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Projective Dimension of Some Graphs

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ABSTRACT. In this paper exact values for the projective dimension of edge ideals associated to some star related graphs and product graphs $G \square P_2$, when $G = C_n, K_n$ and upper bounds for the projective dimension when $G = P_n, W_n$, are obtained. We have proved that $pd(C_{n+1} \square P_2) = 2(n - \lfloor \frac{n}{4} \rfloor)$, $pd(K_n \square P_2) = 2n - 2$ and $pd(P_{n+1} \square P_2) \leq n + 3 + \lfloor \frac{n-3}{2} \rfloor$, $pd(W_n \square P_2) \leq n + 1 + \lceil \frac{2n-1}{3} \rceil$. These values are functions of the number of vertices in the corresponding graphs.

1. INTRODUCTION

In this paper all graphs are finite and simple. Let $V(G)$ denote the vertex set of a graph G and let (u, v) denote an edge of G with end points u and v. For $v \in V(G)$, let $N(v)$ denote the set of all vertices adjacent to v, called the neighbor set of G and $N[v] = N(v) \cup \{v\}$. Let S_n denote the star on $n+1$ vertices $\{u_0, u_1, \ldots, u_n\}$ where u_0 is adjacent to all other vertices. The wheel graph W_n on $n+1$ vertices is a graph obtained by connecting all n vertices of the cycle C_n to an $n+1$-th vertex (called the hub). The edges connecting the hub and the vertices of C_n are called spokes.

The Cartesian product of two graphs G and H is denoted as $G \square H$. It is a graph with vertex set $V(G) \times V(H) = \{(g, h) | g \in G, h \in H\}$ and two vertices (g, h) and (g', h') are adjacent if and only if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$.

Let G is a graph with vertex set $V = \{x_1, x_2, \ldots, x_n\}$ and let $S = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring over the field K. The edge ideal of G is the monomial ideal $I(G) \subseteq S$ generated by $\{x_i x_j : (x_i, x_j) \text{ is an edge of } G\}$. The edge ring of G is the quotient ring $S/I(G)$ [4]. Villarreal introduced the concept of edge ideal of a graph in [6].

Let $U = \{x_1, x_2, \ldots, x_n\}$ be a finite set. A simplicial complex Δ over U is a subset of the powerset U with the property that $\{v_1\}, \{v_2\}, \ldots, \{v_n\}$ belongs to Δ and if $F \in \Delta$ and $J \subseteq F$, then $J \in \Delta$. The elements of Δ are called faces and dimension of a face, $dim\ F = |F| - 1$. The dimension of the simplicial complex Δ, $dim\ \Delta$ is the maximum of the dimensions of its faces [4]. Associated to the edge ideal $I(G)$ of G is its independence complex, $ind(G)$, the simplicial complex on the vertex set V of G which has faces $\{\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\} | \ no\ \{x_{i_j}, x_{i_k}\}\ is\ an\ edge\ of\ G\}$ [3].

The Betti number of an ideal can be defined in terms of its $Stanley - Reisner\ complex$ using the Hochster's Formula.

Theorem 1.1. [3] *Let Δ be the Stanley-Reisner complex of a squarefree monomial ideal $I \subseteq S$ and let $\beta_{i,m}(I)$, where m is a squarefree monomial of degree greater than or equal to i, be the multigraded betti number of I. Then $\beta_{i-1,m}(I) = dim_K \tilde{H}_{deg\ m-i-1}(\Delta_m; K)$, where Δ_m is the subcomplex of Δ consisting of those faces whose vertices correspond to variables occuring in m and $\tilde{H}_k(\Delta)$ is the associated homology group of Δ.*

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For a graph G with vertex set $V = \{x_1, x_2, \ldots, x_n\}$, the projective dimension of G denoted by $pd(S/I(G)$ is defined as the least integer i such that

$$\tilde{H}_{|W|-i-j-1}(ind(G[W])) = 0$$

for all $j > 0$ and $W \subseteq V$, where $G[W]$ is the subgraph of G induced by W [3].

The study of edge ideals and the invariants associated to it connects three branches of mathematics - commutative algebra, graph theory and combinatorial topology. Projective dimension of the ring $S/I(G)$ is one of the central invariants associated to $I(G)$. Finding connections between algebraic properties of an edge ideal and this invariant is interesting.

The properties and bounds of projective dimension for various classes of graphs are studied in [1, 2, 3, 4, 5, 7]. For a graph G, $pd(G)$ denotes $pd(S/I(G))$. The following theorems give some combinatorially constructed bounds for projective dimension and we use these theorems to prove the main results of this paper.

Theorem 1.2. [4] *If G is a graph such that its complement, G^c, is disconnected, then $pd(G) = |V(G)| - 1$.*

Theorem 1.3. [4] *If G is the disjoint union of two graphs G_1 and G_2, then $pd(G) = pd(G_1) + pd(G_2)$.*

Theorem 1.4. [4] *Let T be a forest and v be a vertex of T which has all but at most one of its neighbors of degree 1. If $v_1, v_2, \ldots, v_n$ denote the neighbors of v such that $v_1, v_2, \ldots, v_{n-1}$ all have degree 1, then $pd(T) = max\{pd(T - v_1),\ pd(T - \{v, v_1, v_2, \ldots, v_n\}) + n\}$.*

Theorem 1.5. [3] *Let $x \in V(G)$. Then, $pd(G) \leq max\{pd(G - \{x\}) + 1,\ pd(G - N[x]) + deg(x)\}$.*

Theorem 1.6. [1] *Let x be a vertex of a graph G. Then*

(1) $pd(G) = pd(G - \{x\}) + 1$ or $pd(G - N[x]) + deg(x)$

(2) If $pd(G - N[x]) + deg(x) \geq pd(G - \{x\}) + 1$, then $pd(G) = pd(G - N[x]) + deg(x)$.

Theorem 1.7. [3] *If P_n denotes a path on n vertices and C_n denotes a cycle on n vertices, then $pd(P_n) = \lfloor \frac{2n}{3} \rfloor$ and $pd(C_n) = \lceil \frac{2n-1}{3} \rceil$.*

In this paper we have found the projective dimension of some star related graphs and the product graphs $G \square P_2$, when $G = P_n,\ C_n,\ K_n,\ W_n$. Throughout the paper if G is a graph containing isolated vertices we discard those isolated vertices.

2. Main results

Theorem 2.8. *Let G denote a graph obtained from S_n by adding i_j pendant edges to the vertex u_j for $0 \leq j \leq n$ such that $1 \leq i_0 \leq i_1 \leq \ldots \leq i_n$. Then, $pd(G) = \sum_{j=1}^{n} i_j + 1$.*

Proof. Let $\{u_0, u_1, \ldots, u_n\}$ be the vertex set of S_n and let $\{u_{j1}, u_{j2}, \ldots, u_{ji_j}\}$ be the leaves adjacent with vertex u_j for $0 \leq j \leq n$. In Theorem 1.4, taking $v = u_n$ we get,

$$pd(G) = max\{pd(G - \{u_{n1}\}), pd(G - \{u_{n1}, u_{n2}, \ldots, u_{ni_n}, u_n, u_0\}) + i_n + 1\}.$$

Since $pd(S_i) = i$, $pd(G - \{u_{n1}, u_{n2}, \ldots, u_{ni_n}, u_n, u_0\}) = \sum_{j=1}^{n-1} i_j$. So

$$pd(G) = max\{pd(G - \{u_{n1}\}), \sum_{j=1}^{n} i_j + 1\}.$$

Applying Theorem 1.4 to the graph $G - \{u_{n1}\}$ with $v_1 = u_{n2}$,

$$pd(G - \{u_{n1}\}) = max\{pd(G - \{u_{n1}, u_{n2}\}),\ pd(G - \{u_{n1}, u_{n2}, u_n, u_0\}) + 1\}.$$

Now, $pd(G-\{u_{n1}, u_{n2}, u_n, u_0\}) = \sum_{j=1}^{n-1} i_j$. Thus $pd(G) = max\{pd(G-\{u_{n1}, u_{n2}\}), \sum_{j=1}^{n} i_j+1\}$. Repeatedly applying Theorem 1.4 to $G - \{u_{n1}, u_{n2}\}, G - \{u_{n1}, u_{n2}, u_{n3}\}, \ldots$, we get

$$pd(G) = max\{pd(G - \{u_{n1}, u_{n2}, \ldots, u_{ni_n}\}), \sum_{j=1}^{n} i_j + 1\}.$$

Let $G_1 = G - \{u_{n1}, u_{n2}, \ldots, u_{ni_n}\}$. Then, from Theorem 1.5, we get

$$pd(G_1) \leq max\{pd(G_1 - N[u_0]) + deg(u_0), pd(G_1 - \{u_0\}) + 1\}.$$

Now, $pd(G_1 - N[u_0]) = 0$, $deg(u_0) = n + i_0$, $pd(G_1 - \{u_0\}) = \sum_{j=1}^{n-1} i_j$. Thus

$$pd(G_1) \leq max\Big\{n + i_0, \sum_{j=1}^{n-1} i_j + 1\Big\} \leq \sum_{j=1}^{n} i_j + 1.$$

Hence, $pd(G) = \sum_{j=1}^{n} i_j + 1$. □

Theorem 2.9. *For $n \geq 2$, let G denote the graph obtained by identifying the vertices $u_1, u_2, \ldots, u_n$ of S_n with a vertex of each of the cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_n}$ respectively. Then, $pd(G) = \sum_{j=1}^{n} \left\lfloor \frac{2(i_j-1)}{3} \right\rfloor + n$.*

Proof. Let $x = u_0$. Then, $G - \{x\}$ is the graph $\bigcup_{j=1}^{n} C_{i_j}$. So by Theorem 1.3,

$$pd(G - \{x\}) + 1 = \sum_{j=1}^{n} pd(C_{i_j}) + 1 = \sum_{j=1}^{n} \left\lceil \frac{2i_j - 1}{3} \right\rceil + 1.$$

Also, $G - N[x]$ is the graph $\bigcup_{j=1}^{n} P_{i_j-1}$. So,

$$pd(G - N[x]) + deg(x) = \sum_{j=1}^{n} P_{i_j-1} + n = \sum_{j=1}^{n} \left\lfloor \frac{2(i_j - 1)}{3} \right\rfloor + n$$

Now, $\sum_{j=1}^{n} \left\lceil \frac{2i_j-1}{3} \right\rceil + 1 \leq \sum_{j=1}^{n} \left\lfloor \frac{2(i_j-1)}{3} \right\rfloor + n$. So by Theorem 1.6, $pd(G) = \sum_{j=1}^{n} \left\lfloor \frac{2(i_j-1)}{3} \right\rfloor + n$. □

Consider the graph $P_{n+1} \,\square\, P_2$ and for any n, label its vertices as in Figure 1.

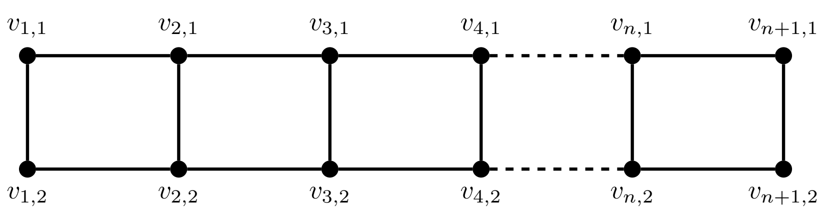


FIGURE 1. $P_{n+1} \,\square\, P_2$

Lemma 2.1. *For $n \in \mathbb{N}$, let A_n denote the graph obtained by adding a pendant edge to the vertex $v_{1,2}$ of $P_{n+1} \,\square\, P_2$. Then $pd(A_n) = n + 2 + \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Consider the graph A_n and let $x = v_{1,2}$. Then $deg(x) = 3$. Proof is by induction on n. When $n = 1$, $A_1 - \{x\} = P_3$ and $A_1 - N[x] = K_1$. So $pd(A_1 - \{x\}) + 1 = 3$ and $pd(A_1 - N[x]) + deg(x) = 3$. By Theorem 1.6, $pd(A_1) = 3 = 1 + 2 + \lfloor \frac{1}{2} \rfloor$. Hence the result holds for $n = 1$.

When $n = 2$, $pd(A_2 - \{x\}) + 1 = pd(A_1) + 1 = 4$ and $pd(A_2 - N[x]) + deg(x) = pd(P_3) + 3 = 5$. By Theorem [1.6], $pd(A_2) = 5 = 2 + 2 + \lfloor \frac{2}{2} \rfloor$. Hence the result holds for $n = 2$.

Suppose that the result holds for all $k \leq n$, $n \geq 2$. Consider the graph A_{n+1}.

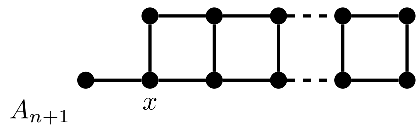


FIGURE 2

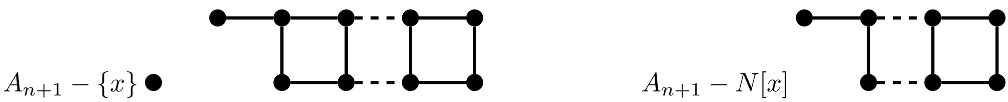


FIGURE 3

Now, $A_{n+1} - \{x\} = A_n$ and $A_{n+1} - N[x] = A_{n-1}$. So by the induction hypothesis,

$$\begin{aligned} pd(A_{n+1} - \{x\}) &= pd(A_n) \\ &= n + 2 + \left\lfloor \frac{n}{2} \right\rfloor \\ pd(A_{n+1} - N[x]) &= pd(A_{n-1}) \\ &= (n-1) + 2 + \left\lfloor \frac{n-1}{2} \right\rfloor. \\ pd(A_{n+1} - N[x]) + deg(x) &= (n-1) + 2 + \left\lfloor \frac{n-1}{2} \right\rfloor + 3 \\ &= n + 3 + (\left\lfloor \frac{n-1}{2} \right\rfloor + 1) \\ &\geq n + 3 + \left\lfloor \frac{n}{2} \right\rfloor \\ &= n + 2 + \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ &= pd(A_{n+1} - \{x\}) + 1. \end{aligned}$$

Hence, by Theorem 1.6,

$$\begin{aligned} pd(A_{n+1}) &= (n-1) + 2 + \left\lfloor \frac{n-1}{2} \right\rfloor + 3 \\ &= (n-1) + 2 + (\left\lfloor \frac{n+1}{2} \right\rfloor - 1) + 3 \\ &= (n+1) + 2 + \left\lfloor \frac{n+1}{2} \right\rfloor. \end{aligned}$$

Therefore, by principle of mathematical induction, $pd(A_n) = n + 2 + \lfloor \frac{n}{2} \rfloor$. □

Theorem 2.10. *For* $n \in \mathbb{N}$, *let* $B_n = P_{n+1} \square P_2$. *Then* $pd(B_n) \le n + 3 + \lfloor \frac{n-3}{2} \rfloor$.

Proof. Let $x = v_{2,2}$ and $y = v_{3,1}$. Then $deg(x) = deg(y) = 3$. B_1 is the cycle on four vertices. So by Theorem 1.7, $pd(B_1) = 3 = 1 + 3 + \lfloor \frac{1-3}{2} \rfloor$. $B_2 - \{x\} = P_5$ and hence by Theorem 1.7 $pd(B_2 - \{x\}) + 1 = 3 + 1 = 4$ and $pd(B_2 - N[x]) + deg(x) = 0 + 3 = 3$. Thus by Theorem 1.5, $pd(B_2) \le 4 = 2 + 3 + \lfloor \frac{2-3}{2} \rfloor$.

Now consider B_3. Then $pd(B_3 - N[x]) = pd(P_3) = 2$. To find $pd(B_3 - \{x\})$, consider the graphs $B_3 - \{x, y\} = 2P_3$ and $B_3 - \{x\}) - N[y] = P_2$. By Theorem 1.3, $pd(B_3 - \{x, y\} = 4$ and by Theorem 1.7, $pd(B_3 - \{x\}) - N[y]) = 1$. So $pd(B_3 - \{x\}) \le \max\{pd(B_3 - \{x, y\}) + 1, pd((B_3 - \{x\}) - N[y]) + deg(y)\} = 5$.
Hence $pd(B_3) \le \max\{pd(B_3 - \{x\}) + 1, pd(B_3 - N[x]) + deg(x)\} = 6 = 3 + 3 + \lfloor \frac{3-3}{2} \rfloor$.

Now, suppose $n \ge 4$. Consider the graphs B_n, $B_n - \{x\}$, $B_n - N[x]$ shown in Figure 4.

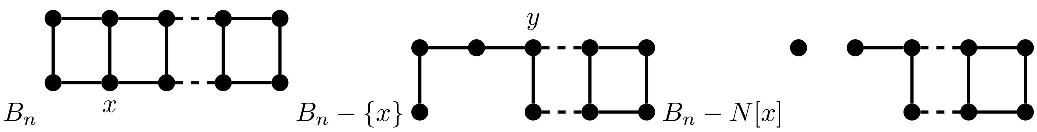


FIGURE 4

By Lemma 2.1, $pd(B_n - N[x]) = pd(A_{n-3}) \le (n-3) + 2 + \lfloor \frac{n-3}{2} \rfloor$. Now to find $pd(B_n - \{x\})$, consider $B_n - \{x, y\}$ and $(B_n - \{x\}) - N[y]$ as in Figure 5.

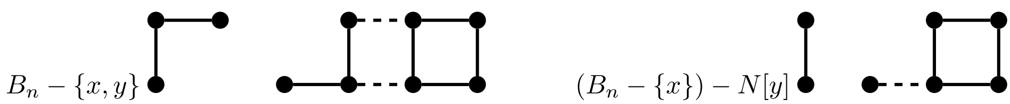


FIGURE 5

$$\begin{aligned} pd(B_n - \{x, y\}) &= pd(P_3) + pd(A_{n-3}) \\ &= 2 + (n-3) + 2 + \left\lfloor \frac{n-3}{2} \right\rfloor \\ &= n + 1 + \left\lfloor \frac{n-3}{2} \right\rfloor \\ pd((B_n - \{x\}) - N[y]) &= pd(P_2) + pd(A_{n-4}) \\ &= 1 + (n-4) + 2 + \left\lfloor \frac{n-4}{2} \right\rfloor \\ &= n - 1 + \left\lfloor \frac{n-4}{2} \right\rfloor . \end{aligned}$$

So $\max\{pd(B_n - \{x,y\}) + 1, pd((B_n - \{x\}) - N[y]) + deg(y)\} = n + 2 + \left\lfloor \frac{n-3}{2} \right\rfloor$.
Thus, from Theorem 1.5, $pd(B_n - \{x\}) \leq n + 2 + \left\lfloor \frac{n-3}{2} \right\rfloor$ and

$$\begin{aligned} pd(B_n) &\leq \max\left\{pd(B_n - \{x\}) + 1, pd(B_n - N[x]) + deg(x)\right\} \\ &\leq \max\left\{n + 3 + \left\lfloor \frac{n-3}{2} \right\rfloor, n + 2 + \left\lfloor \frac{n-3}{2} \right\rfloor\right\} \\ &= n + 3 + \left\lfloor \frac{n-3}{2} \right\rfloor. \end{aligned}$$

Thus, $pd(P_{n+1} \,\square\, P_2) \leq n + 3 + \left\lfloor \frac{n-3}{2} \right\rfloor$. □

Lemma 2.2. *For $n \in \mathbb{N}$, let F_n denote the graph obtained by adding a pendant edge to the vertices $v_{1,2}$ and $v_{n+1,2}$ of $P_{n+1} \,\square\, P_2$. Then, $pd(F_n) = 2n + 1 - 2\left\lfloor \frac{n-1}{4} \right\rfloor$.*

Proof. The proof is by the principle of mathematical induction on n and for that Theorem 1.6 is used. Let $x = v_{1,2}$, $y = v_{2,1}$ and $\alpha(n) = 2n + 1 - 2\left\lfloor \frac{n-1}{4} \right\rfloor$. Then $deg(x) = deg(y) = 3$. Consider F_1. Then $pd(F_1 - \{x\}) = pd(P_4) = 2$ and $pd(F_1 - N[x]) = 0$, since it consists of two isolated vertices. So, by Theorem 1.6, $pd(F_1) = 3 = \alpha(1)$. Thus the result holds for $n = 1$.
For $n = 2$, $F_2 - N[x] = P_4$. To find $F_2 - \{x\}$, consider $F_2 - \{x,y\} = S_3$ and $F_2 - \{x\} - N[y] = P_2$. Then by Theorem 1.6, $pd(F_2 - \{x\}) = 4$ and hence $pd(F_2) = 5 = \alpha(2)$. So the result holds for $n = 2$.
Assume that the result holds for all $k \leq n$, $n \geq 3$. Consider the graphs F_{n+1}, $F_{n+1} - \{x\}$, $F_{n+1} - N[x]$, $F_{n+1} - \{x,y\}$, and $F_{n+1} - N[x] - N[y]$ as in the figures below.

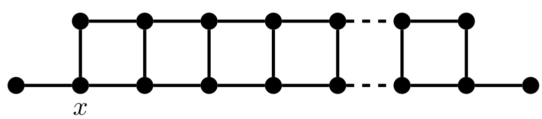


FIGURE 6. F_{n+1}

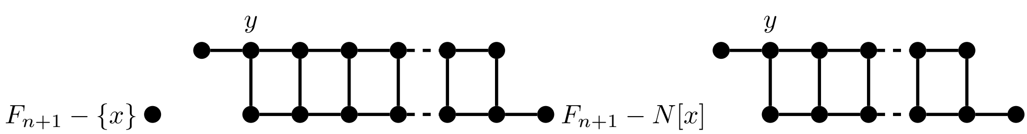


FIGURE 7

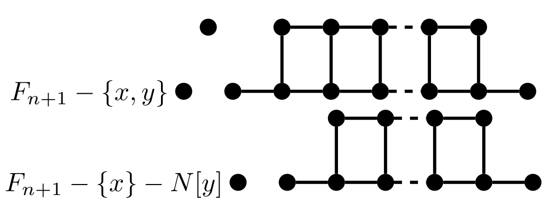


FIGURE 8

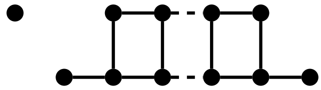
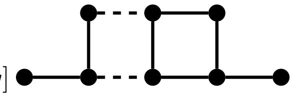
$F_{n+1} - N[x] - \{y\}$  $F_{n+1} - N[x] - N[y]$ 

FIGURE 9

To find $pd(F_{n+1})$, $pd(F_{n+1} - \{x\})$ and $pd(F_{n+1} - N[x])$ must be known.
Now,

$$pd(F_{n+1} - \{x, y\}) = pd(F_{n-1}) = 2n - 1 - 2\left\lfloor \frac{n-2}{4} \right\rfloor$$
$$pd(F_{n+1} - \{x\} - N[y]) = pd(F_{n-2}) = 2n - 3 - 2\left\lfloor \frac{n-3}{4} \right\rfloor$$

Thus, by Theorem 1.6, $pd(F_{n+1} - \{x\}) = 2n - 2\left\lfloor \frac{n-3}{4} \right\rfloor$.
Also,

$$pd(F_{n+1} - N[x] - \{y\}) = pd(F_{n-2}) = 2n - 3 - 2\left\lfloor \frac{n-3}{4} \right\rfloor$$
$$pd(F_{n+1} - N[x] - N[y]) = pd(F_{n-3}) = 2n - 5 - 2\left\lfloor \frac{n-4}{4} \right\rfloor.$$

Thus, $pd(F_{n+1} - N[x]) = 2n - 2 - 2\left\lfloor \frac{n-4}{4} \right\rfloor$. Hence, $pd(F_{n+1}) = 2n + 1 - 2\left\lfloor \frac{n-4}{4} \right\rfloor = \alpha(n+1)$.
So, by induction, the result follows. □

Theorem 2.11. *For $n \geq 2$, let $J_n = C_{n+1} \square P_2$. Then, $pd(J_n) = 2\big(n - \left\lfloor \frac{n}{4} \right\rfloor\big)$.*

Proof. The vertices of J_n are labelled as in B_n. Consider J_2 and let $x = v_{2,1}$, $y = v_{1,2}$. Then $J_2 - \{x, y\} = P_4$ and $J_2 - \{x\} - N[y]$ is an isolated vertex. So $pd(J_2 - \{x, y\}) = 2$ and $pd(J_2 - \{x\} - N[y]) = 0$. Then from Theorem 1.6, $pd(J_2 - \{x\}) = 3$. Now, $pd(J_2 - N[x]) = pd(P_2) = 1$. Hence $pd(J_2) = 4 = 2\big(2 - \left\lfloor \frac{2}{4} \right\rfloor\big)$.

For $n \geq 3$, let $x = v_{1,1}$ and $y = v_{1,2}$. J_n, $J_n - \{x\}$, $J_n - N[x]$, $J_n - \{x, y\}$ and $J_n - \{x\} - N[y]$ are the graphs shown in the Figures 10,11,12.

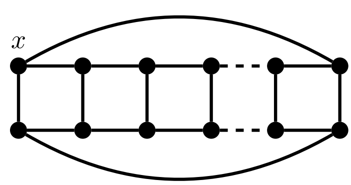
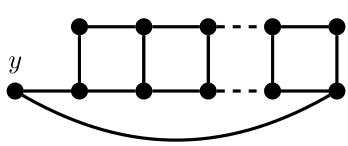


FIGURE 10. J_n

$J_n - \{x\}$ 

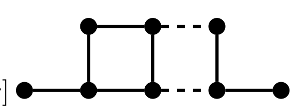
$J_n - N[x]$ 

FIGURE 11

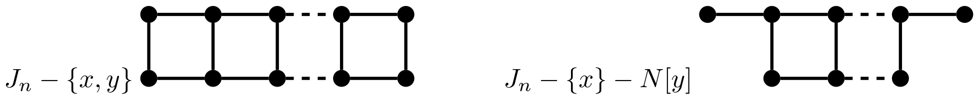


FIGURE 12

Then,

$$pd(J_n - \{x, y\}) = pd(P_2 \times P_n) \leq n + \left\lfloor \frac{n}{2} \right\rfloor$$
$$pd(J_n - \{x\} - N[y]) = pd(F_{n-3}) = 2n - 3 - 2\left\lfloor \frac{n}{4} \right\rfloor$$

So, $pd(J_n - \{x\}) = 2n - 1 - 2\left\lfloor \frac{n}{4} \right\rfloor$. Also, $pd(J_n - N[x]) = pd(F_{n-3}) = 2n - 3 - 2\left\lfloor \frac{n}{4} \right\rfloor$. Hence, $pd(J_n) = 2\big(n - \left\lfloor \frac{n}{4} \right\rfloor\big)$. □

Theorem 2.12. *For $n \geq 3$, $pd(K_n \,\square\, P_2) = 2n-2$ where K_n is the complete graph on n vertices.*

Proof. Let $G_0 = K_n \,\square\, P_2$, which consists of two complete graphs on n vertices that are connected to each other by n edges. Let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of the first and second complete graphs respectively and let $e_i = u_i v_i, 1 \leq i \leq n$ be the n edges connecting the two complete graphs with each other. Let G_0 be represented as $[K_n, K_n, n]$ where K_n's are the two complete graphs and n is the number of e_i's in G_0. By Theorem 1.2, $pd(K_n) = n - 1$. The theorem will be proved using Theorem 1.5 repeatedly.

Consider G_0 and the edge $e_1 = u_1 v_1$. To apply Theorem 1.5 we have to select a vertex $x \in V(G_0)$. If the number of e_i's in G_0 is even, $x = u_1$ or else $x = v_1$. Form the graphs $G_1 = G_0 - \{x\}$ and $H_1 = G_0 - N[x]$. Then $G_1 = [K_{n-1}, K_n, n-1]$ or $[K_n, K_{n-1}, n-1]$ according as number of e_i's in G_0 is even or odd and $H_1 = K_{n-1}$. Now choose the edge $e_2 = u_2 v_2$ in G_1 and let $x = u_2$ or v_2 depending on the number of e_i's in G_1 is even or odd. Then $G_2 = [K_{n-1}, K_{n-1}, n-2]$ and $H_2 = K_{n-2}$. Continuing the above procedure two sequences of graphs $G_1, G_2, \ldots,$ and $H_1, H_2, \ldots,$ are obtained. The vertex $'x'$ is selected according to the number of e_i's in the graph G_j. If the number of e_i's in G_j is even, then $x = u_{j+1}$, otherwise $x = v_{j+1}$. Also in $G_j = [K_r, K_s, n], 0 \leq j \leq n$, either $r - s = 0$ or $r - s = 1$ and to obtain G_{j+1} from G_j the vertex x is chosen from $K_{max(r,s)}$. Now $H_{j+1} = G_j - N[x]$ where $x \in G_j$ and from the above observation $x \in K_{max(r,s)}$. So in G_j, $deg(x) = max(r, s)$. After each deletion of the vertex $'x'$, number of e_i's reduce by one. Thus after n deletions all the e_i's will vanish. Then G_n will be the union of two complete graphs and H_n will be a complete graph.
If n is even,

$$G_n = \big[K_{\frac{n}{2}}, K_{\frac{n}{2}}, 0\big],\ H_n = K_{\frac{n}{2}-1}$$
$$G_{n-t} = \big[K_{\frac{n}{2}+\lfloor\frac{t}{2}\rfloor}, K_{\frac{n}{2}+\lceil\frac{t}{2}\rceil}, t\big],\ H_{n-t} = K_{\frac{n}{2}-1+\lceil\frac{t}{2}\rceil}, 1 \leq t \leq n-1$$

and if n is odd,

$$G_n = \big[K_{\frac{n+1}{2}}, K_{\frac{n-1}{2}}, 0\big],\ H_n = K_{\frac{n-1}{2}}$$
$$G_{n-t} = \big[K_{\frac{n+1}{2}+\lfloor\frac{t}{2}\rfloor}, K_{\frac{n-1}{2}+\lceil\frac{t}{2}\rceil}, t\big],\ H_{n-t} = K_{\frac{n-1}{2}+\lfloor\frac{t}{2}\rfloor}, 1 \leq t \leq n-1$$

By Theorem 1.3, depending on whether n is even or odd,

$$\begin{aligned} pd(G_n) &= pd(K_{\frac{n}{2}}) + pd(K_{\frac{n}{2}}) \text{ or } pd(K_{\frac{n-1}{2}}) + pd(K_{\frac{n+1}{2}}) \\ &= n-2 \\ pd(H_n) &= \frac{n-2}{2} - 1 \text{ or } \frac{n-1}{2} - 1 \end{aligned}$$

$G_{n-1} = \left[K_{\frac{n}{2}}, K_{\frac{n}{2}+1}, 1\right]$ *or* $\left[K_{\frac{n+1}{2}}, K_{\frac{n-1}{2}+1}, 1\right]$ and $H_{n-1} = K_{\frac{n}{2}}$ *or* $K_{\frac{n-1}{2}}$

$$\begin{aligned} pd(G_{n-1}) &\le max\{pd(G_n)+1,\ pd(H_n)+deg(x)\},\ x \in V(G_{n-1}) \\ &= max\{n-1, \frac{n-2}{2} + \frac{n}{2} + 1\} \text{ or } max\{n-1, \frac{n-1}{2} + \frac{n-1}{2} + 1\} \\ &= n-1 \end{aligned}$$

$G_{n-2} = \left[K_{\frac{n}{2}+1}, K_{\frac{n}{2}+1}, 2\right]$ *or* $\left[K_{\frac{n+1}{2}+1}, K_{\frac{n-1}{2}+1}, 2\right]$ and $H_{n-2} = K_{\frac{n}{2}}$ *or* $K_{\frac{n+1}{2}}$

$$\begin{aligned} pd(G_{n-2}) &\le max\{pd(G_{n-1})+1,\ pd(H_{n-1})+deg(x)\},\ x \in V(G_{n-2}) \\ &= max\{n, \frac{n}{2} - 1 + \frac{n}{2} + 1\} \text{ or } max\{n, \frac{n-1}{2} - 1 + \frac{n+1}{2} + 1\} \\ &= n \end{aligned}$$

Now from Theorem 1.5, $pd(G_{n-t}) \le max\{pd(G_{n-(t-1)})+1,\ pd(H_{n-(t-1)})+deg(x)\},\ x \in V(G_{n-t})$. First note that $pd(G_{n-(t-1)})+1 = pd(H_{n-(t-1)})+deg(x)$ for all $t, 1 \le t \le n$. So by Theorem 1.6, $pd(G_{n-t}) = pd(G_{n-(t-1)})+1$. Thus

$$\begin{aligned} pd(G_1) &= pd(G_{n-(n-1)}) \\ &= pd(G_{n-(n-2)}) + 1 \\ &= pd(G_{n-(n-3)}) + 2 \\ &= pd(G_{n-(n-4)}) + 3 \\ &\ \ \vdots \\ &= pd(G_{n-(n-n)}) + n - 1 \end{aligned}$$

So $pd(G_0) = pd(G_1) + 1 = pd(G_n) + n = n - 2 + n = 2n - 2$ □

Theorem 2.13. *For* $n \ge 3$, $pd(W_n \,\square\, P_2) \le n + 1 + \lceil \frac{2n-1}{3} \rceil$ *where* W_n *is the wheel graph on* $n+1$ *vertices.*

Proof. Let $G = W_n \,\square\, P_2$ consists of two wheel graphs W_n, connected by $n+1$ edges and it will be represented as $[W_n, W_n, n+1]$. Let v_1 and v_2 be the hub vertices in G. Let $x = v_1$ and $deg(x) = n+1$ in G. Then $G_1 = G - \{x\} = [C_n, W_n, n]$ and $H_1 = G - N[x] = C_n$. By Theorem 1.7, $pd(H_1) = \lceil \frac{2n-1}{3} \rceil$. Now let $x = v_2$ and $deg(x) = n$ in G_1. Then $G_2 = G_1 - \{x\} = [C_n, C_n, n] = C_n \,\square\, P_2$ and $H_2 = G_1 - N[x] = C_n$. $pd(H_2) = \lceil \frac{2n-1}{3} \rceil$. By

Theorem 1.5,

$$pd(G_1) \leq max\Big\{pd(G_2)+1, pd(H_2)+deg(v_2)\Big\}$$
$$= max\Big\{2n-1-2\left\lfloor\frac{n-1}{4}\right\rfloor, \left\lceil\frac{2n-1}{3}\right\rceil+n\Big\}$$
$$pd(G) \leq max\Big\{pd(G_1)+1, pd(H_1)+deg(v_1)\Big\}$$
$$\leq max\Big\{2n-2\left\lfloor\frac{n-1}{4}\right\rfloor, \left\lceil\frac{2n-1}{3}\right\rceil+n+1, \left\lceil\frac{2n-1}{3}\right\rceil+n+1\Big\}$$
$$= \left\lceil\frac{2n-1}{3}\right\rceil+n+1$$

□

3. Conclusion

We have obtained exact values for the projective dimension of edge ideals associated to some star related graphs and product graphs $G \,\square\, P_2$, when G is a cycle or a complete graph and upper bounds for the projective dimension when G is a path or wheel and these values are functions of the number of vertices in the corresponding graphs. One can try a similar study for Cartesian product of some other graphs and other graph products such as corona product and rooted product of graphs.

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