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A heuristic method for solving linear and quasi-linear first order partial differential equations

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ABSTRACT. We present a heuristic method for finding first integrals for linear and quasi-linear first order partial differential equations, when we are often lead to solve exact ordinary differential equations. The method is inspired by the corresponding one used for integrating first order exact ordinary differential equations of the form

$$P(x,y)dx + Q(x,y)dy = 0$$

and is applied for integrating first order exact ordinary differential equations in the case of three variables:

$$a(x,y,z)dx + b(x,y,z)dy + c(x,y,z)dz = 0.$$

To our best knowledge, the method is new at least from the point of view of its theoretical and applicative features presented in full details in the following.

1. INTRODUCTION

A first order differential equation of the form

$$P(x,y) + Q(x,y)y' = 0 \quad (\Longleftrightarrow P(x,y)dx + Q(x,y)dy = 0) \tag{1.1}$$

is is said to be *exact*, see for example [1], [12], if there exists a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = P(x,y) \text{ and } \frac{\partial u}{\partial y} = Q(x,y), \tag{1.2}$$

which is equivalent to the fact that the differential expression $P(x,y)dx + Q(x,y)dy$ is the exact *differential* of $u(x,y)$, that is,

$$du = P(x,y)dx + Q(x,y)dy.$$

We recall that a necessary and sufficient condition for (1.1) to be exact is that

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y}$$

on the domain $D \subset \mathbb{R}^2$ on which the equation (1.1) is considered.

Under the above condition, it follows that the solution $y(x)$ of (1.1) is given implicitly by

$$u(x,y) = c.$$

There exists various ways to find the solution $u(x,y)$ of (1.1), see for example [1] (Lecture 3, pp. 13), [12] (Chapter 10, pp. 89), [5], [8], [3], [4], [7], [10], [14] and [17]. To our best knowledge, the most extensive treatment of linear first order partial differential equations in the case of two and three independent variables, with both theory and solved exercises, is presented in [18], Chapter 2, pp. 24.

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For the purpose of the current note, we briefly describe the direct integration method [12], which does not require the computation of the corresponding curvilinear integrals independent of the path, like in [1]. The algorithm is the following.

1. We reverse the partial differentiation with respect to x in the first equation of (1.2)

$$\frac{\partial u}{\partial x} = P(x, y)$$

and obtain

$$u(x, y) = \int P(x, y)dx + C(y). \tag{1.3}$$

In order to calculate the "constant" $C(y)$, we first partially differentiate (1.3) with respect to y to get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int P(x, y)dx + \frac{dC}{dy}, \tag{1.4}$$

(we have an ordinary derivative in the last term since C only depends on y).

2. Then we use the second equation in (1.2) to get

$$Q(x, y) = \frac{\partial}{\partial y}\int P(x, y)dx + \frac{dC}{dy},$$

and therefore

$$\frac{dC}{dy} = Q(x, y) - \frac{\partial}{\partial y}\int P(x, y)dx. \tag{1.5}$$

Remark 1.1. It is important to note that the right hand side of (1.5) only depends on y, since

$$\frac{\partial}{\partial x}\left(Q(x, y) - \frac{\partial}{\partial y}\int P(x, y)dx\right) = \frac{\partial Q(x, y)}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\int P(x, y)dx\right)$$

$$= \frac{\partial Q(x, y)}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\left(\int P(x, y)dx\right) = \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} = 0.$$

3. Now, by integrating (1.5) we will find $C(y)$ up to an arbitrary additive constant and so by (1.3) one obtains the solution $u(x, y)$.

Starting from the above presented method, our aim in this note is to design a similar technique for solving linear and quasi-linear first order partial differential equations.

It appears that such a technique has not been considered so far in this context. For documentation, we have consulted some introductory PDE books, where the treatment of linear and quasi-linear first order partial differential equations is included and illustrated by appropriate exercises: [2] (Chapter 2, pp. 19); [5] (Chapter 3, pp. 25); [6] (Chapter 2, pp. 28); [8] (Chapter 2, pp. 59), where quasi-linear first order partial differential equations are solved for discontinuous coefficients in the case of two independent variables; [9] (Chapter 2, pp. 61); [11] (Chapter 2, pp. 23); [16] (§1.2, pp. 6) and [18] (Chapter 2, pp. 24), to mention just a selective list.

In contrast to the case of integrating exact ODE, where the above method *always* works, in the case of linear and quasi-linear first order partial differential equations our method only works if some additional conditions are satisfied. This is the reason why we labeled the method as being "heuristic" in this new context.

2. Linear and quasi-linear first order partial differential equations

Definition 2.1. *A* first order partial differential equation *is a relation of the form*

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right) = 0, \tag{2.6}$$

where $F : \Omega \subset \mathbb{R}^{2n+1} \to \mathbb{R}$ is a given function, $x_1, x_2, \ldots, x_n$ are the independent variables and $u : D \subset \mathbb{R}^n \to \mathbb{R}$ is the unknown function.

If F is such that the equation (2.6) *can be written under the particular form*

$$P_1(x_1, x_2, \ldots, x_n, u)\frac{\partial u}{\partial x_1} + P_1(x_1, x_2, \ldots, x_n, u)\frac{\partial u}{\partial x_2} + \tag{2.7}$$
$$\cdots + P_1(x_1, x_2, \ldots, x_n, u)\frac{\partial u}{\partial x_n} = P_{n+1}(x_1, x_2, \ldots, x_n, u).$$

where $P_k : D_1 \subset \mathbb{R}^{n+1} \to \mathbb{R}$ are in $C^1(D_1)$, $k = 1, 2, \ldots, n+1$, and P_k, $k = 1, 2, \ldots, n+1$, are not simultaneously zero on D_1, then (2.7) *is called* a quasilinear first order partial differential equation.

Remark 2.2.

(1) In literature it is also used the term *semilinear first order partial differential equation* to designate a quasilinear first order partial differential equation;
(2) If P_k, $k = 1, 2, \ldots, n+1$, do not depend on u, then (2.7) is in fact a *linear and nonhomogenous first order partial differential equation*;
(3) If, additionally to the previous item, $P_{n+1} \equiv 0$, then (2.7) is a *linear and homogenous first order partial differential equation*.

In most elementary PDE courses that include the study of quasilinear first order partial differential equations one may find a certain version of the following classical result, which shows that the solution of any quasilinear first order partial differential equation can be obtained by solving a linear and homogeneous first order partial differential equation.

Theorem 2.1. *If $V(x_1, x_2, \ldots, x_n, u)$ is a solution of the linear and homogeneous first order partial differential equation*

$$P_1(x_1, x_2, \ldots, x_n, u)\frac{\partial V}{\partial x_1} + P_1(x_1, x_2, \ldots, x_n, u)\frac{\partial V}{\partial x_2} + \tag{2.8}$$
$$\cdots + P_1(x_1, x_2, \ldots, x_n, u)\frac{\partial V}{\partial x_n} + P_{n+1}(x_1, x_2, \ldots, x_n, u)\frac{\partial V}{\partial u} = 0,$$

then

$$V(x_1, x_2, \ldots, x_n, u) = 0. \tag{2.9}$$

is the solution of the quasilinear first order partial differential equation (2.7).

Proof. We search for a solution of (2.7) defined implicitly by

$$V(x_1, x_2, \ldots, x_n, u) = 0, \tag{2.10}$$

where V is a function assumed to satisfy the following conditions:

(1) V is continuous on a domain $D_2 \subset \mathbb{R}^{n+1}$;
(2) $V \in C^1(D_2)$;
(3)
$$\frac{\partial V}{\partial u}(x_1, x_2, \ldots, x_n, u) \neq 0, \forall (x_1, x_2, \ldots, x_n, u) \in D_2.$$

Now using the chain rule for differentiating the function $u = u(x_1, x_2, \ldots, x_n)$, implicitly defined by (2.10), we get

$$\frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial u} \cdot \frac{\partial u}{\partial x_1} = 0 \Leftrightarrow \frac{\partial u}{\partial x_1} = -\frac{\partial V}{\partial x_1} \Big/ \frac{\partial V}{\partial u}$$

$$\frac{\partial V}{\partial x_2} + \frac{\partial V}{\partial u} \cdot \frac{\partial u}{\partial x_2} = 0 \Leftrightarrow \frac{\partial u}{\partial x_2} = -\frac{\partial V}{\partial x_2} \Big/ \frac{\partial V}{\partial u}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial V}{\partial x_n} + \frac{\partial V}{\partial u} \cdot \frac{\partial u}{\partial x_n} = 0 \Leftrightarrow \frac{\partial u}{\partial x_n} = -\frac{\partial V}{\partial x_n} \Big/ \frac{\partial V}{\partial u},$$

which by (2.7) yield exactly (2.8).

□

3. A heuristic method for solving linear and quasi-linear first order partial differential equations

A course on partial differential equations (PDEs) commonly starts with the theory of first order PDEs, which turns out to be at the beginning a quite difficult topic for students, especially at the moment they need to apply the theory above for solving various linear or quasilinear first order partial differential equations they find in some excellent problems books like [10], [14] and [17] (unfortunately accessible to Romanian readers only).

Apparently, the difficulties come from the heuristic way of solving the *characteristics equations* corresponding to (2.8), that is, the symmetric system

$$\frac{dx_1}{P_1(x_1, x_2, \ldots, x_n, u)} = \cdots = \frac{dx_n}{P_1(x_1, x_2, \ldots, x_n, u)} = \frac{du}{P_{n+1}(x_1, x_2, \ldots, x_n, u)}, \tag{3.11}$$

for which we have to find n functionally independent *first integrals*:

$$\begin{cases} \varphi_1(x_1, x_2, \ldots, x_n, u) = c_1 \\ \varphi_2(x_1, x_2, \ldots, x_n, u) = c_2 \\ \ldots\ldots\ldots\ldots \\ \varphi_n(x_1, x_2, \ldots, x_n, u) = c_n. \end{cases} \tag{3.12}$$

The main aim of this section is to present a method for finding first integrals that works for some particular classes of equations of the form (3.11).

We first illustrate our method for a particular linear and homogenous PDE, when the unknown function depends on three independent variables, and then we shall formulate the principle of the method in the general case.

Example 3.1. Find the solution of the following linear and homogeneous first order partial differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + xy\frac{\partial u}{\partial z} = 0. \tag{3.13}$$

Solution. The characteristics equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy} \tag{3.14}$$

from which we routinely get

$$\frac{dx}{x} = \frac{dy}{y} = \frac{ydx - xdy}{0} \text{ and } \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy} = \frac{ydx + xdy - 2dz}{0}$$

and thus one obtains quite easily the following two first integrals:

$$\frac{x}{y} = c_1; \quad xy - 2z = c_2. \tag{3.15}$$

So, according to Theorem 2.1, the general solution of (3.13) is given by

$$u(x,y,z) = \Phi\left(\frac{x}{y}, xy - 2z\right),\ (x,y,z) \in D \subset \mathbb{R}^3,$$

where Φ is an arbitrary differentiable function with continuous first order partial derivatives on a domain $D_\Phi \subset \mathbb{R}^2$.

Remark 3.3. One can find other first integrals for the characteristic system (3.14). Indeed, we also have

$$\frac{{}^{xy^2)}dx}{x} = \frac{{}^{xz)}dy}{y} = \frac{{}^{-xy)}du}{xy} = \frac{xy^2dx + xzdy - xydz}{xyz}$$

which imply

$$\frac{dx}{x} = \frac{xy^2dx + xzdy - xydz}{xyz} \Longleftrightarrow (xy^2 - yz)dx + xzdy - xydz = 0.$$

From my long term experience in teaching PDEs, I know that most undergraduate students encounter serious difficulties to integrate differential equations similar to the one above, i.e., to integrate the system

$$\begin{cases} \frac{\partial \varphi}{\partial x} = xy^2 - yz \\ \frac{\partial \varphi}{\partial y} = xz \\ \frac{\partial \varphi}{\partial z} = -xy. \end{cases} \tag{3.16}$$

The method that we present in the following could be useful in some of such difficult situations. For reasons that will be explained later, we first multiply the equation

$$(xy^2 - yz)dx + xzdy - xydz = 0$$

by the integrant factor $\frac{1}{y^2} \neq 0$, to obtain

$$\left(x - \frac{z}{y}\right)dx + \frac{xz}{y^2}dy - \frac{x}{y}dz = 0,$$

which leads to the following system of PDEs

$$\begin{cases} \frac{\partial \varphi}{\partial x} = x - \dfrac{z}{y} \\ \frac{\partial \varphi}{\partial y} = \dfrac{xz}{y^2} \\ \frac{\partial \varphi}{\partial z} = -\dfrac{x}{y} \end{cases} \tag{3.17}$$

to be approached by our method. To solve it, we proceed as follows.

Step 1. We integrate the third equation in (3.17) with respect to z and obtain

$$\varphi(x,y,z) = -\frac{xz}{y} + k(x,y), \tag{3.18}$$

where $k(x,y)$ is the "constant" of integration with respect to z.

Step 2. Now we differentiate (3.18) with respect to y to obtain

$$\frac{\partial \varphi}{\partial y} = \frac{xz}{y^2} + \frac{\partial k}{\partial y},$$

which, by using the second equation in (3.17) yields

$$\frac{\partial k}{\partial y} = 0,$$

which means that $k(x, y)$ is in fact depending only of x. Denote this function by $h(x)$. So, at this stage we know that

$$\varphi(x, y, z) = -\frac{xz}{y} + h(x). \tag{3.19}$$

Step 3. We differentiate (3.19) with respect to x and obtain

$$\frac{\partial \varphi}{\partial x} = -\frac{z}{y} + h'(x),$$

which, by using the first equation in (3.17) yields

$$h'(x) = x.$$

So, $h(x) = \dfrac{x^2}{2}$ and from (3.19) we obtain the desired function

$$\varphi(x, y, z) = \frac{x^2}{2} - \frac{xz}{y},$$

which provides a third first integral for the linear partial differential equation (3.13):

$$\frac{x^2}{2} - \frac{xz}{y} = c_3.$$

Remark 3.4. Even in the case of determining some simpler first integrals, like the second one given in (3.15), students may encounter the same kind of difficulties that could be overcome by using the method presented above.

Let us illustrate it for the case of determining the second first integral in (3.15).

We have to integrate the differential expression $ydx + xdy - 2dz$, that is, to determine a function $\varphi(x, y, z)$ such that

$$\frac{\partial \varphi}{\partial x} = y; \quad \frac{\partial \varphi}{\partial y} = x; \quad \frac{\partial \varphi}{\partial z} = -2. \tag{3.20}$$

Step 1. We integrate the first equation in (3.20) with respect to x and obtain

$$\varphi(x, y, z) = xy + k(y, z), \tag{3.21}$$

where $k(y, z)$ is the "constant" of integration with respect to x.

Step 2. Now we differentiate (3.21) with respect to y and obtain

$$\frac{\partial \varphi}{\partial y} = x + \frac{\partial k}{\partial y},$$

which, by using the second equation in (3.20) yields

$$\frac{\partial k}{\partial y} = 0,$$

that is, $k(y, z)$ is in fact depending only on z. Denote this function by $h(z)$. So, at this stage we know that

$$\varphi(x, y, z) = xy + h(z). \tag{3.22}$$

Step 3. Now we differentiate (3.22) with respect to z and obtain

$$\frac{\partial \varphi}{\partial z} = h'(z),$$

which, by using the third equation in (3.20) yields

$$h'(z) = -2$$

and so $h(x) = -2z$. Therefore the desired function φ is given by

$$\varphi(x, y, z) = xy - 2z.$$

Remark 3.5. It is easy to check that the described method does not work for the original differential expression $(xy^2 - yz)dx + xzdy - xydz$. The natural question is "Why?"

Before answering this question, we first present our method in the general case. So, one considers the system

$$\begin{cases} \frac{\partial \varphi}{\partial x} = a(x, y, z) \\ \frac{\partial \varphi}{\partial y} = b(x, y, z) \\ \frac{\partial \varphi}{\partial z} = c(x, y, z). \end{cases} \tag{3.23}$$

that corresponds to the integration of the following differential expression

$$a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz$$

The proposed method runs as follows.

Step 1. We integrate the third equation in (3.23) with respect to z and obtain

$$\varphi(x, y, z) = \int c(x, y, z)dz + k(x, y), \tag{3.24}$$

where $k(y, y)$ is the "constant" of integration with respect to z.

Step 2. Now we differentiate (3.24) with respect to y and obtain

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left(\int c(x, y, z)dz\right) + \frac{\partial k}{\partial y},$$

which, by using the second equation in (3.23) yields

$$\frac{\partial k}{\partial y} = b(x, y, z) - \frac{\partial}{\partial y}\left(\int c(x, y, z)dz\right).$$

Step 3. If

$$b(x, y, z) = \frac{\partial}{\partial y}\left(\int c(x, y, z)dz\right)$$

then k will only depend on x and if, additionally,

$$\frac{\partial}{\partial x}\left(\int c(x, y, z)dz\right) - a(x, y, z) \text{ is a function only of } x,$$

k can be determined by integrating with respect to x the ordinary differential equation

$$k'(x) = a(x, y, z) - \frac{\partial}{\partial x}\left(\int c(x, y, z)dz\right).$$

One can now state

Proposition 3.1. *If $a(x, y, z)$, $b(x, y, z)$ and $c(x, y, z)$ are such that*

$$b(x, y, z) = \frac{\partial}{\partial y}\left(\int c(x, y, z)dz\right)$$

and

$$\frac{\partial}{\partial x}\left(\int c(x, y, z)dz\right) - a(x, y, z) \text{ is a function only of } x,$$

then the function

$$\varphi(x,y,z) = \int a(x,y,z)dx$$

is a solution of the system (3.23).

We illustrate Proposition 3.1 by means of

Example 3.2. Find a function $\varphi(x,y,z)$ such that

$$d\varphi = \left(\frac{y}{z} + \frac{yz}{x^2}\right) dx + \left(\frac{x}{z} - \frac{z}{x}\right) dy + \left(-\frac{xy}{z^2} - \frac{y}{x}\right) dz.$$

Solution. The problem is equivalent to solving the following system of PDEs

$$\begin{cases} \frac{\partial \varphi}{\partial x} = \frac{y}{z} + \frac{yz}{x^2} \\ \frac{\partial \varphi}{\partial y} = \frac{x}{z} - \frac{z}{x} \\ \frac{\partial \varphi}{\partial z} = -\frac{xy}{z^2} - \frac{y}{x}. \end{cases} \tag{3.25}$$

We use Proposition 3.1. First, we note that

$$\int c(x,y,z)dz = \int \left(-\frac{xy}{z^2} - \frac{y}{x}\right) dz = \frac{xy}{z} - \frac{yz}{x}$$

and

$$\frac{\partial}{\partial y}\left(\int c(x,y,z)dz\right) = \frac{\partial}{\partial y}\left(\frac{xy}{z} - \frac{yz}{x}\right) = \frac{x}{z} - \frac{z}{x} = b(x,y,z).$$

Since

$$\frac{\partial}{\partial x}\left(\int c(x,y,z)dz\right) - a(x,y,z) = \frac{y}{z} + \frac{yz}{x^2} - \frac{y}{z} - \frac{yz}{x^2} = 0$$

it follows that the function

$$\frac{\partial}{\partial x}\left(\int c(x,y,z)dz\right) - a(x,y,z)$$

does not depend on y and z and so, one can consider that it depends only on x. Therefore, by Proposition 3.1, the function

$$\varphi(x,y,z) = \int a(x,y,z)dx = \int \left(\frac{y}{z} + \frac{yz}{x^2}\right) dx = \frac{xy}{z} - \frac{yz}{x}$$

is a solution of the problem.

Remark 3.6. At Step 3, we could alternatively ask for the following conditions to hold:

$$a(x,y,z) = \frac{\partial}{\partial x}\left(\int c(x,y,z)dz\right)$$

and

$$\frac{\partial}{\partial y}\left(\int c(x,y,z)dz\right) - b(x,y,z) \text{ is a function only of } y,$$

and the analogous ones when at Step 1 we start by integrating the first or the second equation in (3.23) etc.

The corresponding results for two of these situations are stated by the next propositions.

Proposition 3.2. *If* $a(x,y,z)$, $b(x,y,z)$ *and* $c(x,y,z)$ *are such that*

$$a(x,y,z) = \frac{\partial}{\partial x}\left(\int c(x,y,z)dz\right)$$

and

$$\frac{\partial}{\partial y}\left(\int c(x,y,z)dz\right) - b(x,y,z) \text{ is a function only of } y,$$

then the function

$$\varphi(x,y,z) = \int b(x,y,z)dy$$

is a solution of the system (3.23).

Proposition 3.3. *If* $a(x,y,z)$, $b(x,y,z)$ *and* $c(x,y,z)$ *are such that*

$$b(x,y,z) = \frac{\partial}{\partial y}\left(\int a(x,y,z)dx\right)$$

and

$$\frac{\partial}{\partial y}\left(\int a(x,y,z)dx\right) - b(x,y,z) = \text{ a function only of } z,$$

then the function

$$\varphi(x,y,z) = \int c(x,y,z)dz$$

is a solution of the system (3.23).

Remark 3.7. We are now interested to see whether or not one could apply Proposition 3.2 or Proposition 3.3 to the problem in Example 3.1.

Using the previous calculations, we have

$$\frac{\partial}{\partial x}\left(\int c(x,y,z)dz\right) = \frac{\partial}{\partial x}\left(\frac{xy}{z} - \frac{yz}{x}\right) = \frac{y}{z} + \frac{yz}{x^2} = a(x,y,z),$$

and so the first assumption in Proposition 3.2 is satisfied.

Next,

$$\frac{\partial}{\partial y}\left(\int c(x,y,z)dz\right) = \frac{\partial}{\partial y}\left(\frac{xy}{z} - \frac{yz}{x}\right) = \frac{x}{z} - \frac{z}{x}$$

and hence

$$\frac{\partial}{\partial y}\left(\int c(x,y,z)dz\right) - b(x,y,z) = 0,$$

which means that

$$\frac{\partial}{\partial y}\left(\int c(x,y,z)dz\right) - b(x,y,z)$$

does not depend on x and z (so, it can be considered as depending only on y).

Now, by applying Proposition 3.2, we determine the same solution as the one obtained in Example 3.1:

$$\varphi(x,y,z) = \int b(x,y,z)dy = \int \left(\frac{x}{z} - \frac{z}{x}\right) dy = \frac{xy}{z} - \frac{yz}{x}.$$

4. Integrant factor

First, let us answer the question raised in Remark 3.5. Indeed, in this particular case we have $c(x, y, z) = -xy$, $b(x, y, z) = xz$, and so

$$b(x, y, z) = xz \neq -xz = \frac{\partial}{\partial y}\left(\int c(x, y, z)dz\right),$$

which shows why Proposition 3.1 does not work here. We have a similar situation when we start by integrating the first or the second equation in (3.23).

As we have seen in the previous section, we may encounter situations when neither Proposition 3.1 nor Proposition 3.2 or Proposition 3.3 applies. In such cases we could try to find an integrant factor. We note that we actually used an *a priori* integrant factor in the solution we have presented above for Example 3.1, but we did not indicate how we obtained it.

Here, we illustrate the method of finding and integrant factor in the case corresponding to Proposition 3.1 (the other cases could be approached similarly).

If the first condition in Proposition 3.1 is not satisfied, that is,

$$b(x, y, z) \neq \frac{\partial}{\partial y}\left(\int c(x, y, z)dz\right),$$

then we search for a function $\eta(y)$ such that by replacing $b(x, y, z)$ by $\eta(y)b(x, y, z)$ and $c(x, y, z)$ by $\eta(y)c(x, y, z)$ the above condition is satisfied. In this case, by integrating the equation

$$\frac{\partial \varphi}{\partial z} = \eta(y)c(x, y, z)$$

with respect to z, we get

$$\varphi(x, y, z) = \int \eta(y)c(x, y, z)dz + k(x, y) = \eta(y) \cdot \int c(x, y, z)dz + k(x, y), \tag{4.26}$$

where $k(x, y)$ is the "constant" of integration with respect to z.

Then, by partially differentiating (4.26) with respect to y, we have

$$\frac{\partial \varphi}{\partial y} = \eta'(y) \cdot \int c(x, y, z)dz + \eta(y) \cdot \frac{\partial}{\partial y}\left(\int c(x, y, z)dz\right) + \frac{\partial k}{\partial y}$$

and hence by using the fact that

$$\frac{\partial \varphi}{\partial y} = \eta(y)b(x, y, z),$$

one obtains

$$\frac{\partial k}{\partial y} = \eta(y)b(x, y, z) - \eta'(y) \cdot \int c(x, y, z)dz - \eta(y) \cdot \frac{\partial}{\partial y}\left(\int c(x, y, z)dz\right).$$

Now, by imposing $\frac{\partial k}{\partial y} = 0$, we are led to the next ordinary differential equation

$$\eta'(y) \cdot \int c(x, y, z)dz - \left(b(x, y, z) - \frac{\partial}{\partial y}\int c(x, y, z)dz\right) \cdot \eta(y) = 0 \tag{4.27}$$

Therefore, in order to solve (4.27) with respect to $\eta(y)$, it is necessary to have satisfied a second condition, i.e.,

$$\frac{\int c(x, y, z)dz}{b(x, y, z) - \frac{\partial}{\partial y}\int c(x, y, z)dz} \text{ is a function only of } y. \tag{4.28}$$

Under these circumstances, the integrant factor $\eta(y)$ is obtained by solving the ordinary differential equation with separable variables (4.27).

Thus, we can state the above results as

Proposition 4.4. *If $b(x, y, z)$ and $c(x, y, z)$ are such that* (4.28) *is satisfied, then there exists an integrant factor $\eta(y)$ for the system* (3.23)*, which is obtained by solving the ordinary differential equation* (4.27)*.*

For the other remaining cases we shall proceed similarly.

Example 4.3. Let us illustrate Proposition 4.4 for the case of the system (3.16) that has been included in Example 3.1.

According to Remark 3.7, the first condition in Proposition 3.1 is not satisfied but condition (4.28) is satisfied, as we have

$$\frac{\int c(x,y,z)dz}{b(x,y,z) - \frac{\partial}{\partial y}\int c(x,y,z)dz} = \frac{-xyz}{2xz} = -\frac{y}{2},$$

and hence the integrant factor exists and will be obtained by solving the ordinary differential equation

$$\eta' = -2y\eta \Longleftrightarrow \frac{d\eta}{\eta} = -2ydy,$$

with respect to η, which provides exactly the integrant factor $\eta(y) = \dfrac{1}{y^2}$ that has been used in Example 3.1.

We close this paper by suggesting the readers to practice the method presented and illustrated above for solving other (quasi-)linear first order partial differential equations that can be found for example in the following problem books: [3], [4], [7], [10], [14] and [17]. Usually, these problems lead to the integration of some first order exact ordinary differential equations depending on three variables of the form

$$a(x,y,z)dx + b(x,y,z)dy + c(x,y,z)dz = 0.$$

A sample example, adapted from [17], is the following

Exercise. Find a function $\varphi(x, y, z)$ such that

$$d\varphi = \frac{2x}{z}\left(\sqrt{x^2+y^2-a^2} + \sqrt{x^2+y^2}\right)dx + \frac{2y}{z}\left(\sqrt{x^2+y^2-a^2} + \sqrt{x^2+y^2}\right)dy$$
$$-\frac{1}{z^2}\left(\sqrt{x^2+y^2-a^2} + \sqrt{x^2+y^2}\right)dz.$$

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