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The Radon-Nikodým property for the Fourier space of some hypergroups

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ABSTRACT. In this paper, we study the Radon-Nikodỳm Property for the Fourier space of a commutative compact hypergroup and that of a compact (non necessarily commutative) hypergroup. We prove the coincidence of the weak-* topology and the norm topology on the unit sphere of the subset $A_K(H)$ of the Fourier space $A(H)$ of a commutative hypergroup H consisting of elements that have support in a fixed compact subset K of the hypergroup H. Finally, we derive the fact that $A_K(H)$ has the Radon-Nikodým property.

1. INTRODUCTION

In their various works on locally compact spaces, researchers in harmonic analysis have noticed the existence of some topological spaces which, although not being groups, possess some of their characteristic structures, such as the possibility that a convolution on the corresponding space of all the finite regular Borel measures can be defined similar to the group case. Among the authors who have taken a very close interest in this class of spaces that have such convolutions are Dunkl [6], Jewett [13] and Spector [22] who succeeded in introducing, independently in the 1970s, axiomatic formalizations with small differences of these spaces. Dunkl and Spector called these spaces ”hypergroups” and Jewett refered to them as ”convos”. The term ”hypergroup” was adopted subsequently by the majority of authors. More details of the theory of hypergroups and standard examples can be found in W. R. Bloom and H. Heyer's monograph [1].

Since their introduction, hypergroups have received a good deal of attention from harmonic analysis reseachers due to the fact that hypergroups generalize, in many ways, locally compact groups. Analogues of many important results in harmonic analysis on groups can be shown for hypergroups, especially commutative hypergroups. In [15], Muruganandam studied several hypergroups the Fourier spaces of which are Banach algebra under the pointwise multiplication. For a commutative hypergroup, the author succeeded in giving sufficient conditions so that its Fourier space forms a Banach algebra. On the other hand, many authors studied conditions under which some Banach spaces related to the Fourier algebra of a locally compact group (the definition of which was clarified by Eymard [8]) possesses some geometric properties such as the Radon-Nikodým property (RNP). In this context, we can cite the works of E. Granirer [11], M. Leinert [12] and K. Taylor [25]. In [9], Finet studied the Radon-Nikodým property for some subspace of $L^1(K)$ where K is a compact hypergroup. See the book [4] by J. Diestel and J.J. Uhl, Jr. where various aspects of the Radon-Nikodým property are discussed. Recent papers on the subject are [20, 10, 18, 14].

The aim of this paper is to investigate the Radon-Nikodým property for the Fourier spaces and for other spaces related to commutative hypergroups and compact hypergroups.

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The rest of the paper is organized as follows. In Section 2 we recapitulate some definitions, facts and notations used throughout this work. In Section 3, we state our main results. Here, we investigate the Radon-Nikodým property for the Fourier space of a compact commutative hypergroup and that of a compact (non necessarily commutative) hypergroup. We end the section proving the Radon-Nikodým property for the space $A_K(H) = \{u \in A(H), supp(u) \subset K\}$ where K is a compact subsets of the hypergroup H by using a method of Granirer and Leinert in [12].

2. Preliminaries

2.1. **Definition of Hypergroups.** Useful references are [1, 13, 16]. Let H be a nonempty locally compact Hausdorff topological space and let $C(H), C_c(H), C_b(H)$ as usual denote respectively the space of all complex continuous functions, the space of complex continuous functions with compact support and the space of complex continuous bounded functions on H. Also $C_c^+(H)$ denotes the subspace of $C_c(H)$ consisting of positive functions. Furthermore, let $M(H)$ denote the Banach space of all bounded Radon measures on H. Let the topology on $M(H)$ be given by the weak topology $\sigma(M(H), C_b(H))$. For any μ belonging to $M(H)$, let $supp(\mu)$ denote the support of μ. For every $x \in H$, let δ_x denote the Dirac measure at x.

Let $M^1(H)$ denote the space of all probability measures on H equipped with the weak topology and let $M^+(H)$ denote the subspace of $M(H)$ consisting of positive functions.

Let $\mathcal{K}(H)$ denote the space of all not empty compact subsets of H. For subsets U and V of H, set

$$\mathcal{K}_U(V) = \{A \in \mathcal{K}(H) : A \cap U \neq \emptyset, A \subset V\}.$$

Then $\mathcal{K}(H)$ can be given the topology generated by the sub-basis of all set $\mathcal{K}_U(V)$ for which U and V are open subsets of H. This topology is called the Michael topology on $\mathcal{K}(H)$ [1, page 7].

We recall the following definition of hypergroup from [16] using Jewett's axioms [13].

Definition 2.1. H is said to be a *hypergroup*, if H is a nonempty locally compact Hausdorff topological space which satisfies the following conditions.

- **H1:** There exists a binary operation $*$ called *convolution* on $M(H)$ under which $M(H)$ is an algebra. Moreover, for every $x, y \in H$, $\delta_x * \delta_y$ is a probability measure and the mapping $(x, y) \to \delta_x * \delta_y$ is continuous from $H \times H$ into $M^1(H)$.
- **H2:** There exists an element (necessarily unique) e in H such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in H$.
- **H3:** There exists a (necessarily unique) homeomorphism $x \to x^-$ of H called *involution* satisfying the following :
  1. $(x^-)^- = x$ for all $x \in H$.
  2. If μ^- is defined by $\int_H f(x) d\mu^-(x) = \int_H f(x^-) d\mu(x)$ for all $f \in C_c(H)$, then $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$ for all $x, y \in H$.
  3. e belongs to $supp(\delta_x * \delta_y)$ if and only if $y = x^-$.
- **H4:** For every $x, y \in H$, $supp(\delta_x * \delta_y)$ is compact. Moreover, the mapping $(x, y) \to supp(\delta_x * \delta_y)$ is continuous from $H \times H$ into $\mathcal{K}(H)$, with respect to the Michael topology.

We shall denote the probability measure $\delta_x * \delta_y$ simply by $x * y$. That is, for every continuous function f on H:

$$f(x * y) = \langle f, \delta_x * \delta_y \rangle = \int_H f(z) d(\delta_x * \delta_y)(z).$$

The element e will be called the *identity* of H. For every $x \in H$, x^- will be called the *adjoint* of x.

Definition 2.2. [1, page 10] A hypergroup H is said to be *commutative* if its convolution is commutative. If the involution is the identity mapping on H, that is: $x^- = x$ for every $x \in H$, then H is said to be a *symmetric (hermitian)* hypergroup.

Symmetric hypergroups are commutative. Note that the study of hypergroups is generally done through the study of their associated measure algebras, since the properties of hypergroups are given through these algebras.

In this article, all the hypergroups H that we will study have a Haar measure which we denote by m. For commutative and compact hypergroups the existence of a Haar measure is a fact [1, page 33]. We will use $\int_H ...dx$ to denote $\int_H ...dm(x)$, the integration with respect to m when there is no risk of confusion. The Banach spaces $L^p(H)$, $1 \leq p \leq \infty$, are understood as usual with respect to m.

2.2. **Involutive Banach algebras $M(H)$ and $L^1(H)$.** Let H be a hypergroup.

With the involution $\mu^* = \overline{(\mu^-)}$ for all $\mu \in M(H)$, the space $M(H)$ is an involutive Banach algebra with unit, [13, Theorem 6.1G]. The space $L^1(H)$ is an involutive Banach algebra with the convolution

$$(f * g)(x) = \int_H f(x * y)g(y^-)dy$$

and the involution $f^*(x) = \Delta(x^-)\overline{f(x^-)}$ for all $f, g \in L^1(H)$ and for every $x \in H$ where Δ stands for the modular function of H defined by $m * \delta_{x^-} = \Delta(x)m$. It is a homomorphism from H into the multiplicative group of positive real numbers [13, Section 5.3].

Definition 2.3. [1, Definition 1.6.14]A net $(k_\alpha) \subset L^1(H)$ is called a *bounded approximate unit* for $L^1(H)$ if $lim\|k_\alpha * f - f\|_1 = 0$ for all $f \in L^1(H)$.

Bounded approximate units act as substitute for identity in $L^1(H)$ (unless H is discrete).

Before going further on hypergroups, let us recall some important results about involutive Banach algebra which will be used in the sequel.

Fact 2.1. *Positive linear functional on an involutive algebra.*

Let A be an involutive Banach algebra having an approximate identity and f a continuous positif linear form on A, then

$$\|f\| = f(1),\ |f(x)|^2 \leq \|f\|f(xx^*) \text{ and } |f(yxy^*)| \leq \|x\|f(yy^*) \text{ for all } x, y \in A.$$

If (k_α) is an approximate identity for A, then $f(k_\alpha) \to f(1) = \|f\|$ [5, Section 2.1]. If f is a continuous linear functional on A, we denoted by $|f|$ the positive linear functional determined by the conditions $\||f|\| = \|f\|$ and $|f(x)|^2 \leq \|f\||f|(xx^*)$ for every $x \in A$. The extension of $|f|$ to the algebra with adjoined unit is again denoted by $|f|$.

2.3. **Representations of hypergroups.** Let A be an involutive Banach algebra, $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the involutive algebra of all bounded linear operators on $\mathcal{H}$, and let I be the identity operator on $\mathcal{H}$. A $*$-representation of A in $\mathcal{H}$ is a $*$-homorphism of the involutive algebra A into the involutive algebra $B(\mathcal{H})$ [5, Section 2.2].

Definition 2.4. [1, Definition 2.1.1]We refer to π as a *representation* of the hypergroup H in some Hilbert space $\mathcal{H}_\pi$ if

(1) π is a $*$-representation of the involutive Banach algebra $M(H)$ into $B(\mathcal{H}_\pi)$.

(2) $\pi(\delta_e) = I$

(3) For elements $\xi, \eta \in \mathcal{H}_\pi$, the mapping $\mu \to \langle \pi(\mu)\xi, \eta\rangle_{\mathcal{H}_\pi}$ is continuous on $M^+(H)$ with respect to the weak topology.

The Hilbert space $\mathcal{H}_\pi$ is called the *representation space* of H. A representation π is said to be *unitary* if for all $\mu \in M(H)$, the operator $\pi(\mu)$ is a unitary operator on $\mathcal{H}_\pi$. The representation π is said to be *irreducible* if there is no closed proper subspace of $\mathcal{H}_\pi$ that is invariant by $\pi(\mu)$ for all $\mu \in M(H)$.

It follows from this definition that each representation π is norm decreasing. In the sequel, $\pi(x)$ will denote $\pi(\delta_x)$.
The representations π_1 and π_2 of H, with representation spaces $\mathcal{H}_{\pi_1}$ and $\mathcal{H}_{\pi_2}$ respectively are said to be equivalent if there exists an isomorphism U from $\mathcal{H}_{\pi_1}$ onto $\mathcal{H}_{\pi_2}$ such that $\pi_2(\mu)U = U\pi_1(\mu)$ for all $\mu \in M(H)$. The set of equivalence classes of unitary irreducible representations of H is denoted by $\tilde{H}$ and one may denote by the same symbol a representation and its equivalence class.

2.4. **The Fourier space of a hypergroup.** This part describes briefly the Fourier space of a hypergroup, we refer to [15, Section 2] for more details. Let λ denote the left regular representation of H on $L^2(H)$ given by

$$\lambda(x)f(y) = f(x^- * y)$$

where $x, y \in H$ and $f \in L^2(H)$. This can be extended to $L^1(H)$ by setting

$$\lambda(f)(g) = f * g$$

for $f \in L^1(H)$ and $g \in L^2(H)$. Let $C^*(H)$ denote the enveloping C^*-algebras of the hypergroup algebra $L^1(H)$ and let $C^*_\lambda(H)$ denote the reduced C^*-algebras of H. That is, $C^*_\lambda(H)$ is the norm closure of the space $\{\lambda(f) : f \in L^1(H)\}$ in the algebra $B(L^2(H))$ of bounded linear operators on $L^2(H)$. The norm on $C^*(H)$ is given by

$$\|f\|_{C^*(H)} = \sup\{\|\pi(f)\| : \pi \in \tilde{H}\}.$$

Definition 2.5. [15, Definition 2.2] The Banach space dual of the full C^*-algebra $C^*(H)$ is called the *Fourier-Stieltjes space* of H and is denoted by $B(H)$.

The space $B(H)$ is contained in $L^\infty(H)$ and the Banach space dual of $C^*_\lambda(H)$ is denoted by $B_\lambda(H)$, which can be considered as a closed subspace of $B(H)$.

For $f \in C_c(H)$, for every $x, y \in H$, set $f^-(x) = f(x^-)$, $\tilde{f}(x) = \overline{f(x^-)}$ and $f^*(x) = \tilde{f}(x)\Delta(x^-)$. We have

$$f * \tilde{g}(x) = \int_H f(x * y)\overline{g(y)}dy.$$

Definition 2.6. [15, Section 2.3] The closed subspace spanned by $\{f * \tilde{f} : f \in C_c(H)\}$ in $B_\lambda(H)$ is called the Fourier space of H and is denoted by $A(H)$.

$A(H)$ is also the closure of the span of the set $\{f * \tilde{f} : f \in L^2(H)\}$ in $B_\lambda(H)$.

Definition 2.7. [15, Definition 2.17] The von Neumann algebra $[\lambda(H)]''$ associated to the left regular representation λ of H is called the von Neumann algebra of H and is denoted by $VN(H)$.

$VN(H)$ is the same as $[\lambda(L^1(H))]''$. Notice that $C^*_\lambda(H)$ is contained in $VN(H)$, as $VN(H)$ is the σ-weakly closed sub-algebra of $B(L^2(H))$ containing $\{\lambda(f) : f \in C_c(H)\}$. $VN(H)$ is the dual space of $A(H)$ [15, Theorem 2.19]. The space $VN(H)$ is called an *atomic* von Neumann algebra if the representation λ is atomic, that is, the direct sum of irreducible representations [24, Section I.9 and Section III.6].

2.5. **Commutative Hypergroups.** We recall some properties of commutative hypergroups. Let H be a commutative hypergroup. In [23], the author proved that a left Haar measure m exists on every commutative hypergroup.
A complex function χ on H is said to be *multiplicative* if $\chi(x*y) = \chi(x)\chi(y)$ for all $x, y \in H$. The dual $\widehat{H}$ of H is the space of hermitian characters of H, that is, the space of multiplicative continuous functions χ on H such that $\chi(x^-) = \overline{\chi(x)}$ for all $x \in H$. It is well-known that for a compact hypergroup H the dual $\widehat{H}$ is discrete [1, Theorem 2.2.9].
For μ in $M(H)$, the *Fourier-Stieltjes* transform $\mathcal{F}(\mu)$ of μ is defined on $\widehat{H}$ by

$$\mathcal{F}(\mu)(\chi) = \int_H \overline{\chi} d\mu, \ \chi \in \widehat{H}.$$

For any f in $L^1(H)$, the Fourier transform $\mathcal{F}(f)$ of f with respect to m is defined on $\widehat{H}$ by

$$\mathcal{F}(f)(\chi) = \int_H f\overline{\chi} dm, \ \chi \in \widehat{H}.$$

Let us denote by S the subset of $\widehat{H}$ given by :

$$S = \{\chi \in \widehat{H} : |\mathcal{F}(\mu)(\chi)| \leq \|\lambda(\mu)\|, \forall \mu \in M(H)\}.$$

The unique non-negative measure π on $\widehat{H}$ such that

$$\int_H |f|^2 dm = \int_{\widehat{H}} |\mathcal{F}(f)|^2 d\pi$$

for all $f \in L^1(H) \cap L^2(H)$, is called the *Plancherel-Levitan* measure associated with m; moreover S is exactly the support of π.
When H is a commutative hypergroup, $A(H) = \{f * \tilde{g} : f, g \in L^2(H)\}$ and $A(H)$ is isometrically isomorphic to $L^1(S, d\pi)$ [15, page 69].

2.6. **The Radon-Nikodým Property (RNP).** Several equivalent formulations of the Radon-Nikodým Property (RNP) exist; the book [4] is an excellent source with details on various aspects of the RNP. There is a summary on pages 217 and 218 of equivalent properties, which are equivalent to the Radon-Nikodým Property. Here we recall the following geometric definition.

Definition 2.8. A Banach space X has the Radon-Nikodým property (RNP) if every bounded subset D of X is dentable; that is, for each $\varepsilon > 0$, there is some $x_\varepsilon \in D$ such that $x_\varepsilon \notin \overline{co}\big(D \backslash B_\varepsilon(x_\varepsilon)\big)$ where $B_\varepsilon(x_\varepsilon) = \{y \in X : \|y - x_\varepsilon\| < \varepsilon\}$ and $\overline{co}\big(D \backslash B_\varepsilon(x)\big)$ is the norm closed convex-hull of $D \backslash B_\varepsilon(x)$.
A point $x \in D$ which satisfies this property for each $\varepsilon > 0$ is said to be a denting point of D.

For every discrete set Γ, $l^1 := l^1(\Gamma)$ has the RNP and every Banach space which is norm isomorphic to l^1 has the RNP [4].
If M is a von Neumann algebra with predual M_*, then "M is an atomic von Neumann algebra" is equivalent to "M_* has the RNP" [25, Theorem 3.5].

3. Main results

Some of the results on hypergroups in this section are almost the same as in the case of groups, with appropriate modifications when needed. For instance, in the proof of Theorem 3.2, we use the fact that $A(H)$ is isometrically isomorphic to $L^1(S, d\pi)$ where S is the subset of $\widehat{H}$ that supports the Plancherel-Levitan measure on $\widehat{H}$. However, for a compact abelian group G, $A(G)$ is directly isometrically isomorphic to $l^1(\widehat{G})$, where $\widehat{G}$ is the dual group of G and $\widehat{G}$ is discrete since G is compact. Also to obtain some results in Subsection 3.2 we assumed that H is a commutative hypergroup, while the analogous results for a locally compact group G do not require the commutativity assumption for G [12].

3.1. RNP for the Fourier space of compact commutative and compact Hypergroups.

Theorem 3.2. *If H is a compact commutative hypergroup, then $A(H)$ has the Radon-Nikodým property.*

Proof. If H is a commutative hypergroup then $A(H)$ is isometrically isomorphic to $L^1(S, d\pi)$. If H is compact then S is a discrete subset of $\widehat{H}$. Therefore $L^1(S, d\pi)$ is isomorphic to l^1. It follows that $A(H)$ has the RNP. □

Theorem 3.3. *If H is a compact hypergroup, then $A(H)$ has the Radon-Nikodým property.*

Proof. If H is a compact hypergroup, then the left regular representation λ of H can be written as a direct sum of continuous irreductible subrepresentations, see [13, Theorem 7.2C] and [26, page 243]. Thus λ is atomic and $VN(H) = [\lambda(H)]''$ is an atomic von Neumman algebra. Since $A(H)$ is the predual of $VN(H)$, then $A(H)$ has the RNP. □

3.2. Coincidence of the weak-* topology and the norm topology on the unit sphere of $A_K(H)$ and RNP.

In the following, H is assumed to be a commutative hypergroup with Haar measure m. Since a commutative hypergroup H is unimodular [13, Section 7.3], we have, for every $x \in H$

$$f^*(x) = \tilde{f}(x) = \overline{f(x^-)} = \overline{f^-(x)}.$$

Let K be closed subset of H and set $A_K(H) = \{f \in A(H), supp(f) \subset K\}$. We are now going to investigate the Radon-Nikodým property for the spaces $A_K(H)$ when K is compact. We borrow methods from [12] which dealt with the group case. Hereafter are some facts that we may need.

Fact 3.4. If X and Y are normed spaces in duality and if for each $y \in Y$, $\|y\| = sup\{|\langle y, x\rangle|, x \in X, \|x\| \leq 1\}$, then y_α converges to y in τ_{w^*}, the weak-* topology of Y (i.e. $\langle y_\alpha, x\rangle$ converges to $\langle y, x\rangle, \forall x \in X$) implies *lim inf* $\|y_\alpha\| \geq \|y\|$. If in addition $sup\|y_\alpha\| < \infty$, then $|\langle y_\alpha - y, x\rangle|$ converges to 0 uniformly on norm compact subsets of X, see [12, page 460] and [2, Chapter 3].

Fact 3.5. Let H be a commutative hypergroup. For $1 \leq p < \infty$, the space $C_c(H)$ is dense in $L^p(H)$. Moreover $L^q(H)$ is the dual space of $L^p(H)$ where $1/p + 1/q = 1$ [3, page 5]. This duality is defined by $\langle f, g\rangle = \int_H f(x)g(x)dx$ for every $f \in L^p(H)$ and every $g \in L^q(H)$.

Since H is a locally compact topological space, let U_α be a basis of relatively compact neighbourhoods at $e \in H$. For each α, choose a symmetric neighbourhood V_α at e such that $V_\alpha * V_\alpha^- = V_\alpha^2 \subset U_\alpha$ and write $h_\alpha = m(V_\alpha)^{-1} 1_{V_\alpha}$, where 1_{V_α} denotes the characteristic function of V_α. $h_\alpha \in C_c^+(H)$ with $supp(h_\alpha) \subset V_\alpha$ and $\|h_\alpha\|_1 = 1$. [1, pages 66 and 67]. $k_\alpha = h_\alpha * h_\alpha^-$ is a bounded approximate unit for $L^1(H)$ and we also have $k_\alpha \in C_c^+(H)$ with $limsupp(k_\alpha) = \{e\}$ and $\|k_\alpha\|_1 = 1$ [1, page 88]. The functions h_α and k_α are real valued positive functions.

Proposition 3.1. *Let $k_\alpha = h_\alpha * h_\alpha^-$ in $L^1(H)$ be as above. Let (u_β) be a net in $B(H)$ such that u_β converges to u_0 in $\sigma(B(H), C^*(H))$ and $\|u_\beta\|_{B(H)}$ converges to $\|u_0\|_{B(H)}$, (we write $u_\beta \to u_0$ in τ_{nw^*} of $B(H)$). Then, for any $\varepsilon > 0$, there exists α_0 such that $\|k_{\alpha_0} * u_0 - u_0\|_{B(H)} < \varepsilon$ and there exists β_0 such that $\|k_{\alpha_0} * u_\beta - u_\beta\|_{B(H)} < \varepsilon$ for all $\beta \geq \beta_0$.*

Proof. We may adjoin a unit 1 to $L^1(H)$ when H is non-discrete and we may assume that $u_0 \neq 0$.

$(k_\alpha)^* = (h_\alpha * h_\alpha^-)^* = (h_\alpha * h_\alpha^*)^*$ since H is commutative and k_α is real.

$(h_\alpha * h_\alpha^*)^* = (h_\alpha^*)^* * (h_\alpha)^* = h_\alpha * h_\alpha^* = k_\alpha$. Thus $(k_\alpha - 1)^* = k_\alpha - 1$.

For $u \in B(H)$ and $f \in L^1(H)$, we have

$$\begin{aligned}
|\langle k_\alpha * u - u, f\rangle|^2 &= |\langle (k_\alpha - 1) * u, f\rangle|^2 \\
&= |\langle u, (k_\alpha - 1) * f\rangle|^2 \\
&\leq \|u\|_{B(H)} |\langle |u|, (k_\alpha - 1) * f * ((k_\alpha - 1) * f)^*\rangle| \text{ using Fact 2.1} \\
&\leq \|u\|_{B(H)} |\langle |u|, (k_\alpha - 1) * f * f^* * (k_\alpha - 1)^*\rangle| \\
&\leq \|u\|_{B(H)} |\langle |u|, (k_\alpha - 1) * (f * f^*) * (k_\alpha - 1)^*\rangle| \\
&\leq \|u\|_{B(H)} \|f * f^*\|_{C^*(H)} |\langle |u|, (k_\alpha - 1) * (k_\alpha - 1)\rangle| \text{ using Fact 2.1} \\
&\leq \|u\|_{B(H)} \|f * f^*\|_{C^*(H)} |\langle |u|, (1 - k_\alpha) * (1 - k_\alpha)\rangle|,
\end{aligned}$$

Since any representation σ on H is norm decreasing, we have $\|\sigma(f)\|_1 \leq \|f\|_1$ for all $f \in L^1(H)$. Thus for all $k_\alpha \in L^1(H)$, we have $\|k_\alpha\|_{C^*(H)} \leq \|k_\alpha\|_1$.

It follows that $0 \leq k_\alpha \leq 1$, $0 \leq 1 - k_\alpha \leq 1$, $0 \leq (1 - k_\alpha) * (1 - k_\alpha) \leq 1 - k_\alpha$ in $C^*(H)$ with adjoined identity 1 and

$$\begin{aligned}
|\langle k_\alpha * u - u, f\rangle|^2 &\leq \|u\|_{B(H)} \|f * f^*\|_{C^*(H)} |\langle |u|, (1 - k_\alpha)\rangle| \\
&\leq \|u\|_{B(H)} \|f\|^2_{C^*(H)} |\langle |u|, (1 - k_\alpha)\rangle|
\end{aligned}$$

since $C^*(H)$ is a C^*-algebra. Therefore

$$\|k_\alpha * u - u\|^2_{B(H)} \leq \|u\|_{B(H)} |\langle |u|, (1 - k_\alpha)\rangle|.$$

Since k_α is an approximate identity for the involutive Banach algebra $L^1(H)$ and for $u \in B(H)$, $|u|$ is a positive linear functional on $C^*(H)$ then we conclude that $|u|(k_\alpha)$ converges to $|u|(1) = \|u\|$, according to Fact 2.1. Therefore $\langle |u|, k_\alpha\rangle$ converges to $\langle |u|, 1\rangle$; finally, $\langle |u|, 1 - k_\alpha\rangle$ converges to 0.

One can choose an α_0 such that for any $\varepsilon > 0$,

$$\|u_0\|_{B(H)} |\langle |u_0|, (1 - k_{\alpha_0})\rangle| < \varepsilon^2.$$

The latter estimation leads to

$$\|k_{\alpha_0} * u_0 - u_0\|_{B(H)} < \varepsilon.$$

Since u_β converges to u_0 in τ_{nw^*}, then by [7, Lemma 3.5], $|u_\beta|$ converges to $|u_0|$. Thus $\|k_{\alpha_0} * u_\beta - u_\beta\|^2_{B(H)} \leq \|u_\beta\|_{B(H)} |\langle |u_\beta|, (1 - k_{\alpha_0})\rangle|$ and the right hand side of the latter inequality converges to $\|u_0\|_{B(H)} |\langle |u_0|, (1 - k_{\alpha_0})\rangle| < \varepsilon^2$.

One can chose β_0 such that $\|u_\beta\|_{B(H)} |\langle |u_\beta|, (1 - k_{\alpha_0})\rangle| < \varepsilon^2$ if $\beta \geq \beta_0$. Then for all $\beta \geq \beta_0$,

$$\|k_{\alpha_0} * u_\beta - u_\beta\|_{B(H)} < \varepsilon.$$

□

Proposition 3.2. *Let A be a norm bounded subset of $L^\infty(H)$. Let $f \in L^1(H)$ and let (ϕ_α) be a net in A which weak-* converges to $\phi_0 \in A$. Then $\phi_\alpha * f$ converges to $\phi_0 * f$ in the topology of compact convergence.*

Proof. For $\phi \in L^\infty(H)$ and $f \in L^1(H)$, we have

$$(\phi * f)(x) = \int_H f(x * y)\phi(y^-)dy = \int_H {}_x f(y)\phi^-(y)dy = \langle \phi^-, {}_x f\rangle,$$

where

$$_xf(y) = f(x * y) = \int_H f(z)d(\delta_x * \delta_y)(z).$$

As $f \in L^1(H)$ then $_xf \in L^1(H)$ because $\|_xf\|_1 \leq \|f\|_1$ [13, Lemma 3.3B].

The net (ϕ_α) weak-* converges to ϕ_0 in A, this implies that $\langle \phi_\alpha, f\rangle$ converges to $\langle \phi_0, f\rangle, \forall f \in L^1(H)$, thus $(\phi_\alpha * f)(x) = \langle \phi_\alpha^-, _x f\rangle$ converges to $\langle \phi_0^-, _x f\rangle = (\phi_0 * f)(x)$.

If K is a compact subset of H, then the set $\{_xf : x \in K\}$ is a compact subset of $L^1(H)$ since the mapping $x \longmapsto {_xf}$ from H into $L^1(H)$ is continuous [1, page 15]. Therefore by Fact 3.4, $\phi_\alpha * f$ converges to $\phi_0 * f$ in the topology of compact convergence. □

For $u \in B(H)$ and $\mu \in M(H)$. The convolution product of μ by u is defined by

$$(\mu * u)(x) = \int_H u(y^- * x)d\mu(y).$$

Proposition 3.3. *If $u \in B(H)$ and $\mu \in M(H)$ then $\mu * u \in B(H)$. Moreover,*

$$\|\mu * u\|_{B(H)} \leq \|\mu\|_{\tilde{H}} \|u\|_{B(H)}.$$

Proof. Let $u \in B(H)$. Then there exists, by [15, Proposition 2.8], a representation π in $\tilde{H}$, such that $u(x) = \langle \pi(x)\xi, \eta\rangle_{\mathcal{H}_\pi}$ with $\|u\|_{B(H)} = \|\xi\|_{\mathcal{H}_\pi}\|\eta\|_{\mathcal{H}_\pi}$ for some $\xi, \eta \in \mathcal{H}_\pi$ and every $x \in H$. Note that $\pi(y^-) = \pi^-(y) = \pi^*(y)$. Then

$$\begin{aligned}(\mu * u)(x) &= \int_H u(y^- * x)d\mu(y)\\ &= \int_H \langle \pi(y^- * x)\xi, \eta\rangle_{\mathcal{H}_\pi} d\mu(y)\\ &= \int_H \langle \pi(y^-)\pi(x)\xi, \eta\rangle_{\mathcal{H}_\pi} d\mu(y)\\ &= \int_H \langle \pi(x)\xi, \pi(y)\eta\rangle_{\mathcal{H}_\pi} d\mu(y)\\ &= \langle \pi(x)\xi, \pi(\mu)\eta\rangle_{\mathcal{H}_\pi}.\end{aligned}$$

Since $\pi(\mu)\eta \in \mathcal{H}_\pi$, then the function $(\mu * u)(\cdot) = \langle \pi(\cdot)\xi, \pi(\mu)\eta\rangle_{\mathcal{H}_\pi}$ belongs to $B(H)$. Moreover,

$$\begin{aligned}|\langle \pi(x)\xi, \pi(\mu)\eta\rangle_{\mathcal{H}_\pi}| &\leq \|\pi(x)\xi\|_{\mathcal{H}_\pi}\|\pi(\mu)\eta\|_{\mathcal{H}_\pi}\\ &\leq \|\xi\|_{\mathcal{H}_\pi}\|\pi(\mu)\|_{B(\mathcal{H}_\pi)}\|\eta\|_{\mathcal{H}_\pi}\\ &\leq \|\mu\|_{\tilde{H}}\|\xi\|_{\mathcal{H}_\pi}\|\eta\|_{\mathcal{H}_\pi}.\end{aligned}$$

Thus $\|\mu * u\|_{B(H)} \leq \|\mu\|_{\tilde{H}}\|u\|_{B(H)}$.

□

Definition 3.9. [15, Definition 4.7]Let H be a commutative hypergroup. We say that H satisfies condition (F) if there exists a constant $M > 0$ such that

If $\chi_1, \chi_2 \in S$ then $\chi_1\chi_2 \in B_\lambda(H)$ and $\|\chi_1\chi_2\|_{B_\lambda(H)} \leq M$.

In what follows, we assume that H is a commutative hypergroup which satisfies condition (F) with $M = 1$.

A net $(u_\alpha) \subset B(H)$ converges to u in the multiplier topology τ_M if $\|(u_\alpha - u)v\|_{A(H)}$ converges to 0 for every $v \in A(H)$.

Proposition 3.4. *Let $f, g \in C_c(H)$. Let (u_α) be a net that converges to u in $\sigma(B(H), C^*(H))$ such that*

$$\sup_\alpha \|u_\alpha\|_{B(H)} < C < \infty.$$

Then the mapping $u \longmapsto (f * g) * u$ is continuous from $(B(H), \tau_{nw^*})$ into $(B(H), \tau_M)$.

Proof. Let $v \in A(H) \cap C_c(H)$, $w \in L^\infty(H)$ and $h \in L^1(H)$. We have $(f * w)v \in L^\infty(H)$ and thus:

$$\begin{aligned}\langle (f * w)v, h\rangle &= \int_H \Big((f * w)(x)v(x)\Big)h(x)dx\\ &= \int_H \Big(\int_H w(x * y)f(y^-)dy\Big)v(x)h(x)dx\\ &= \int_H \Big(\int_H w(x * y)f(y^-)v(x)dy\Big)h(x)dx,\end{aligned}$$

Let $K \subset H$ be compact such that $supp(v)supp(f)^- \subset K$:

$$\begin{aligned}\int_H \Big(\int_H w(x * y)f(y^-)v(x)dy\Big)h(x)dx &= \int_H \Big(\int_H 1_K w(x * y)f(y^-)v(x)dy\Big)h(x)dx\\ &= \int_H \Big(\int_H 1_K w(x * y)f(y^-)dy\Big)v(x)h(x)dx\\ &= \int_H \Big((f * (1_K w))(x)\Big)v(x)h(x)dx\\ &= \langle (f * (1_K w))v, h\rangle\end{aligned}$$

Hence

$$\langle (f * w)v, h\rangle = \langle (f * (1_K w))v, h\rangle.$$

The hypothesis on f implies $f \in L^2(H)$. Also $(1_K w)^* \in L^2(H)$. Therefore $f * (1_K w) = f * ((1_K w)^*)^* \in L^2(H) * L^2(H)^*$. Since H is commutative, then $\tilde{g} = g^*$ and $A(H) = L^2(H) * L^2(H)^*$. We have $f * (1_K w) \in A(H)$.

If H satisfies condition (F) with $M = 1$, then by [15, Corollary 4.13], $A(H)$ is a Banach algebra under pointwise product and we have:

$$\begin{aligned}\|(f * w)v\|_{A(H)} = \|f * (1_K w)v\|_{A(H)} &\le \|f * (1_K w)\|_{A(H)}\|v\|_{A(H)}\\ &\le \|f\|_2 \|(1_K w)^*\|_2 \|v\|_{A(H)}.\end{aligned}$$

Now let $w = g * (u_\alpha - u)$. Then

$$\|(f * w)v\|_{A(H)} = \|(f * g * (u_\alpha - u))v\|_{A(H)} \le \|f\|_2 \|(1_K(g * (u_\alpha - u)))^*\|_2 \|v\|_{A(H)}.$$

By Proposition 3.2, $g*(u_\alpha - u)$ converges to 0 uniformly on K, since $g \in C_c(H) \subset L^1(H)$ and $u_\alpha, u \in B(H) \subset L^\infty(H)$ and $\sup_\alpha \|u_\alpha\|_{B(H)} < C < \infty$.

Hence $\|((f * g) * (u_\alpha - u))v\|_{A(H)} \le \|f\|_2 \|(1_K(g * (u_\alpha - u)))^*\|_2 \|v\|_{A(H)}$ converges to 0 for any $v \in A(H) \cap C_c(H)$ which is dense in $A(H)$. Since $C_c(H)$ is dense in $L^1(H)$ (which is isomorphic to a closed ideal of $M(H)$), for every $f, g \in C_c(H)$ and $u \in B(H)$ by Proposition 3.3, we have

$$(f * g) * u \in B(H) \text{ and } \|(f * g) * u\|_{B(H)} \le \|f * g\|_{\tilde{H}} \|u\|_{B(H)} \le \|f * g\|_1 \|u\|_{B(H)}.$$

Thus $\|(f*g)*u_\alpha\|_{B(H)} \le \|f*g\|_1 \|u_\alpha\|_{B(H)} \le \|f*g\|_1 C$ and therefore $\sup_\alpha \|(f*g)*u_\alpha\|_{B(H)} < \infty$. We conclude that $(f*g)*u_\alpha$ converges to $(f*g)*u$ in the multiplier topology of $B(H)$. Hence $u \longmapsto (f * g) * u$ is continuous from $(B(H), \tau_{nw^*})$ to $(B(H), \tau_M)$. □

The following theorem is fundamental in our work.

Theorem 3.6. *Let H be a commutative hypergroup satisfying condition (F) with $M = 1$. If u_β is a net in $B(H)$ such that u_β converges to u_0 in $\sigma(B(H), C^*(H))$, the weak-* topology of $B(H)$ and if $\|u_\alpha\|_{B(H)}$ converges to $\|u_0\|_{B(H)}$, then u_β converges to u_0 in the multiplier topology of $B(H)$.*

Proof. Consider the bounded approximate unit (k_α) for $L^1(H)$ as in Proposition 3.1. Choose $\varepsilon > 0$. By Proposition 3.1, there exists α_0 and β_0 such that $\|k_{\alpha_0} * u - u\|_{B(H)} < \varepsilon$ and $\|k_{\alpha_0} * u_\beta - u_\beta\|_{B(H)} < \varepsilon$ for all $\beta \geq \beta_0$.

For $v \in A(H)$ and $\beta \geq \beta_0$,

$$\begin{aligned}\|(u_\beta - u)v\|_{A(H)} &= \|(k_{\alpha_0} * u_\beta - k_{\alpha_0} * u_\beta + u_\beta - u + k_{\alpha_0} * u - k_{\alpha_0} * u)v\|_{A(H)} \\ &\leq \|k_{\alpha_0} * u_\beta v - u_\beta v\|_{B(H)} + \|[k_{\alpha_0} * (u_\beta - u)]v\|_{A(H)} + \|k_{\alpha_0} * uv - uv\|_{B(H)}.\end{aligned}$$

Since $u_\beta v \to uv$ satisfies also the conditions in Proposition 3.1, we have

$$\|k_{\alpha_0} * u_\beta v - u_\beta v\|_{B(H)} < \varepsilon \text{ and } \|k_{\alpha_0} * uv - uv\|_{B(H)} < \varepsilon.$$

By taking $k_{\alpha_0} = h_{\alpha_0} * h^-_{\alpha_0}$ as $f * g$ in Proposition 3.4, there exists some $\beta_1 \geq \beta_0$ such that $\|[k_{\alpha_0} * (u_\beta - u)]v\|_{A(H)} < \varepsilon$ if $\beta \geq \beta_1$. There exists α_0, β_0 and β_1, such that for all $\beta \geq \beta_1 \geq \beta_0$, $\|(u_\beta - u)v\|_{A(H)} \leq 3\varepsilon$. □

Now, let us recall this fact which we will use in the proof of our next result.

Fact 3.7. [16, Proposition 2.22]. *Let H be a hypergroup. If K is a nonempty compact subset of H and U is a neighborhood of K, then there exists $w \in A(H)$ such that*

$$0 \leq w(x) \leq 1, w_{|K} = 1 \textit{ and } supp(w) \subset U.$$

Let K be a compact nonempty subset of H. Set

$$A_K(H) = \{u \in A(H), supp(u) \subset K\}.$$

Now consider the unit sphere of $A_K(H)$:

$$S_{A_K(H)} = \{v \in A_K(H) : \|v\|_{A(H)} = 1\}.$$

We have the following result.

Theorem 3.8. *If Let K be a compact nonempty subset of H, then the weak-* topology coincides with the norm topology on $S_{A_K(H)}$.*

Proof. $S_{A_K(H)}$ is a subset of $A_K(H) \subset B(H)$. Let $v_\alpha, v \in S_{A_K(H)}$ such that v_α converges to v in the weak-* topology of $B(H)$. We have $\|v_\alpha\|_{B(H)} = \|v\|_{B(H)} = 1$. Thus v_α is as in Theorem 3.6 and for every $w \in A(H)$, $\|(v_\alpha - v)w\|_{A(H)}$ converges to 0. By Fact 3.7, we can chose $w \in A(H)$ such that $w_{|K} = 1$, thus $\|(v_\alpha - v)w\|_{A(H)} = \|v_\alpha - v\|_{B(H)}$ converges to 0. □

Let us recall these propositions of Namioka and Phelps. Consider the following condition.
(C): "On the unit sphere, the weak topology coincides with the norm topology."*

Proposition 3.5. [17, Corollary 4.9 and Proposition 4.11] *Let E be a normed linear space such that the norm of E^*(the dual of E) satisfies the condition (C). Then each norm-closed, convex, bounded subset of E^* is contained in the weak-*-closed convex hull of its denting points.*

Proposition 3.6. [19, Corollary 14] *If every bounded closed nonempty convex subset of the Banach space E has a denting point, then every such set is the closed convex hull of its strongly exposed points.*

These two propositions 3.5 and 3.6 implie that every dual Banach space that satisfies condition *(C)* has the RNP.

Proposition 3.7. *$A_K(H)$ is weak-* closed in $B(H)$ and it is a dual Banach space.*

Proof. $A_K(H) = \{u \in A(H), supp(u) \subset K\}$. If the interior $int(K) = \emptyset$ then $A_K(H) = \{0\}$, so we may assume that $int(K) \neq \emptyset$.

For any $x \notin K$, there is some $v \in A(H)$ such that $v(K) = 0$ and $v(x) \neq 0$. (Such v can be taken as $1 - w$ with w as in Fact 3.7).

Now let $(u_\alpha) \in A_K(H)$ and $v \in B(H)$ such that $v = 0$ on K. Since $(u_\alpha) \in A_K(H)$ then $u_\alpha(x) = 0$ for $x \notin K$.

If u_α converges to u in τ_{w^*}, then $0 = u_\alpha v$ converges to uv in τ_{w^*}.

Hence $uv = 0$ and $u(x) = 0$ if $x \notin K$, thus $\{y \in H; u(y) \neq 0\} \subset K$. Thus $u \in A_K(H)$. It follows that $A_K(H)$ is weak-* closed in $B(H)$. Let us denote by M the set $(A_K(H))_\perp = \{\phi \in C^*(H); \langle \phi, v\rangle = 0 \text{ for all } v \in A_K(H)\}$. Using [21, Theorem 4.7 and Theorem 4.9(b)], the Banach space $(C^*(H)/M)^*$ is isometric to

$$M^\perp = \{u \in B(H); \langle u, \phi\rangle = 0 \text{ for all } \phi \in M\}$$

and $A_K(H)$ is the dual of a Banach space, since

$$M^\perp = ((A_K(H))_\perp)^\perp = A_K(H).$$

□

Theorem 3.9. *Let H be a commutative hypergroup and let K be a compact subset of H. Then $A_K(H)$ is a dual Banach space with the RNP.*

Proof. $A_K(H)$ is weak-* closed in $B(H)$ and it is a dual Banach space by Proposition 3.7. By Theorem 3.8, $A_K(H)$ satisfies condition *(C)*. Thus, by Proposition 3.5, each bounded norm closed convex subset $\mathcal{K}$ of $A_K(H)$ has a denting point and hence by Proposition 3.6 every bounded norm closed convex subset $\mathcal{K}$ of $A_K(H)$ is the closed convex hull of its strongly exposed points. Hence $A_K(H)$ has the RNP. □

4. Conclusion

We have studied the RNP for the Fourier space of a compact commutative or non necessary commutative hypergroup. In both cases, the Radon-Nikodým property holds. For a general commutative hypergroup, we proved the $A_K(H)$ also has the Radon-Nikodým property under additionnal conditions. It may be interesting to study the RNP or other geometric properties for the Fourier space of structures of other categories such as groupoids, n-groups, etc.

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