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The combinatorial nature of some trigonometric integrals

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ABSTRACT. The combinatorial nature of the trigonometric integrals (2.8) is discussed in connection to the partition of multisets with equal sums. Computational aspects are highlighted for special parameter values.

1. INTRODUCTION

At the District Round of the Romanian Mathematics Olympiad held on 26 March 2022, the third problem for the 12th grade was proposed by Vasile Pop, and stated the following: *For every positive integer $n \in \mathbb{N}^*$ define*

$$I_n = \int_0^\pi \cos(x) \cdot \cos(2x) \cdots \cos(nx) \,\mathrm{d}x. \tag{1.1}$$

Determine the values of n for which $I_n = 0$.

We mention that this problem is closely related to Problem A5 in the William Putnam Competition held in 1985, which asks to determine all positive integers $n \leq 10$ for which $I_n \neq 0$ (see [13]). In fact, this statement coincides with Problem $4.e)$, page 17 in [16].

The solution of the problem uses the trigonometric identity

$$\cos x_1 \cdot \cos x_2 \cdots \cos x_n = \frac{1}{2^n} \sum \cos(\pm x_1 \pm x_2 \pm \cdots \pm x_n), \tag{1.2}$$

where $x_1, \ldots, x_n \in \mathbb{R}^*$ and the sum is considered for all 2^n choices of signs $+$ and $-$. In fact, the first use of this identity (which can be proved by mathematical induction) which we could trace in the literature goes back to Problem 4 proposed at the Final Round of the Romanian Mathematics Olympiad in 1971 (see [10], p. 82, 311–312). This identity is also used in [1], [2] and [5], for solving problems related to the product of more derivatives.

Considering $x_k = kx, x \in \mathbb{R}^*$, $k = 1, 2, \ldots, n$ in the formula (1.2), we obtain

$$\cos x \cdot \cos 2x \cdots \cos nx = \frac{1}{2^n} \sum \cos(\pm 1 \pm 2 \pm \cdots \pm n)x,$$

where the sum is again considered over all 2^n choices of signs $+$ and $-$.

Integrating over the interval $[0, \pi]$, and since for $m \in \mathbb{Z}$ we have

$$\int_0^\pi \sin(mx) \,\mathrm{d}x = \begin{cases} \pi & \text{if } \ m = 0 \\ 0 & \text{if } \ m \neq 0, \end{cases}$$

we obtain the following formula

$$I_n = \frac{S(n)\pi}{2^n}, \tag{1.3}$$

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where the natural number $S(n)$ represents the number of choices of the signs $+$ and $-$ for which we have

$$\pm 1 \pm 2 \pm \cdots \pm n = 0. \tag{1.4}$$

The relation (1.4) has been called *the signum equation* of level n for the sequence $1,\ 2,\ 3,\dots$ by S. R. Finch [12]. A choice of the signs $+$ and $-$ for which (1.4) holds is called a solution of this equation. Hence, a first interpretation is that $S(n)$ represents the number of solutions of the signum equation (1.4).

On the other hand, it is clear that $S(n)$ is the number of ordered bipartitions (C_1, C_2) with equal sums if the set $\{1, 2, \dots, n\}$, that is which satisfy the relation

$$\sum_{a\in C_1} a = \sum_{b\in C_2} b = \frac{n(n+1)}{4}. \tag{1.5}$$

This interpretation of $S(n)$ is purely combinatorial.

In order to finalise the discussion related to the initial problem, we have $I_n = 0$ if and only if there are no bipartitions (C_1, C_2) satisfying (1.5), that is if and only if 4 does not divide $n(n+1)$, hence $n \equiv 1, 2 \pmod 4$.

There still remains to study the problem of the numbers $S(n)$ when $n \equiv 0, 3 \pmod 4$, that is the study of the sequence $(S(n))_{n\geq 1}$, indexed as $A063865$ in the On-Line Encyclopedia of Integer Sequences (OEIS) [15]. This is a nontrivial problem, since there are no known explicit or recurrence formulae for this sequence. A natural problem is to find an efficient method to compute its terms. Such a result can be obtained by noticing that the number of solutions for the signum equation (1.4), is the free term in the expression

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\cdots\left(x^n + \frac{1}{x^n}\right). \tag{1.6}$$

Using this method, the first 50 terms of the sequence are computed as

$0, 0, 2, 2, 0, 0, 8, 14, 0, 0, 70, 124, 0, 0, 722, 1314, 0, 0, 8220, 15272, 0, 0, 99820, 187692, 0, 0, 1265204, 2399784, 0, 0, 16547220, 31592878, 0, 0, 221653776, 425363952, 0, 0, 3025553180, 5830034720, 0, 0, 41931984034, 81072032060, 0, 0.$

By the same calculation, it was found that

$$S(100) = 173102400594872501663378632\,4,$$

far greater than $S(48) = 81072032060$, which suggests a sharp increase of the non-zero terms of the sequence. In the absence of an explicit calculation formula, one may attempt to establish an asymptotic formula for $S(n)$ when $n \equiv 0, 3 \pmod 4$. Following numerous simulations, in the paper [7], the following formula was proposed in 2002:

$$\lim_{n\to\infty} \frac{S(n)}{\frac{2^n}{n\sqrt{n}}} = \sqrt{\frac{6}{\pi}}. \tag{1.7}$$

The relation (1.7), initially known as the Andrica-Tomescu conjecture, was proved in 2013 by B. D. Sullivan [17]. The proof uses analytical tools and the integral formula (1.3).

Remark 1.1. By (1.7) and (1.3) one can deduce the asymptotic limit $\lim_{n\to\infty} n\sqrt{n}\cdot I_n = \sqrt{6\pi}$, which implies that $\lim_{n\to\infty} I_n = 0$. A direct proof of this weaker result is given below.

Indeed, since the function $\cos : \left[0, \frac{\pi}{2}\right] \to [0,1]$ is strictly decreasing, we have

$$\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \cos^n(x)\,\mathrm{d}x = 0. \tag{1.8}$$

An alternative proof of (1.8) follows from (3.11) (see also Problem $8.e$, page 19 in [16]).

On the other hand, as the function $x \mapsto |\cos x|$ has period π, the following holds

$$\begin{aligned}\int_0^{\pi} |\cos(kx)|^n \, \mathrm{d}x &= \frac{1}{k}\int_0^{k\pi} |\cos t|^n \, \mathrm{d}t \\ &= \frac{1}{k}\left(\int_0^{\pi} |\cos t|^n \, \mathrm{d}t + \cdots + \int_{(k-1)\pi}^{k\pi} |\cos t|^n \, \mathrm{d}t\right) = \int_0^{\pi} |\cos t|^n \, \mathrm{d}t.\end{aligned}$$

From the definition of I_n in (1.1), by the modulus and AM-GM inequalities, we have

$$\begin{aligned}\left|\int_0^{\pi} \cos(x)\cdot\cos(2x)\cdots\cos(nx)\,\mathrm{d}x\right| &\le \int_0^{\pi} |\cos(x)|\cdot|\cos(2x)|\cdots|\cos(nx)| \, \mathrm{d}x \\ &\le \int_0^{\pi} \frac{|\cos(x)|^n + |\cos(2x)|^n + \cdots + |\cos(nx)|^n}{n} \, \mathrm{d}x \\ &= \frac{1}{n}\cdot n\int_0^{\pi} |\cos(x)|^n \, \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} |\cos(x)|^n \, \mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} |\cos(x)|^n \, \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} |\cos(x)|^n \, \mathrm{d}x + \int_0^{\frac{\pi}{2}} |\sin(x)|^n \, \mathrm{d}x \\ &= 2\int_0^{\frac{\pi}{2}} \cos^n(x)\,\mathrm{d}x.\end{aligned}$$

Taking the limit in the above inequality, by (1.8) one deduces that $\lim_{n\to\infty} I_n = 0$.

2. The main result

Let $k \ge 2$ and $n_1, \ldots, n_k$ be natural numbers and let M be the multiset

$$M = \{\underbrace{\alpha_1, \ldots, \alpha_1}_{n_1 \text{ times}}, \ldots, \underbrace{\alpha_k, \ldots, \alpha_k}_{n_k \text{ times}}\},$$

where $\alpha_1, \ldots, \alpha_k$ are real numbers. Denote by $S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k)$ the number of ordered partitions of M having equal sums, i.e., the number of pairs (C_1, C_2) such that

(i) $C_1 \cup C_2 = M$;

(ii) $\sigma(C_1) = \sigma(C_2) = \frac{1}{2}\sum_{j=1}^{k} n_j\alpha_j$.

If we assume that M is a multiset of integers, inspiring from formula (1.6) it follows that $S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k)$ represents the free term (independent of z) in the expansion

$$F(z) = \left(z^{\alpha_1} + \frac{1}{z^{\alpha_1}}\right)^{n_1}\left(z^{\alpha_2} + \frac{1}{z^{\alpha_2}}\right)^{n_2}\cdots\left(z^{\alpha_k} + \frac{1}{z^{\alpha_k}}\right)^{n_k}.$$

Let us observe that we can write $F(z) = S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k) + \sum_{l\in\mathbb{Z}^*} c_l z^l$, for some constants $c_l \in \mathbb{Z}$. Setting $z = \cos t + i\sin t$, $t \in [0, 2\pi]$, we get the equivalent form

$$2^{n_1+\cdots+n_k}\prod_{j=1}^{k}(\cos\alpha_j t)^{n_j} = S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k) + \sum_{l\in\mathbb{Z}^*} c_l\left(\cos lt + i\sin lt\right).$$

Integrating this last identity on the interval $[0, 2\pi]$ we obtain the main result of this paper, involving the trigonometric integral

$$I(n_1,\ldots,n_k;\alpha_1,\ldots,\alpha_k) = \int_0^{2\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x. \tag{2.9}$$

Theorem 2.1. *The following formula holds:*

$$I(n_1,\ldots,n_k;\alpha_1,\ldots,\alpha_k) = \frac{2\pi}{2^{n_1+\cdots+n_k}} S(n_1,\ldots,n_k;\alpha_1,\ldots,\alpha_k). \tag{2.10}$$

Remark 2.2. An alternate proof based on Euler's formula $\cos t + i\sin t = e^{it}$, $t \in \mathbb{R}$ for complex numbers, is the following:

$$\begin{aligned}
&\int_0^{2\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x \\
&= \int_0^{2\pi} \left(\frac{e^{i\alpha_1 x}+e^{-i\alpha_1 x}}{2}\right)^{n_1}\cdots\left(\frac{e^{i\alpha_k x}+e^{-i\alpha_k x}}{2}\right)^{n_k}\ \mathrm{d}x \\
&= \frac{1}{2^{n_1+\cdots+n_k}}\int_0^{2\pi}\left(S(n_1,\ldots,n_k;\alpha_1,\ldots,\alpha_k) + \sum_{j\in\mathbb{Z}^*} d_j\left(e^{ijx}+e^{-ijx}\right)\right)\mathrm{d}x \\
&= \frac{1}{2^{n_1+\cdots+n_k}}\int_0^{2\pi}\left(S(n_1,\ldots,n_k;\alpha_1,\ldots,\alpha_k) + 2\sum_{j\in\mathbb{Z}^*} d_j\cos jx\right)\mathrm{d}x,
\end{aligned}$$

where the coefficients d_j are zero except for a finite number, hence (2.10) holds.

We have the following natural consequences.

Corollary 2.1. 1) $I(n_1,\ldots,n_k;\alpha_1,\ldots,\alpha_k) = 0$ *if and only if* $n_1\alpha_1+\cdots+n_k\alpha_k$ *is odd.*
2) *If* $n_1\alpha_1+\cdots+n_k\alpha_k$ *is even, then*

$$\int_0^{\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x = \frac{\pi}{2^{n_1+\cdots+n_k}} S(n_1,\ldots,n_k;\alpha_1,\ldots,\alpha_k).$$

Proof. 1) The property follows directly by formula (2.10).
2) We have

$$\begin{aligned}
\int_0^{2\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x &= \int_0^{\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x \\
&\quad + \int_\pi^{2\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x,
\end{aligned}$$

and by considering $x = \pi + t$ in the last integral one obtains

$$\begin{aligned}
\int_0^{2\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x &= \int_0^{\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x \\
&\quad + (-1)^{n_1\alpha_1+\cdots+n_k\alpha_k}\int_0^{\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x \\
&= 2\int_0^{\pi} (\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k}\ \mathrm{d}x,
\end{aligned}$$

and the conclusion follows by formula (2.10). $\square$

Remark 2.3. Formula (2.10) can be proved by writing

$$(\cos\alpha_1 x)^{n_1}\cdots(\cos\alpha_k x)^{n_k} = \underbrace{\cos\alpha_1 x\cdots\cos\alpha_1 x}_{n_1\ times}\cdots\underbrace{\cos\alpha_k x\cdots\cos\alpha_k x}_{n_k\ times},$$

and then using (1.2). Considering similar arguments in [3], [4] and [9], the first two authors study 3-partitions, and extensions to k-partitions of multisets with equal sums.

3. Two examples

Example 3.1. An immediate application of Theorem 2.1 is an alternative calculation of the trigonometric integral $I(n;1) = \int_0^{2\pi}\cos^n x\,\mathrm{d}x$. Clearly, from formula (2.10) we have

$$\int_0^{2\pi}\cos^n x\,\mathrm{d}x = \frac{2\pi}{2^n}S(n;1),$$

where $S(n;1)$ is the number of choices of signs $+$ and $-$ such that $\underbrace{\pm 1\pm\cdots\pm 1}_{n\ times} = 0$. If n is even then we have $S(n;1) = \binom{n}{\frac{n}{2}}$, hence one obtains the formula

$$\int_0^{2\pi}\cos^n x\,\mathrm{d}x = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2\pi}{2^n}\binom{n}{\frac{n}{2}} & \text{if } n \text{ is even.}\end{cases} \tag{3.11}$$

Usually, the standard calculation of this integral in textbooks uses the recursive relation

$$I(n+2;1) = \frac{n+1}{n}\,I(n;1), \quad n\geq 1,$$

which can be easily obtained by integration by parts.

Example 3.2. Consider the trigonometric integral

$$J_n = \int_0^{\pi}\cos^2(x)\cdot\cos^2(2x)\cdots\cos^2(nx)\,\mathrm{d}x.$$

First notice that for all integers $n\geq 1$, we have $J_n\neq 0$ and

$$\begin{aligned} J_n &= \frac{1}{2}\int_0^{2\pi}\cos^2 x\cdot\cos^2(2x)\cdots\cos^2(nx)\,\mathrm{d}x \\ &= \frac{1}{2}I(2,\ldots,2;1,\ldots,n) = \frac{\pi}{2^{2n}}S(2,\ldots,2;1,\ldots,n).\end{aligned}$$

We shortly denote $S^{(2)}(n) = S(2,\ldots,2;1,\ldots,n)$ and obtain a formula analogous to (1.3):

$$J_n = \frac{S^{(2)}(n)\pi}{2^{2n}}. \tag{3.12}$$

Using the fact that in this case $S^{(2)}(n)$ is the free term of the expansion

$$\left(z+\frac{1}{z}\right)^2\left(z^2+\frac{1}{z^2}\right)^2\cdots\left(z^n+\frac{1}{z^n}\right)^2,$$

by numerical calculations (here in Matlab) one check that $S^{(2)}(n)$ recovers the sequence

$$2, 4, 10, 26, 76, 236, 760, 2522, 8556, 29504, \ldots.$$

This sequence is indexed as A047653 in OEIS [15], for which Kotesovec in 2014 conjectured the following asymptotic expansion

$$\lim_{n\to\infty}\frac{S^{(2)}(n)}{\frac{4^n}{n\sqrt{n}}} = \sqrt{\frac{3}{\pi}}. \tag{3.13}$$

The first ten values of the integral J_n are

$$\frac{\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{32}, \frac{13\pi}{128}, \frac{19\pi}{256}, \frac{59\pi}{1024}, \frac{95\pi}{2048}, \frac{1261\pi}{32768}, \frac{2139\pi}{65536}, \frac{461\pi}{16384}, \ldots.$$

Conjecture 3.1. *By* (3.12) *and* (3.13), *the following asymptotic limit holds*

$$\lim_{n\to\infty} n\sqrt{n}\cdot J_n = \sqrt{3\pi}.$$

Remark 3.4. 1) With a very similar argument as the one used in Remark 1, one can prove the weaker result $\lim_{n\to\infty} J_n = 0$.

2) The connection between some trigonometric integrals and the signum equation for Erdős-Surányi sequences was investigated in paper [6]. The integral representation in connection to almost unimodal sequences is given in [8].

3) In the recent paper [14], the authors using the ideas in [4] and [7], investigate trigonometric integrals of the form

$$\int_0^{2\pi} (\sin\alpha_1 x)^{n_1}\cdots(\sin\alpha_k x)^{n_k}\,\mathrm{d}x,$$

where $n_1,\ldots,n_k$ and $\alpha_1,\ldots,\alpha_k$ are as in Theorem 2.1. In the same paper the improper integral

$$\int_0^{\infty} \frac{(\cos\alpha x-\cos\beta x)^p}{x^q}\,\mathrm{d}x,$$

is considered for α, β real numbers and p, q positive integers. Some special cases of this integral have been studied in [11], using Fourier transform techniques.

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