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Some Hermite-Hadamard type Inequalities for E-preinvex Functions

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ABSTRACT. In this paper, first we establish Hermite-Hadamard type inequalities for a class of preinvex functions, called E-preinvex function. Later, we develop Hermite-Hadamard type inequalities for the functions, whose first and second order derivatives absolute values are E-preinvex.

1. INTRODUCTION

A well known class of non-convexity, namely invexity has acquired an endless range of advantages in both applied and pure mathematics. Mond and Ben-Israel (1986) proposed the notion of invexity. Weir and Mond (1988) introduced preinvex function. Later, Bector and Singh (1991) defined B-vex function. Mohan and Neogy (1995) applied the class of invexity in producing some certain conditions of invex and quasiinvex functions on invex set, that was defined by Hanson. Youness (1999) deduced E-convexity and gained some related results of E-convexity in mathematical programming, later Yang (2001) refined these results. Yang also produced some properties of preinvex functions together with Li. Fulga and Preda (2009) extended the class of invexity and defined E-preinvex and local E-preinvex functions from the ideology of E-convexity and also obtained some properties of E-invex functions and E-prequasiinvex functions.
Today, significance of invexity and preinvexity is not limited to optimization theory, recently researchers have utilized this concept for obtaining new integral inequalities, like Hermite-Hadamard inequality. Noor (2007) derived log-preinvexity from preinvexity and procured Hermite-Hadamard type inequalities for log preinvex functions and two log preinvex functions. Dragomir et al. (2012) revised preinvexity for obtaining Hermite-Hadamard inequality of the functions whose derivatives absolute values are preinvex. Here, we obtain Hermite-Hadamard type inequalities for the generalised class of preinvexity, called E-preinvexity.

2. PRELIMINARIES

First we consider that $E : R^n \to R^n$ and $\gamma : R^n \times R^n \to R^n$ be two mappings, where R^n be an n-dimensional Euclidean space, then we recall some following definitions

Definition 2.1. [4] A set $P \subseteq R^n$ be an invex set with respect to the map $\gamma : R^n \times R^n \to R^n$, if for all $s, t \in P$ and $k \in [0, 1]$, $s + k\gamma(t, s) \in P$.

Definition 2.2. [4, 9] Let $P \subseteq R^n$ be an invex set with respect to the map $\gamma : R^n \times R^n \to R^n$, then function $\psi : P \to R$ is said to be a preinvex function, if for all $s, t \in P$ and $k \in [0, 1]$,

$$\psi(s + k\gamma(t, s)) \leq (1 - k)\psi(s) + k\psi(t)$$

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Definition 2.3. [8] Let $P \subseteq R^n$ is said to be an E-invex set with respect to the map $\gamma : R^n \times R^n \to R^n$, if for all $s, t \in P$ and $k \in [0,1]$, $E(s) + k\gamma(E(t), E(s)) \in P$.

Definition 2.4. [8] Let $P \subseteq R^n$ be an E-invex set with respect to the map $\gamma : R^n \times R^n \to R^n$, then function $\psi : P \to R$ is said to be an E-preinvex function, if for all $s, t \in P$ and $k \in [0,1]$,

$$\psi(E(s) + k\gamma(E(t), E(s))) \leq (1-k)\psi(E(s)) + k\psi(E(t)) \tag{2.1}$$

If we consider that $\gamma(t, s) = t - s$, for all $s, t \in R^n$, then definitions 2.3 and 2.4 reduce to the definitions of E-convex set [20] and E-convex function [20], respectively. Similarly, if the map $E : R^n \to R^n$ is an identity map, then definitions 2.3 and 2.4 reduce to the definitions of invex set and preinvex function, respectively.
If $P \subseteq R^n$ is an E-invex set, then $E(P) \subseteq P$ [8].
If $P_j \subseteq R^n$ and $\forall j \in I$, $\{P_j\}_{j\in I}$ be an arbitrary collection of E-invex (invex) sets, then $\cap_{j\in I} P_j$ also be an E-invex (invex) set. Similarly, if $P \subseteq R^n$ be an E-invex set and $\forall j \in I$, $\{\psi_j\}_{j\in I}$ be a collection of real valued E-preinvex functions, then $sup_{j\in I}\{\psi_j\}(s)$, $\forall s \in P$ is E-preinvex function on E-invex set [8].

Example 2.1. Let $P = [-3, -1] \cup [1, 3]$ and the map $E : R \to R$ is defined as

$$E(s) = \begin{cases} s^2, & \text{if } |s| \leq \sqrt{3} \\ -1, & \text{if } |s| > \sqrt{3} \end{cases}$$

and $\gamma : P \times P \to R$ is defined as

$$\gamma(t, s) = \begin{cases} t - s, & \text{if } t \leq s \\ -3 - s, & \text{if } t > s \end{cases}$$

we see that, set P is E-invex and invex with respect to $\gamma : P \times P \to R$.

Example 2.2. Let the map $E : R \to R$ is defined as

$$E(s) = \begin{cases} 1, & s \leq 0 \\ s, & 1 < s \leq 2 \\ 0, & 0 < s \leq 1 \text{ or } s > 2 \end{cases}$$

and function $\gamma : R \times R \to R$ is defined as

$$\gamma(t, s) = \begin{cases} t - s, & s = t \\ 1 - t, & t > s \\ t - s - 1, & t < s \end{cases}$$

we see that, set R is E-invex and invex with respect to $\gamma : R \times R \to R$. Now we define function $\psi : R \to R$ as

$$\psi(s) = \begin{cases} 0, & s \leq 0 \\ -s + 1, & 0 < s < 1 \\ s, & 1 \leq s \leq 2 \\ 1, & s > 2 \end{cases}$$

we see that function ψ is E-preinvex and preinvex function on set R with respect to $\gamma : R \times R \to R$.

Example 2.3. Let the map $E : R \to R$ is defined as $E(s) = -s^2, \forall s \in R$ and function $\gamma : R \times R \to R$ is defined as

$$\gamma(t,s) = \begin{cases} t-s, & \text{for } t,s \geq 0 \text{ or } t,s \leq 0 \\ s-t, & \text{for } t>0, s<0 \text{ or } t<0, s>0 \end{cases}$$

we see that, set R is E-invex and invex with respect to $\gamma : R \times R \to R$. Now we define function $\psi : R \to R$ as

$$\psi(s) = \begin{cases} 1, & s>0 \\ -s^2, & s \leq 0 \end{cases}$$

we see that function ψ is E-preinvex, $\forall s \in R$ but it is not a preinvex function, if $s,t \leq 0$ and $s<0, t>0$ on set R with respect to $\gamma : R \times R \to R$.

Definition 2.3 follows that any path is started form $E(s)$ to the end point $E(t)$ contains in P, where end point may be any point other than $E(t)$. Now we recall a well known integral inequality, called Hermite-Hadamard inequality.
If $\psi : P \to R$ be a convex function (where J be a sub interval of $(0,\infty)$) and $s,t \in P$ (with $t>s$). Then the following inequality

$$\psi\left(\frac{s+t}{2}\right) \leq \frac{1}{t-s}\int_s^t f(u)\,du \leq \frac{\psi(s)+\psi(t)}{2} \tag{2.2}$$

is hold. In literature, this inequality is known as Hermite-Hadamard integral inequality.

3. Main results

In this section, we always consider that the map $E : R \to R$ be a continuous map and $P \subseteq R$ be an E-invex set with respect to the continuous map $\gamma : P \times P \to R$, then for any $s \in P$, $E(s) \in E(P) \subseteq P$ [8]. If $\psi : P \to R$ be a function defined on E-invex set P then, we obtain Hermite-Hadamard inequality for E-preinvex functions.

Theorem 3.1. *Let P be an open E-invex set and $\psi : P \to (0,\infty)$ be an E-preinvex function on P with respect to $\gamma : P \times P \to R$. Then for any $s,t \in P$ $(E(s) \neq E(t))$ with $E(s) < E(s) + \gamma(E(t),E(s))$, the following inequality holds*

$$\psi\left(\frac{2E(s)+\gamma(E(t),E(s))}{2}\right) \leq \frac{1}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))} \psi(u)\,du \leq \frac{\psi(E(s))+\psi(E(t))}{2}. \tag{3.3}$$

Proof. Let P be an E-invex set, then for any $s,t \in P$ $(E(s) \neq E(t))$ and $k \in [0,1]$, we have $E(s) + k\gamma(E(t),E(s)) \in P$. Now we assume that

$$I = \int_{E(s)}^{E(s)+\gamma(E(t),E(s))} \psi(u)\,du,$$

where $u = E(s) + k\gamma(E(t),E(s)) \in P$. Then

$$I = \gamma(E(t),E(s)) \int_0^1 \psi\left(E(s) + k\gamma(E(t),E(s))\right)\,dk. \tag{3.4}$$

Since $\psi : P \to R$ be an E-preinvex function on E-invex set P, then for any $s,t \in P$ $(E(s) \neq E(t))$ and $k \in [0,1]$

$$I \leq \gamma(E(t),E(s)) \int_0^1 (1-k)\psi(E(s)) + k\psi(E(t))\,dk \leq \gamma(E(t),E(s)) \frac{\psi(E(s))+\psi(E(t))}{2}. \tag{3.5}$$

This gives the right hand side of inequality (3.3). Similarly, we can obtain the left hand side of inequality (3.3) □

Remark 3.1. If $E(s) = s$, $\forall s \in R$, then inequality (3.3) reduces to Hermite-Hadamard inequality (3.1) of [13] for preinvex function.

Remark 3.2. If $E(s) = s, \forall s \in R$ and $\gamma(t, s) = t - s, \forall s, t \in R$, then inequality (3.3) gives the same Hermite-Hadamard inequality (2.2).

Theorem 3.2. *Let P be an open E-invex set and $\psi, \phi : P \to (0, \infty)$ be two E-preinvex functions on E-invex set P with respect to $\gamma : P \times P \to R$. Then for any $s, t \in P$ $(E(s) \neq E(t))$ with $E(s) < E(s) + \gamma(E(t), E(s))$, the following inequality holds*

$$\frac{6}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\phi(u)\,du \leq \left[\nu_1^2+\nu_2^2-\psi(E(s))\psi(E(t))-\phi(E(s))\phi(E(t))\right], \quad (3.6)$$

where $\nu_1 = \psi(E(s)) + \psi(E(t))$ and $\nu_2 = \phi(E(s)) + \phi(E(t))$.

Proof. Let P be an E-invex set, then for any $s, t \in P$ $(E(s) \neq E(t))$ and $k \in [0, 1]$, we have $E(s) + k\gamma(E(t), E(s)) \in P$. Now $u = E(s) + k\gamma(E(t), E(s))$ and ψ, ϕ be two E-preinvex functions, then for any $s, t \in P$ $(E(s) \neq E(t))$ and $k \in [0, 1]$, we get

$$\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\phi(u)\,du$$

$$= \gamma(E(t),E(s))\int_0^1 \psi(E(s)+k\gamma(E(t),E(s))).\phi(E(s)+k\gamma(E(t),E(s)))\,dk$$

$$\leq \frac{\gamma(E(t),E(s))}{2}\int_0^1\left[(\psi(E(s)+k\gamma(E(t),E(s))))^2+(\phi(E(s)+k\gamma(E(t),E(s))))^2\right]dk$$

$$\leq \frac{\gamma(E(t),E(s))}{2}$$

$$\int_0^1\left[((1-k)\psi(E(s))+k\psi(E(t)))^2+((1-k)\phi(E(s))+k\phi(E(t)))^2\right]dk.$$

After solving the right hand side of above inequality, finally we get

$$\frac{6}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\phi(u)\,du \leq \left[\nu_1^2+\nu_2^2-\psi(E(s)).\psi(E(t))-\phi(E(s)).\phi(E(t))\right],$$

where $\nu_1 = \psi(E(s)) + \psi(E(t))$ and $\nu_2 = \phi(E(s)) + \phi(E(t))$. □

Now we deduce Hermite-Hadamard type inequalities for the functions whose first derivatives about values are E-preinvex functions. First, we prove the following lemma:

Lemma 3.1. *Let P be an open E-invex set and $\psi : P \to R$ be a differentiable function on E-invex set P and ψ' is integrable on P. Then for any $s, t \in P$ $(E(s) \neq E(t))$ with $E(s) <$*

$E(s)+\gamma(E(t),E(s))$, the following inequality holds

$$\frac{\gamma(E(t),E(s))}{2}\int_0^1(1-2k)\psi'(E(s)+k\gamma(E(t),E(s)))\,dk = -\frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s))}{2} + \frac{1}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du. \quad (3.7)$$

Proof. Let P be an E-invex set, then for any $s,t\in P$ $(E(s)\neq E(t))$ and $k\in[0,1]$, we have $E(s)+k\gamma(E(t),E(s))\in P$. Then

$$\int_0^1(1-2k)\psi'(E(s)+k\gamma(E(t),E(s)))\,dk = \left[\frac{(1-2k)\psi(E(s)+k\gamma(E(t),E(s)))}{\gamma(E(t),E(s))}\right]_0^1 + \frac{2}{\gamma(E(t),E(s))}\int_0^1\psi(E(s)+k\gamma(E(t),E(s)))\,dk$$
$$= -\frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s))}{\gamma(E(t),E(s))} + \frac{2}{\gamma(E(t),E(s))^2}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du.$$

This completes the proof. □

Theorem 3.3. *Let P be an open E-invex set and $\psi : P\to R$ be a differentiable function P and $|\psi'|$ is E-preinvex on E-invex set P with respect to $\gamma : P\times P\to R$. Then for any $s,t\in P$ $(E(s)\neq E(t))$ with $E(s)<E(s)+\gamma(E(t),E(s))$, the following inequality holds*

$$\left|\frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s)))}{2} - \frac{1}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du\right| \le \frac{|\gamma(E(t),E(s))|}{8}\left[|\psi'(E(s))|+|\psi'(E(t))|\right]. \quad (3.8)$$

Proof. Let P be an E-invex set, then for any $s,t\in P$ $(E(s)\neq E(t))$ and $k\in[0,1]$, we have $E(s)+k\gamma(E(t),E(s))\in P$. Since ψ' is E-preinvex on P, then for any $s,t\in P$ $(E(s)\neq E(t))$ and $k\in[0,1]$, lemma 3.1 follows that

$$\left|\frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s)))}{2} - \frac{1}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du\right|$$
$$\le \left|\frac{\gamma(E(t),E(s))}{2}\int_0^1(1-2k)\psi'(E(s)+k\gamma(E(t),E(s)))\,dk\right|$$
$$\le \frac{|\gamma(E(t),E(s))|}{2}\int_0^1|(1-2k)|\left[(1-k)|\psi'(E(s)|+k|\psi'(E(t))|\right]\,dk$$
$$= \frac{|\gamma(E(t),E(s))|}{8}\left[|\psi'(E(s))|+|\psi'(E(t))|\right],$$

where$\int_0^1|1-2k|.(1-k)\,dk = \int_0^1|1-2k|.k\,dk = 1/4.$ □

Theorem 3.4. *Let P be an open E-invex set and $\psi : P\to R$ be a differentiable function on P and $|\psi'|^{\frac{q}{q-1}}$ is E-preinvex on E-invex set P with respect to $\gamma : P\times P\to R$. Then for any $s,t\in P$*

$(E(s) \neq E(t))$ with $E(s) < E(s) + \gamma(E(t), E(s))$, the following inequality holds

$$\left|\frac{\psi(E(s)) + \psi(E(s) + \gamma(E(t), E(s)))}{2} - \frac{1}{\gamma(E(t), E(s))} \int_{E(s)}^{E(s)+\gamma(E(t),E(s))} \psi(u)\, du\right| \leq \frac{|\gamma(E(t), E(s))|}{2} \frac{\left[|\psi'(E(s))|^{q/q-1} + |\psi'(E(t))|^{q/q-1}\right]^{q-1/q}}{2}. \quad (3.9)$$

Proof. Let P be an E-invex set, then for any $s, t \in P$ $(E(s) \neq E(t))$ and $k \in [0, 1]$, we have $E(s) + k\gamma(E(t), E(s)) \in P$. Since $|\psi'|^{\frac{q}{q-1}}$, $q \neq 1$ is E-preinvex on P, then for any $s, t \in P$ $(E(s) \neq E(t))$ and $k \in [0, 1]$, we get the following inequality from Holder inequality and the proof of theorem 3.3

$$\begin{aligned}
&\left|\frac{\psi(E(s)) + \psi(E(s) + \gamma(E(t), E(s)))}{2} - \frac{1}{\gamma(E(t), E(s))} \int_{E(s)}^{E(s)+\gamma(E(t),E(s))} \psi(u)\, du\right| \\
&\leq \left|\frac{\gamma(E(t), E(s))}{2} \int_0^1 (1 - 2k)\psi'(E(s) + k\gamma(E(t), E(s)))\, dk\right| \\
&\leq \frac{|\gamma(E(t), E(s))|}{2} \left[\int_0^1 |(1 - 2k)|^q\, dk\right]^{1/q} \left(|\psi'(E(s) + k\gamma(E(t), E(s)))|^r\, dk\right)^{1/r} \\
&= \frac{|\gamma(E(t), E(s))|}{2(q + 1)^{1/q}} \left(|\psi'(E(s) + k\gamma(E(t), E(s)))|^r\, dk\right)^{1/r} \\
&\leq \frac{|\gamma(E(t), E(s))|}{2(q + 1)^{1/q}} \left(\int_0^1 \left[(1 - k)|\psi'(E(s))|^r + k|\psi'(E(t))|^r\right] dk\right)^{1/r} \\
&= \frac{|\gamma(E(t), E(s))|}{2(q + 1)^{1/q}} \frac{\left[|\psi'(E(s))|^r + |\psi'(E(t))|^r\right]^{1/r}}{2},
\end{aligned}$$

where $r = q/q - 1$. □

Now we obtain Hermite-Hadamard type inequalities for the functions whose second order derivatives absolute values are E-preinvex. For this, first we prove the following lemma

Lemma 3.2. *Let P be an open E-invex set and $\psi : P \to R$ be a differentiable function on E-invex set P and ψ" is integrable on P. Then for any $s, t \in P$ $(E(s) \neq E(t))$ with $E(s) < E(s) + \gamma(E(t), E(s))$, the following inequality holds*

$$\frac{\psi(E(s)) + \psi(E(s) + \gamma(E(t), E(s)))}{2} - \frac{1}{\gamma(E(t), E(s))} \int_{E(s)}^{E(s)+\gamma(E(t),E(s))} \psi(u)\, du = \frac{(\gamma(E(t), E(s)))^2}{2} \int_0^1 k(1 - k)\psi"(E(s) + k\gamma(E(t), E(s)))\, dk. \quad (3.10)$$

Proof. Let P be an E-invex set, then for any $s,t \in P$ $(E(s) \neq E(t))$ and $k \in [0,1]$, we have $E(s) + k\gamma(E(t),E(s)) \in P$. Then

$$\int_0^1 k(1-k)\psi"(E(s)+k\gamma(E(t),E(s)))\,dk = \left[\frac{k(1-k)\psi'(E(s)+k\gamma(E(t),E(s)))}{\gamma(E(t),E(s))}\right]_0^1$$

$$- \frac{1}{\gamma(E(t),E(s))}\int_0^1 (1-2k)\psi'(E(s)+k\gamma(E(t),E(s)))\,dk$$

$$= -\frac{1}{\gamma(E(t),E(s))}\int_0^1 (1-2k)\psi'(E(s)+k\gamma(E(t),E(s)))\,dk$$

$$= \frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s)))}{\gamma(E(t),E(s))^2} - \frac{2}{(\gamma(E(t),E(s)))^3}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du.$$

This completes the proof. □

Theorem 3.5. *Let P be an open E-invex set and $\psi : P \to R$ be a differentiable function on P and ψ'' is integrable on P. If $|\psi"|$ is E-preinvex on E-invex set P with respect to $\gamma : P \times P \to R$, then for any $s,t \in P$ $(E(s) \neq E(t))$ with $E(s) < E(s)+\gamma(E(t),E(s))$, the following inequality holds*

$$\left|\frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s)))}{2} - \frac{1}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du\right|$$

$$\leq \frac{\gamma(E(t),E(s))^2}{24}\left[|\psi"(E(s))| + |\psi"(E(t))|\right]. \quad (3.11)$$

Proof. Let P be an E-invex set, then for any $s,t \in P$ $(E(s) \neq E(t))$ and $k \in [0,1]$, we have $E(s) + k\gamma(E(t),E(s)) \in P$. Now by lemma 3.2 and E-preinvexity for $|\psi"|$, we get for any $s,t \in P$ $(E(s) \neq E(t))$ and $k \in [0,1]$

$$\left|\frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s)))}{2} - \frac{1}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du\right|$$

$$= \left|\frac{(\gamma(E(t),E(s)))^2}{2}\int_0^1 k(1-k)\psi"(E(s)+k\gamma(E(t),E(s)))\,dk\right|$$

$$\leq \frac{(\gamma(E(t),E(s)))^2}{2}\int_0^1 \left((k-k^2)\left((1-k)|\psi"(E(s)| + k|\psi"(E(t))|\right)\right)\,dk$$

$$= \frac{\gamma(E(t),E(s))^2}{24}\left[|\psi"(E(s))| + |\psi"(E(t))|\right].$$

This completes the proof. □

Theorem 3.6. *Let P be an open E-invex set and $\psi : P \to R$ be a differentiable function on P. If $|\psi"|^{\frac{q}{q-1}}$ is E-preinvex on E-invex set P with respect to $\gamma : P \times P \to R$, $q \neq 1$ and $\psi"$ is integrable on P, then for any $s,t \in P$ $(E(s) \neq E(t))$ with $E(s) < E(s)+\gamma(E(t),E(s))$, the following inequality holds*

$$\left|\frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s)))}{2} - \frac{1}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du\right|$$

$$\leq \frac{\gamma(E(t),E(s))^2}{16}(\sqrt{\pi})^{1/q}\left(\frac{\Gamma(1+q)}{\Gamma(\frac{3}{2}+q)}\right)^{1/q}\left[|\psi"(E(s))|^r + |\psi"(E(t))|^r\right]^{1/r}. \quad (3.12)$$

Proof. Let P be an E-invex set, then for any $s,t \in P$ $(E(s) \neq E(t))$ and $k \in [0,1]$, we have $E(s)+k\gamma(E(t),E(s)) \in P$. Now by E-preinvexity for $|\psi"|$ and Holder integral inequality, for any $s,t \in P$ $(E(s) \neq E(t))$ and $k \in [0,1]$, lemma 3.2 gives

$$\left|\frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s)))}{2} - \frac{1}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du\right|$$
$$= \frac{\gamma(E(t),E(s))^2}{2}\int_0^1 k(1-k)\,|\psi"(E(s)+k\gamma(E(t),E(s)))|\,dk$$
$$\leq \frac{\gamma(E(t),E(s))^2}{2}\left(\int_0^1 (k-k^2)^q\,dk\right)^{1/q}\left(\int_0^1 |\psi"(E(s)+k\gamma(E(t),E(s)))|^r\,dk\right)^{1/r}$$
$$\leq \frac{\gamma(E(t),E(s))^2}{2}\left(\frac{2^{-1-2q}\sqrt{\pi}\Gamma(1+q)}{\Gamma(\frac{3}{2}+q)}\right)^{1/q}$$
$$\left(\int_0^1 ((1-k)|\psi"(E(s))|^r + k|\psi"(E(t))|^r)\,dk\right)^{1/r}$$
$$= \frac{\gamma(E(t),E(s))^2}{16}(\sqrt{\pi})^{1/q}\left(\frac{\Gamma(1+q)}{\Gamma(\frac{3}{2}+q)}\right)^{1/q}\left[|\psi"(E(s))|^r + |\psi"(E(t))|^r\right]^{1/r}.$$

This completes the proof. □

Theorem 3.7. *Let P be an open E-invex set and $\psi : P \to R$ be a differentiable function on P. If $|\psi"|^{\frac{q}{q-1}}$ is E-preinvex on E-invex set P with respect to $\gamma : P \times P \to R$, $(q \neq 1)$ and $\psi"$ is integrable on P, then for any $s,t \in P$ $(E(s) \neq E(t))$ with $E(s) < E(s)+\gamma(E(t),E(s))$, the following inequality holds*

$$\left|\frac{\psi(E(s))+\psi(E(s)+\gamma(E(t),E(s)))}{2} - \frac{1}{\gamma(E(t),E(s))}\int_{E(s)}^{E(s)+\gamma(E(t),E(s))}\psi(u)\,du\right|$$
$$\leq \frac{\gamma(E(t),E(s))^2}{12}\left(\frac{1}{2}\right)^r\left[|\psi"(E(s))|^r + |\psi"(E(t))|^r\right]^{1/r}. \quad (3.13)$$

Proof. Let P be an E-invex set, then for any $s,t \in P$ $(E(s) \neq E(t))$ and $k \in [0,1]$, we have $E(s)+k\gamma(E(t),E(s)) \in P$. Now by E-preinvexity for $|\psi"|$ and weighted power mean inequality, for any $s, t \in P$ $(E(s) \neq E(t))$ and $k \in [0,1]$, lemma 3.2 gives

$$\left| \frac{\psi(E(s)) + \psi(E(s) + \gamma(E(t), E(s)))}{2} - \frac{1}{\gamma(E(t), E(s))} \int_{E(s)}^{E(s)+\gamma(E(t),E(s))} \psi(u)\, du \right|$$

$$= \frac{\gamma(E(t), E(s))^2}{2} \int_0^1 k(1-k)\, |\psi"(E(s) + k\gamma(E(t), E(s)))|\, dk$$

$$\leq \frac{\gamma(E(t), E(s))^2}{2} \left(\int_0^1 (k-k^2)\, dk \right)^{1-1/r} \left(\int_0^1 (k-k^2)\, |\psi"(E(s) + k\gamma(E(t), E(s)))|^r\, dk \right)^{1/r}$$

$$\leq \frac{\gamma(E(t), E(s))^2}{2} \left(\frac{1}{6}\right)^{(r-1)/r} \left(\int_0^1 (k-k^2)\, [(1-k)|\psi"(E(s))|^r + k|\psi"(E(t))|^r]\, dk \right)^{1/r}$$

$$\leq \frac{\gamma(E(t), E(s))^2}{2} \left(\frac{1}{6}\right)^{(r-1)/r} \left[\frac{1}{12} |\psi"(E(s))|^r + |\psi"(E(t))|^r \right]^{1/r}$$

$$= \frac{\gamma(E(t), E(s))^2}{12} \left(\frac{1}{2}\right)^r [|\psi"(E(s))|^r + |\psi"(E(t))|^r]^{1/r}.$$

This completes the proof. □

4. Conclusions

In this paper, if we take $E(s) = s, \forall s \in R$, then we obtain some results of [2, 13, 14] of Hermite-Hadamard inequality for preinvex functions in a special case. This shows that the class of E-preinvex functions is more general than that of preinvex functions in the establishment of Hermite-Hadamard type inequalities.

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