) : x \in f^{-1}(y)\} : m > 0\} \\
&\leq \vee\{Sup\{\mu_A((x\gamma)^m x) : (x\gamma)^m x \in f^{-1}((y\gamma)^m y)\} : m > 0\} \\
&= \vee\{Sup\{\mu_A((z\gamma)^m z) : (z\gamma)^m z \in f^{-1}((y\gamma)^m y)\} : m > 0\} \\
&= \vee\{\mu_{f(A)}((y\gamma)^m y) : m > 0\} \\
&= \mu_{\sqrt{f(A)}}(y).
\end{aligned}$$

Similarly, we can show that $\nu_{f(\sqrt{A})}(y) \geq \nu_{\sqrt{f(A)}}(y)$. Thus $f(\sqrt{A}) \subseteq \sqrt{f(A)}$.

Further, if A is constant on $Kerf$ and $x_0 \in f^{-1}(y)$ be a fixed element of M. Then $\mu_A((x\gamma)^m x) = \mu_A((x_0\gamma)^m x_0)$ and $\nu_A((x\gamma)^m x) = \nu_A((x_0\gamma)^m x_0)$ for all $x \in f^{-1}(y), \gamma \in \Gamma$, $m \in \mathbf{N}$ and $\mu_A(x) = \mu_A((x_0\gamma)^m x_0)$ and $\nu_A(x) = \nu_A((x_0\gamma)^m x_0)$ for all $x \in f^{-1}(y), \gamma \in \Gamma$, $m \in \mathbf{N}$. Hence

$$\begin{aligned}
\mu_{\sqrt{f(A)}}(y) &= \vee\{\mu_{f(A)}((y\gamma)^m y) : m > 0\} \\
&= \vee\{Sup\{\mu_A((x\gamma)^m x) : (x\gamma)^m x \in f^{-1}((y\gamma)^m y)\} : m > 0\} \\
&= Sup\{\vee\{\mu_A((x\gamma)^m x) : m > 0\} : (x\gamma)^m x \in f^{-1}((y\gamma)^m y)\} \\
&\geq Sup\{\vee\{\mu_A((x_0\gamma)^m x_0) : m > 0\} : x \in f^{-1}(y)\} \\
&= Sup\{\vee\{\mu_A((x\gamma)^m x) : m > 0\} : x \in f^{-1}(y)\} \\
&= Sup\{\mu_{\sqrt{A}}(x) : x \in f^{-1}(y)\} \\
&= \mu_{f(\sqrt{A})}(y).
\end{aligned}$$

Similarly, we can show that $\nu_{\sqrt{f(A)}}(y) \leq \nu_{f(\sqrt{A})}(y)$. Thus $\sqrt{f(A)} \subseteq f(\sqrt{A})$.
Hence $\sqrt{f(A)} = f(\sqrt{A})$ proved. □

Theorem 5.16. *Let f be a homomorphism of a Γ-ring M into a Γ-ring $M^{'}$. If B is an IFI of M , then $\sqrt{f^{-1}(B)} = f^{-1}(\sqrt{B})$.*

Proof. Clearly, $\sqrt{f^{-1}(B)}$ and $f^{-1}(\sqrt{B})$ are IFIs of M. Let $x \in M, \gamma \in \Gamma$ be any element, then

$$\begin{aligned}
\mu_{f^{-1}(\sqrt{B})}(x) &= \mu_{\sqrt{B}}(f(x)) = \vee\{\mu_B((f(x)\gamma)^m f(x)) : m > 0\} \\
&= \vee\{\mu_B(f((x\gamma)^m x)) : m > 0\} \\
&= \vee\{\mu_{f^{-1}(B)}((x\gamma)^m x) : m > 0\} \\
&= \mu_{\sqrt{f^{-1}(B)}}(x).
\end{aligned}$$

Similarly, we can show that $\nu_{f^{-1}(\sqrt{B})}(x) = \nu_{\sqrt{f^{-1}(B)}}(x)$, for all $x \in M, \gamma \in \Gamma$.
Hence $\sqrt{f^{-1}(B)} = f^{-1}(\sqrt{B})$. □

6. Conclusion

In this paper we have explored the fundamental ideas of intuitionistic fuzzy primary ideal and intuitionistic fuzzy prime radical in Γ-ring M. We proved that intuitionistic fuzzy primary ideal of Γ-ring is a two valued intuitionistic fuzzy set and base set is a primary ideal (The base set of intuitionistic fuzzy set Q is defined as the set $\{x \in M : \mu_Q(x) = 1, \nu_Q(x) = 0\}$). We also defined the notion of intuitionistic fuzzy prime radical in Γ-ring M and have shown that the intuitionistic fuzzy prime radical of an intuitionistic fuzzy primary ideal is an intuitionistic fuzzy prime ideal. We also investigate the homomorphic behaviour of intuitionistic fuzzy primary ideal as well as intuitionistic fuzzy prime radical in Γ-ring. The results obtained in this paper are more advanced than that in the classical ring under intuitionistic fuzzy environments. Moreover these results not only enrich the previous studies but also lay down the foundation of more powerful result which is under study for future work as the decomposition of ideal in terms of primary ideals in intuitionistic fuzzy environment (A generalization of prime factorization of number in number theory).

Acknowledgements. We thank the anonymous reviewer(s) for the constructive and insightful comments, which have helped us to substantially improve our manuscript. The second author takes this opportunity to express her gratitude to Lovely Professional University, Phagwara, for giving the platform for the research work to be conducted.

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