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Generalized Laguerre transforms of sequences

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ABSTRACT. In our present research, we investigate a new class of polynomials which we call the associated polynomials. Firstly, we drive a general identity which generalizes the Laguerre transform of a sequence. More precisely, we give a three-term recurrence formula for calculating the associated polynomials. Secondly, as an application, we define the associated Fibonacci polynomials.

1. INTRODUCTION

The classical Laguerre polynomials $L_n(x)$ [2, 5] are usually defined by the following generating function

$$\frac{1}{1-z}\exp\left(-\frac{xz}{1-z}\right)=\sum_{n\geq 0}L_n(x)\,z^n$$

and can be computed explicitly by

$$L_n(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k!}.$$

The associated Laguerre polynomials $L_n^m(x)$ [1, 2] are defined by

$$L_n^m(x)=(-1)^m\frac{d^m}{dx^m}L_{n+m}(x) \tag{1.1}$$

and can be computed recursively by

$$(n+1)\,L_{n+1}^{m-1}(x)=(m+n)\,L_n^{m-1}(x)-xL_n^m(x).$$

These polynomials can be put in an infinite matrix $\mathcal{L}=(L_n^m(x))_{n,m\geq 0}$ in which the first column $L_n^0(x)=L_n(x)$ is the classical Laguerre polynomials.

$$\mathcal{L}=\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1-x & 2-x & 3-x & 4-x & \cdots \\ \frac{1}{2}x^2-2x+1 & \frac{1}{2}x^2-3x+3 & \frac{1}{2}x^2-4x+6 & \vdots & \\ -\frac{1}{6}x^3+\frac{3}{2}x^2-3x+1 & -\frac{1}{6}x^3+2x^2-6x+4 & \vdots & & \\ \frac{1}{24}x^4-\frac{2}{3}x^3+3x^2-4x+1 & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ L_n^0(x):=L_n & L_n^1(x) & L_n^2(x) & & \end{pmatrix}. \tag{1.2}$$

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Let $(a_n(x))_{n\geq 0}$ be any sequence of polynomials of the form

$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k x^k,$$

where $(\lambda_k)_{k\geq 0}$ a sequence depend on k.

The aim of this paper is to study the properties of the associated polynomials $a_{n,m}(x)$ satisfying the following conditions:

$$a_{n,m}(x) = (-1)^m \frac{d^m}{dx^m} a_{n+m}(x) \tag{1.3}$$

and

$$a_{n+1,m}(x) = \left(1 + \frac{m}{n+1}\right) a_{n,m}(x) - \frac{x}{n+1} a_{n,m+1}(x). \tag{1.4}$$

The idea is to construct an infinite matrix $\mathcal{M} := (a_{n,m}(x))_{n,m\geq 0}$ in which the first row $\alpha_m := a_{0,m}$ of the matrix is the initial sequence and the first column $\beta_n := a_{n,0}(x)$ is the final sequence. More precisely, for x nonzero complex number, we propose to study the three-term recurrence relation (1.4) or more simply, we propose to generalize the following Laguerre transformation. Recall that the Laguerre transform [8] of a sequence $(\alpha_n)_{n\geq 0}$ is the sequence $(\beta_n(x))_{n\geq 0}$ given by

$$\beta_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!} \alpha_k \tag{1.5}$$

and the inverse transform is

$$\alpha_n = n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{-n} \beta_k(x).$$

We note that if $a_{n,m}(x)$ satisfies (1.3) then the initial sequence α_m does not depend on x. For similar studies see [6, 7, 9].

2. The generalized Laguerre transform

Theorem 2.1. *Given an initial sequence $\alpha_m := a_{0,m}$, define the matrix $\mathcal{M}$ associated with the initial sequence by (1.4) then*

i) The entries of the matrix $\mathcal{M}$ are given by

$$a_{n,m}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+m}{n-k} \frac{x^k}{k!} \alpha_{m+k}. \tag{2.6}$$

ii) Suppose that the initial sequence α_{m+s} has the following exponential generating function $A_s(z) = \sum_{k\geq 0} \alpha_{k+s} \frac{z^k}{k!}$. Then, the sequence $(a_{n,s}(x))_{n\geq 0}$ of the sth columns of the matrix $\mathcal{M}$ has an ordinary generating functions $B_s(z;x) = \sum_{n\geq 0} a_{n,s}(x) z^n$, given by

$$B_s(z;x) = \frac{1}{(1-z)^{s+1}} A_s\left(-\frac{xz}{1-z}\right). \tag{2.7}$$

Proof. From

$$a_{n,m}(x) = (-1)^m \frac{d^m}{dx^m} a_{n+m}(x)$$

and using the Laguerre transform (1.5) we obtain

$$a_{n,m}(x) = (-1)^m \frac{d^m}{dx^m} \sum_{k=0}^{n+m} \binom{n+m}{k} \frac{(-x)^k}{k!} \alpha_k$$
$$= (-1)^m \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} \frac{x^{k-m}}{(k-m)!} \alpha_k.$$

After some rearrangement, we find (2.6).

Now by using (2.6), we obtain

$$B_s(z;x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} (-1)^k \binom{n+s}{n-k} \frac{x^k}{k!} \alpha_{s+k} \right) z^n$$
$$= \sum_{k\geq 0} (-1)^k \frac{x^k}{k!} \alpha_{k+s} \sum_{n\geq 0} \binom{n+s}{n-k} z^n.$$

Since

$$\sum_{n\geq 0} \binom{n+s}{n-k} z^n = \frac{z^k}{(1-z)^{s+k+1}},$$

we get

$$B_s(z;x) = \frac{1}{(1-z)^{s+1}} \sum_{k\geq 0} \frac{1}{k!} \alpha_{k+s} \left(-\frac{xz}{1-z} \right)^k.$$

This evidently completes the proof of Theorem. □

Theorem 2.2. *Given final sequence $\beta_n := a_{n,0}(x)$, define the matrix $\mathcal{M}$ associated with the final sequence by*

$$a_{n,m+1}(x) = \left(\frac{n+m+1}{x} \right) a_{n,m}(x) - \left(\frac{n+1}{x} \right) a_{n+1,m}(x), \tag{2.8}$$

then

i) The entries of the matrix $\mathcal{M}$ are given by

$$a_{n,m}(x) = x^{-m} \sum_{k=0}^{m} (-1)^k \frac{(n+m)!}{n!} \binom{m}{m-k} \beta_{n+k}(x). \tag{2.9}$$

ii) Suppose that the final sequence β_{n+s} has the following ordinary generating function $\mathcal{B}_s(z) = \sum_{k\geq 0} \beta_{k+s} z^k$. Then, the sequence $(a_{s,m}(x))_{m\geq 0}$ of the sth row of the matrix $\mathcal{M}$ has an exponential generating function $\mathcal{A}_s(z;x) = \sum_{m\geq 0} a_{s,m}(x) \frac{z^m}{m!}$, given by

$$\mathcal{A}_s(z;x) = \frac{(-1)^s x}{s!(x-z)^{s+1}} \left((x-z)^2 \frac{d}{dz} \right)^s \left(\left(\frac{z}{z-x} \right)^s \mathcal{B}_s\left(\frac{z}{z-x} \right) \right). \tag{2.10}$$

Proof. The verification of (2.9) follows by induction on m. According to (2.9), we have

$$\begin{aligned}\mathcal{A}_s(z;x) &= \sum_{m\geq 0}\left(x^{-m}\sum_{k=0}^{m}(-1)^k\frac{(s+m)!}{s!}\binom{m}{m-k}\beta_{s+k}\right)\frac{z^m}{m!}\\ &= \sum_{k\geq 0}(-1)^k\beta_{s+k}\sum_{m\geq 0}\frac{(s+m)!}{s!m!}\binom{m}{m-k}\left(\frac{z}{x}\right)^m\\ &= \sum_{k\geq 0}(-1)^k\beta_{s+k}\binom{s+k}{s}\sum_{m\geq 0}\binom{s+m}{s+k}\left(\frac{z}{x}\right)^m.\end{aligned}$$

Using the fact that

$$\sum_{m\geq 0}\binom{s+m}{s+k}\left(\frac{z}{x}\right)^m = \frac{x^{s+1}}{(x-z)^{k+s+1}}z^k$$

and

$$\binom{k+s}{s}t^k = \frac{1}{s!}\frac{d^s}{dt^s}t^{k+s}$$

we obtain

$$\begin{aligned}\mathcal{A}_s(z;x) &= \frac{x^{s+1}}{(x-z)^{s+1}}\sum_{k\geq 0}\beta_{s+k}\binom{s+k}{s}\left(-\frac{z}{x-z}\right)^k\\ &= \frac{x^{s+1}}{(x-z)^{s+1}}\sum_{k\geq 0}\beta_{s+k}\frac{1}{(-x)^s s!}\left((x-z)^2\frac{d}{dz}\right)^s\left(\frac{z}{z-x}\right)^{k+s}\\ &= \frac{(-1)^s x}{s!(x-z)^{s+1}}\left((x-z)^2\frac{d}{dz}\right)^s\left[\left(\frac{z}{z-x}\right)^s\mathcal{B}_s\left(\frac{z}{z-x}\right)\right]\end{aligned}$$

which completes the proof. □

Corollary 2.1. *For $n, m \geq 0$, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n+m}{n-k}\frac{x^k}{k!}\alpha_{m+k}=x^{-m}\sum_{k=0}^{m}(-1)^k\frac{(n+m)!}{n!}\binom{m}{k}\beta_{n+k}. \tag{2.11}$$

The identity (2.11) can be viewed as the generalized Laguerre transform which can be reduced, for $m=0$ to the Laguerre transform of the sequence α_n, and for $n=0$ to the inverse Laguerre transform of the sequence β_m.

Example 2.1. Setting the initial sequence $a_{0,m}=1$ in (1.4), we get (1.2). Since $A_m(z) = \exp(z)$, it follows from (2.7) that the final sequence has an ordinary generating function given by

$$\sum_{n\geq 0}L_n^m(x)z^n = \frac{1}{(1-z)^{m+1}}\exp\left(-\frac{xz}{1-z}\right). \tag{2.12}$$

Now, from Corollary 2.1, we have

$$\begin{aligned}L_n^m(x) &= \sum_{k=0}^{n}(-1)^k\binom{n+m}{n-k}\frac{x^k}{k!}\\ &=x^{-m}\sum_{k=0}^{m}(-1)^k\frac{(n+m)!}{n!}\binom{m}{k}L_{n+k}(x).\end{aligned}$$

Theorem 2.3. *We consider a matrix* $\mathcal{L} = \left(L_n^m(x)\right)_{n,m\geq 0}$ *of order* $n+1$ *arising from (1.4) with initial sequence* $a_{0,m} = 1$, *then*

$$\det \mathcal{L} = 1,$$

where det *denotes determinant.*

Proof. Let C_j denote the jth column of $\mathcal{L}$. By applying repeatedly the following operation from $1 \leq r \leq n$

$$C_{n-j} \longleftarrow C_{n-j} - C_{n-j-1} \ (0 \leq j \leq n-r),$$

we get a lower triangular matrix with diagonal equal to 1

$$\mathcal{L} = \left(\sum_{k=0}^{i} (-1)^k \binom{i}{k+j} \frac{x^k}{k!}\right)_{0\leq i,j\leq n}.$$

Since the value of a determinant is unaltered by adding to any one column a linear combination of all the other columns, we get the desired result. □

3. On the associated Fibonacci polynomials

The classical Fibonacci numbers F_n [4] are given by the ordinary generating function

$$\frac{z}{1-z-z^2} = \sum_{n\geq 0} F_n z^n \tag{3.13}$$

and the general term can be expressed as

$$F_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right), \tag{3.14}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Setting the initial sequence $a_{0,m} := -m!F_m$ in (1.4), we get the following matrix

$$\mathcal{F} = \begin{pmatrix} 0 & -1 & -2 & -12 & -72 \\ x & 2x-2 & 12x-6 & 72x-48 & \cdots \\ -x^2+2x & -6x^2+6x-3 & -36x^2+48x-12 & \vdots & \\ 2x^3-3x^2+3x & 12x^3-24x^2+12x-4 & \vdots & & \\ -3x^4+8x^3-6x^2+4x & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ \mathbb{F}_n^0(x) := \mathbb{F}_n(x) & \mathbb{F}_n^1(x) & \mathbb{F}_n^2(x) & & \end{pmatrix}$$

Let denotes the final sequence $a_{n,0} := \mathbb{F}_n(x)$. Now, we can introduce the $m-$associated Fibonacci polynomials.

Definition 3.1. The m-associated Fibonacci polynomials $\mathbb{F}_n^m(x)$ are defined by the following generating function:

$$\frac{m!}{\alpha-\beta}\left(\frac{\beta^m}{(1-z+\beta xz)^{m+1}} - \frac{\alpha^m}{(1-z+\alpha xz)^{m+1}}\right) = \sum_{n\geq 0} \mathbb{F}_n^m(x)\, z^n.$$

For $x = 1$ and $m = 0$, we obtain the classical Fibonacci numbers (3.13). Now, from Corollary 2.1, we have the following explicit formulas

Theorem 3.4. *For* $m \geq 0$

$$\begin{aligned}\mathbb{F}_n^m(x) &= (-1)^m \frac{d^m}{dx^m}\mathbb{F}_{n+m}(x)\\ &= \sum_{k=0}^{n}(-1)^{k+1}\binom{n+m}{n-k}\frac{x^k}{k!}(m+k)!F_{m+k}\\ &= x^{-m}\sum_{k=0}^{m}(-1)^k\frac{(n+m)!}{n!}\binom{m}{k}\mathbb{F}_{n+k}(x).\end{aligned}$$

It it easy to show that $\mathbb{F}_n(x)$ obey the recurrence relation

$$\mathbb{F}_{n+2}(x) = (2-x)\mathbb{F}_{n+1}(x) + \left(x^2+x-1\right)\mathbb{F}_n(x), \tag{3.15}$$

with initial conditions $\mathbb{F}_0(x) = 0$ and $\mathbb{F}_1(x) = x$. From (3.14), we get the Binet's formula for $\mathbb{F}_n(x)$

$$\mathbb{F}_n(x) = \frac{1}{\sqrt{5}}\left(\beta^n(x) - \alpha^n(x)\right),$$

with $\alpha(x) = 1-\alpha x$ and $\beta(x) = 1-\beta x$ denote the roots of the characteristic equation of (3.15).

Conclusions

In this paper, we have presented an investigation of a certain family of polynomials which have the same properties as the associated Laguerre polynomials. The results of the generalized Laguerre transform generalize several well-known results and can be specialized to yield many known identities. More precisely, we have introduced new ways to study several associated polynomials, which have similar properties as those derived from associated Laguerre polynomials. Using a particular sequence, we developed a new extension of Fibonacci polynomials.

Dattoli and Torre in [3] defined two-variable Laguerre polynomials. For future direction, it is interesting to generalize our work to the two-variable cases and deepen the study of the associated Fibonacci polynomials. The applications of specific numbers and polynomials produced by the initial sequence in this paper are also planned to be studied and investigated soon.

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