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A Jensen-type inequality in the framework of 2-convex systems

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ABSTRACT. Let A_S be the solution set of the system $x_1+x_2+\ldots+x_n = ns$, $e(x_1)+e(x_2)+\ldots+e(x_n) = nk$, $x_1 \geq x_2 \geq \ldots \geq x_n$, where $e : I \to \mathbb{R}$ is a (fully extended) strictly convex or concave function. We call such a system 2–convex and prove the existence of two special points $\omega, \Omega \in A_S$ such that for all $x \in A_S$ and for all $f : I \to \mathbb{R}$ strictly 3-convex with respect to e, the following inequality holds: $\forall x \in A_S \Rightarrow E_f(\omega) \leq E_f(x) \leq E_f(\Omega)$ where $E_f(x) = f(x_1)+f(x_2)+\ldots+f(x_n)$. This may be seen as a broader version of the equal variable method of V. Cîrtoaje. It follows that ω and Ω have at most three distinct components and we also give a detailed analysis of their structure.

1. INTRODUCTION

Let $I \subseteq \mathbb{R}$ be an interval. For any function $f : I \to \mathbb{R}$ we define $E_f : I^n \to \mathbb{R}$ by

$$E_f(x) = f(x_1) + f(x_2) + \ldots + f(x_n) \;\; \forall x = (x_1, \ldots, x_n) \in I^n \tag{1.1}$$

If $s \in I$, $\bar{s} = (s, \ldots, s)$ and $A = \{(x_1, \ldots, x_n) \in I^n | x_1 + x_2 + \ldots + x_n = ns\}$ then the well-known Jensen's inequality states that for any convex function $f : I \to \mathbb{R}$

$$x \in A \Rightarrow E_f(x) \geq E_f(\bar{s}) \tag{1.2}$$

Our main objective is to get inequalities of type 1.2 when A is the solution set of a system defined by two equations (not only one, as in the above case of Jensen's inequality). For this, we define here both a general type of two equations system (2–convex systems) and a suitable class of functions f that satisfy the corresponding inequalities of type 1.2.

Such extensions of Jensen inequality have been previously studied by V. Cîrtoaje in [2] and [3] under the name of *equal variable method*. See also [4] for many applications and examples of the same author. Our main result 3.5 is a direct generalization of V. Cîrtoaje results to a broader type of systems (see Remark 1.2).

For $A \subseteq \mathbb{R}$ we denote by $\overline{A}$ and $\mathring{A}$ the closure set and, respectively, the interior set of A.

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval. A continuous, convex function $e : I \to \mathbb{R}$ is called fully extended on I if it can no more be extended by continuity at any point of $\overline{I} \setminus I$.

Let $m = \inf(I) \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $M = \sup(I) \in \overline{\mathbb{R}}$ and $e : I \to \mathbb{R}$ fully extended on I. Using known properties of convex functions, we infer from the above definition that, if $m \notin I$, then either $m = -\infty$, or m is finite but $\lim_{x\to m} e(x) = +\infty$ (and similarly for M).

Definition 1.2. A 2–convex system is a system of the form $\begin{cases} x_1 + x_2 + \ldots + x_n = ns \\ e(x_1)+e(x_2)+\ldots+e(x_n)=nk \\ x_1 \geq x_2 \geq \ldots \geq x_n \end{cases}$

where $n \geq 3$, $e : I \to \mathbb{R}$ is a continuous, *strictly* convex, **fully extended on I** function and

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$s,k \in \mathbb{R}$, $s \in \mathring{I}$. We also denote it by $S(e,s,k,n)$ and the solution set by A_S. We consistently use the notation $I = I_S$ and $m = \inf(I_S) \in \overline{\mathbb{R}}$, $M = \sup(I_S) \in \overline{\mathbb{R}}$.

Remark 1.1. We also consider 2-concave systems $S(e,s,k,n)$ (for which the function e is strictly concave). For each system $S(e,s,k,n)$ we associate a dual one $S'(-e,s,-k,n)$ and, clearly, $A_{S'} = A_S$. The dual of a 2-concave system is a 2–convex system (and vice versa).

Remark 1.2. V. Cîrtoaje's original theorems correspond to the particular case of a system $S(e,s,k,n)$ where e is of the form $e(x) = x^r$ or $e(x) = \ln(x)$ and I_S is an appropriate interval of the type $[0,\infty)$, $(0,\infty)$ or $\mathbb{R}$ (see [2], [3]).

Definition 1.3. Let $f, e : I \subseteq \mathbb{R} \to \mathbb{R}$ be to functions continuous on I and differentiable on $\mathring{I}$. We say that f is (strictly) 3–convex with respect to e if there exists a (strictly) convex function $g : J \to \mathbb{R}$ with $e'(\mathring{I}) \subseteq J$ such that $f' = g \circ e'$ on $\mathring{I}$.

Remark 1.3. In the particular case of $e(x) = x^2$ we get the definition of the usual 3-convex functions (in an equivalent form). See for example [8].

Remark 1.4. If f is 3–convex with respect to e, then it is also 3–convex with respect to $h = -e$. Indeed, we know that there exists a function $g : J \to \mathbb{R}$ strictly convex with $e'(\mathring{I}_S) \subseteq J$ such that $f' = g \circ e'$. Let $g_1 : -J \to \mathbb{R}$, $g_1(y) = g(-y)$ and it's clear that g_1 is also strictly convex and $f'(x) = g(e'(x)) = g_1(-e'(x)) = g_1(h'(x))$, hence $f' = g_1 \circ h'$.

For $x \in \mathbb{R}^n$ and $1 \le i \le n$ we define $\begin{cases} T_i(x) = x_1 + \ldots + x_i \\ B_i(x) = x_i + \ldots + x_n \end{cases}$ (the top and bottom sums). Using these notations, we can define the classical majorization relation $\preccurlyeq$ like this:

$$x \preccurlyeq y \Leftrightarrow \begin{cases} T_n(x) = T_n(y) \\ T_i(x) \le T_i(y) \ \forall i \in \{1,2,\ldots,n-1\} \end{cases} \quad \text{(for any two decreasing n-tuples } x, y\text{)}.$$

Remark 1.5. The above condition $T_i(x) \le T_i(y) \ \forall i \in \{1,2,\ldots,n-1\}$ can be replaced with:

$$\exists p \in \{1,2,\ldots,n\} \text{ such that } \begin{cases} T_i(x) \le T_i(y) & \forall i \in \{1,2,\ldots p-1\} \\ B_i(x) \ge B_i(y) & \forall i \in \{p+1,\ldots,n\} \end{cases}$$

because for $p+1 \le i \le n$ we have $B_i(x) \ge B_i(y) \Leftrightarrow T_n(x) - T_{i-1}(x) \ge T_n(y) - T_{i-1}(y) \Leftrightarrow T_{i-1}(x) \le T_{i-1}(y)$. Hence $T_i(x) \le T_i(y) \ \forall i \in \{p,\ldots,n-1\}$ and these inequalities, together with $T_i(x) \le T_i(y) \ \forall i \in \{1,2,\ldots,p-1\}$, give us $T_i(x) \le T_i(y) \ \forall i \in \{1,2,\ldots,n-1\}$.

We state here the classical result of Hardy-Littlewood-Polya (HLP theorem – see [5]), also called the majorization inequality or Karamata inequality (see [6]):

Theorem 1.1. *(HLP) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous convex function and $x, y \in I^n$. Then*

$$x \preccurlyeq y \Rightarrow E_f(x) \le E_f(y)$$

Moreover, if f is strictly convex, then the equality occurs if and only if $x = y$.

In the following, we will use this theorem extensively and, typically, the justification for the majorization step $x \preccurlyeq y$ will be based on the Remark 1.5.

2. Preliminary results

Lemma 2.1. *Let $S(e,s,k,n)$ be a non-empty 2–convex system, $m = \inf(I_S)$, $M = \sup(I_S)$.*

(a) If $M \notin I_S$ then there exists an $M_0 \in I_S$ such that $\forall (x_1,\ldots,x_n) \in A_S \Rightarrow x_1 \le M_0$
(b) If $m \notin I_S$ then there exists an $m_0 \in I_S$ such that $\forall (x_1,\ldots,x_n) \in A_S \Rightarrow x_n \ge m_0$

Proof. (a) *Case 1.* M is finite, hence $\lim_{t\to M} e(t) = +\infty$. We consider two subcases.

Subcase 1.1 m is finite. First, we will show that e is bounded below on I_S.

Assume $m \in I_S$. Because $\lim_{t\to M} e(t) = +\infty$ we find an $\varepsilon > 0$ with $e(t) \geq 1\,\forall t \in (M - \varepsilon, M)$. Let $C = \inf_{t\in[m,M-\varepsilon]} e(t)$. Because e is continuous on the compact set $[m, M-\varepsilon]$ it follows that $C \in \mathbb{R}$. Thus $e(t) \geq C_0 = \min\{1, C\}$ on I_S.

Assume now $m \notin I_S$. Because $\lim_{t\to m} e(t) = \lim_{t\to M} e(t) = +\infty$ there is an $\varepsilon > 0$ with $e(t) \geq 1\,\forall t \in (m, m+\varepsilon) \cup (M-\varepsilon, M)$ (and $m+\varepsilon < M-\varepsilon$). Let $C = \inf_{t\in[m+\varepsilon,M-\varepsilon]} e(t)$ so $C \in \mathbb{R}$ and $e(t) \geq C_0 = \min\{1, C\}$ on I_S. Thus, e is bounded below on I_S in all situations.

Now, since $\lim_{t\to M} e(t) = +\infty$ there is an $M_0 < M$ such that $e(t) > nk - (n-1)C_0$ $\forall t \in (M_0, M)$. But $e(x_1) = nk - [e(x_2) + \ldots + e(x_n)] \leq nk - (n-1)C_0$ and so $x_1 \leq M_0$.

Subcase 1.2 $m = -\infty$. This subcase can be reduced to the previous one. Observe first that $x_n = ns - (x_1 + \ldots + x_{n-1}) \geq ns - (n-1)M \overset{def}{=} m_0$ and, obviously, the system $S'(e_{\big|[m_0,M)}, s, k, n)$ has $A_{S'} = A_S$. But for S' we can apply the subcase 1.1 because m_0 is finite etc.

Case 2. $M = +\infty$. Fix $t_1 > s > t_2 > m$ and consider the support lines given by $\varphi_1(t) = \alpha_1 t + \beta_1$, $\varphi_2(t) = \alpha_2 t + \beta_2$ where $\alpha_1 = e'_+(t_1)$, $\alpha_2 = e'_+(t_2)$. From the strict convexity of e we infer that $\alpha_1 > \alpha_2$ and $e(t) \geq \varphi_1(t)$, $e(t) \geq \varphi_2(t)\,\forall t \in \mathbb{R}$. Thus,

$$\begin{aligned} nk &= e(x_1) + [e(x_2) + \ldots + e(x_n)] \geq \varphi_1(x_1) + [\varphi_2(x_2) + \ldots + \varphi_2(x_n)] \\ &= \alpha_1 x_1 + \beta_1 + \alpha_2(x_2 + \ldots + x_n) + (n-1)\beta_2 = \alpha_1 x_1 + \beta_1 + \alpha_2(ns - x_1) + (n-1)\beta_2 \end{aligned}$$

Hence, $nk \geq x_1(\alpha_1 - \alpha_2) + C$ where $C = ns\alpha_2 + \beta_1 + (n-1)\beta_2$ and so $x_1 \leq M_0 \overset{def}{=} \frac{nk-C}{\alpha_1-\alpha_2}$.

(b) The proof is similar to (a). □

Theorem 2.2. *Let $S(e, s, k, n)$ be a 2–convex system. Then*

(a) There exists a compact interval $I_0 = [m_0, M_0] \subseteq I_S$ such that $A_S \subseteq I_0^n$.
(b) A_S is a compact set.

Proof. If A_S is empty the theorem is trivially true, hence we can suppose in the following that A_S is non-empty. If $M \in I_S$ we choose $M_0 = M$. If not, we use Lemma 2.1 to find such an M_0 and for the left side we proceed similarly. Next, we write A_S as $A_1 \cap A_2 \cap E_1 \ldots \cap E_{n-1}$ where

$E_p = \{x \in \mathbb{R}^n | x_{p+1} - x_p \leq 0\} \quad \forall 1 \leq p \leq n-1$
$A_1 = \{x \in \mathbb{R}^n | x_1 + x_2 + \ldots x_n = ns\}$
$A_2 = \{x \in I_0^n | e(x_1) + e(x_2) + \ldots e(x_n) = nk\}$

and, because all these sets are closed, we conclude that A_S is a compact set. □

Remark 2.6. Hence, for every system $S(e, s, k, n)$ we can find an equivalent "compact" system $S_0(e_{|I_{S_0}}, s, k, n)$ with $I_{S_0} = [m_0, M_0] \subseteq I_S$ and $A_{S_0} = A_S$.

3. Main results

Lemma 3.2. *Let $S(e, s, k, 3)$ be a non-empty 2–convex system and $x, y \in A_S$ such that $y_1 > x_1$. Then*

$$y_1 > x_1 \geq x_2 > y_2 \geq y_3 > x_3$$

Proof. We only show that $x_2 > y_2$ and $y_3 > x_3$, the other inequalities being obvious. If $x_3 \geq y_3$ then, using the fact that $x_1 < y_1$, we deduce that $x \prec y$ (strictly majorization) and from HLP theorem we get $e(x_1) + e(x_2) + e(x_3) < e(y_1) + e(y_2) + e(y_3)$ so $3k < 3k$, a contradiction. Thus $y_3 > x_3$. Next, if $x_2 \leq y_2$, then using $x_1 < y_1$ we infer that $x_1 + x_2 < y_1 + y_2$ so $x \prec y$ (strictly majorization) and applying HLP theorem we get a contradiction exactly as above. So we also have $x_2 > y_2$. □

The next theorem is an extension of an interesting result from [1] (see also [9], [7], [8]).

Theorem 3.3. *Let $S(e, s, k, 3)$ be a non-empty 2–convex system with $e \in C^1(\overset{\circ}{I_S})$ and $f : I_S \to \mathbb{R}$ strictly 3–convex with respect to e. Then*

$$\forall x, y \in A_S, \;\; x_1 < y_1 \Rightarrow f(x_1) + f(x_2) + f(x_3) < f(y_1) + f(y_2) + f(y_3)$$

Proof. f is strictly 3–convex with respect to e and so there exists $g : J \to \mathbb{R}$ strictly convex with $e'(\overset{\circ}{I_S}) \subseteq J$ such that $f' = g \circ e'$. According to Lemma 3.2, if $y_1 > x_1$ then $y_1 > x_1 \geq x_2 > y_2 \geq y_3 > x_3$ and, for enough large integers $p \geq p_0$, we define the intervals

$$A_1^p = [x_1, y_1 - \frac{1}{p}], \; A_2 = [y_2, x_2], \; A_3^p = [x_3 + \frac{1}{p}, y_3] \subset \overset{\circ}{I_S}$$

Because e' is continuous strictly increasing and A_1^p, A_2, A_3^p are compact sets with disjoint interiors we get also that $B_1^p = e'(A_1^p), \; B_2 = e'(A_2), \; B_1^p = e'(A_1^p)$ are compact intervals with disjoint interiors and their ordering on x-axis is exactly that of A_1^p, A_2, A_3^p.

Next, we consider the linear function $L : \mathbb{R} \to \mathbb{R}, \; L(r) = \alpha + \beta r$ that agree with g at the endpoints of B_2 and, because g is convex, we have

$$\begin{aligned} g(r) &\geq L(r) \; \forall r \in B_1^p \cup B_3^p \\ g(r) &\leq L(r) \; \forall r \in B_2 \end{aligned} \tag{3.3}$$

Since g is *strictly* convex we also have strict versions of these inequalities, for example

$$g(r) < L(r) \; \forall r \in \overset{\circ}{B_2} \tag{3.4}$$

Using 3.3 we infer that

$$E_1^p \overset{def}{=} \int_{A_1^p} g(e'(t))dt + \int_{A_3^p} g(e'(t))dt \geq \int_{A_1^p} L(e'(t))dt + \int_{A_3^p} L(e'(t))dt \overset{def}{=} E_2^p \tag{3.5}$$

But $g(e'(t) = f'(t)$ so $E_1^p = f(y_1 - \frac{1}{p}) - f(x_1) + f(y_3) - f(x_3 + \frac{1}{p})$ and because f is continuous on I_S it follows that

$$\lim_{p \to \infty} E_1^p = f(y_1) - f(x_1) + f(y_3) - f(x_3)$$

On the other hand, $E_2^p = \int_{A_1^p} [\alpha + \beta e'(t)]dt + \int_{A_3^p} [\alpha + \beta e'(t)]dt$

$$= \alpha(l(A_1^p) + l(A_3^p)) + \beta(e(y_1 - \frac{1}{p}) - e(x_1)) + \beta(e(y_3) - e(x_3 + \frac{1}{p}))$$

and using the continuity of e and the initial $\begin{cases} x_1 + x_2 + x_3 = y_1 + y_2 + y_3 \\ e(x_1) + e(x_2) + e(x_3) = e(y_1) + e(y_2) + e(y_3) \end{cases}$ conditions, we infer that

$$\begin{aligned} \lim_{p \to \infty} E_2^p &= \alpha(y_1 - x_1 + y_3 - x_3) + \beta(e(y_1) - e(x_1) + e(y_3) - e(x_3)) \\ &= \alpha(x_2 - y_2) + \beta(e(x_2) - e(y_2)) \\ &= \alpha l(A_2) + \beta(e(x_2) - e(y_2)) \\ &= \int_{A_2} L(e'(t))dt \end{aligned}$$

But using 3.4 we can write further

$$\int_{A_2} L(e'(t))dt > \int_{A_2} g(e'(t)) = \int_{A_2} f'(t)dt = f(x_2) - f(y_2)$$

Thus, passing to the limit in 3.5 we get $\lim_{p\to\infty} E_1^p \geq \lim_{p\to\infty} E_2^p$, that is

$$f(y_1) - f(x_1) + f(y_3) - f(x_3) \geq \int_{A_2} L(e'(t))dt > f(x_2) - f(y_2)$$

and the conclusion follows. □

Theorem 3.4. *Let $S(e,s,k,3)$ be a non-empty 2–convex system and a point $(x_0,y_0,z_0) \in A_S$.*

(a) If $M > x_0 \geq y_0 > z_0 \geq m$ then there is $x_0' \in I_S,\ x_0' > x_0$ such that

$$\forall x \in (x_0, x_0')\ \exists y, z \in I_S \text{ with } (x,y,z) \in A_S$$

(b) If $M \geq x_0 > y_0 \geq z_0 > m$ then there exists $z_0' \in I_S,\ z_0' < z_0$ such that

$$\forall z \in (z_0', z_0)\ \exists x, y \in I_S \text{ with } (x,y,z) \in A_S$$

where $m = \inf(I_S)$, $M = \sup(I_S)$.

Proof. (a) Let $\varepsilon_0' = \min(M - x_0, y_0 - z_0)$. We see that $M \geq x_0 + \varepsilon \geq \frac{y_0+z_0-\varepsilon}{2} \geq m$ for all $\varepsilon \in [0, \varepsilon_0']$ and thus we can define the function $R : [0, \varepsilon_0'] \to \mathbb{R}$ given by

$$R(\varepsilon) = e\left(x_0 + \varepsilon\right) + e\left(\frac{y_0 + z_0 - \varepsilon}{2}\right) + e\left(\frac{y_0 + z_0 - \varepsilon}{2}\right)$$

By Jensen's inequality we get $R(0) = e(x_0) + 2e\left(\frac{y_0+z_0}{2}\right) < e(x_0) + e(y_0) + e(z_0) = 3k$ (the inequality being strict because $y_0 \neq z_0$) and, using the continuity of R, we can fix an $0 < \varepsilon_0 \leq \varepsilon_0'$ such that $R(\varepsilon) < 3k\ \forall \varepsilon \in [0, \varepsilon_0]$.

Now, for every fixed $0 < \varepsilon \leq \varepsilon_0$ we define $I_\varepsilon = \left[0, \frac{y_0-z_0-\varepsilon}{2}\right]$ and observe that for $\theta \in I_\varepsilon$ we have $M \geq x_0 + \varepsilon \geq y_0 - \varepsilon - \theta \geq z_0 + \theta \geq m$. Let $H_\varepsilon : I_\varepsilon \to \mathbb{R}$ given by

$$H_\varepsilon(\theta) = e(x_0 + \varepsilon) + e(y_0 - \varepsilon - \theta) + e(z_0 + \theta)$$

and using HLP theorem for the strictly convex function e we get

$$H_\varepsilon(0) = e(x_0 + \varepsilon) + e(y_0 - \varepsilon) + e(z_0) > e(x_0) + e(y_0) + e(z_0) = 3k$$

(the inequality being strict because $\varepsilon > 0$). On the other hand,

$$H_\varepsilon\left(\frac{y_0 - z_0 - \varepsilon}{2}\right) = e\left(x_0 + \varepsilon\right) + e\left(\frac{y_0 + z_0 - \varepsilon}{2}\right) + e\left(\frac{y_0 + z_0 - \varepsilon}{2}\right) = R(\varepsilon) < 3k$$

and using the continuity of H_ε there exists $\theta = \theta_\varepsilon \in I_\varepsilon$ with $H_\varepsilon(\theta) = 3k$, that is

$$(x_0 + \varepsilon, y_0 - \varepsilon - \theta, z_0 + \theta) \in A_S$$

and if we define $x_0' = x_0 + \varepsilon_0$ the conclusion follows.

(b) (sketch) As above, let $\varepsilon_0' = \min(z_0 - m, x_0 - y_0)$ and $R : [0, \varepsilon_0'] \to \mathbb{R}$ given by $R(\varepsilon) = e\left(\frac{x_0+y_0+\varepsilon}{2}\right) + e\left(\frac{x_0+y_0+\varepsilon}{2}\right) + e\left(z_0 - \varepsilon\right)$. It follows that $R(0) < 3k$ and so we can fix an $0 < \varepsilon_0 \leq \varepsilon_0'$ such that $R(\varepsilon) < 3k\ \forall \varepsilon \in [0, \varepsilon_0]$. For every fixed $0 < \varepsilon \leq \varepsilon_0$ we define $I_\varepsilon = \left[0, \frac{x_0-y_0-\varepsilon}{2}\right]$ and let $H_\varepsilon : I_\varepsilon \to \mathbb{R},\ \ H_\varepsilon(\theta) = e(x_0 - \theta) + e(y_0 + \varepsilon + \theta) + e(z_0 - \varepsilon)$.

As above, we get $H_\varepsilon(0) > 3k,\ H_\varepsilon(\frac{x_0-y_0-\varepsilon}{2}) = R(\varepsilon) < 3k$. Using the continuity of H_ε there exists $\theta = \theta_\varepsilon \in I_\varepsilon$ with $H_\varepsilon(\theta) = 3k$, that is $(x_0 - \theta, y_0 + \varepsilon + \theta, z_0 - \varepsilon) \in A_S$ etc. □

Corollary 3.1. *Let $S(e,s,k,3)$ be a non-empty 2–convex system with $e \in C^1(\mathring{I}_S)$ and $m = \inf(I_S)$, $M = \sup(I_S)$. Let $f : I_S \to \mathbb{R}$ be a strictly 3–convex function with respect to e.*

(a) If E_f has a maximum value at the point $(c_1,c_2,c_3) \in A_S$ then $c_1 = M$ or $c_2 = c_3$.

(b) If E_f has a minimum value at the point $(c_1,c_2,c_3) \in A_S$ then $c_1 = c_2$ or $c_3 = m$.

Proof. We prove only (a), the (b) being similar. Assume that $M > c_1 \geq c_2 > c_3$. Then, according to the Theorem 3.4, there exist solutions $(c_1', c_2', c_3') \in A_S$ with $c_1' > c_1$. On the other hand, by Theorem 3.3, it follows that $E_f(c') > E_f(c)$ and so we get a contradiction. □

Theorem 3.5. *Let $S(e, s, k, n)$ be a non-empty 2–convex (or 2-concave) system with $e \in C^1(\mathring{I_S})$. If $m = \inf(I_S)$, $M = \sup(I_S)$ then*

(a) There is an unique point $\Omega \in A_S$ of the form $(\underbrace{M, \ldots, M}_{0 \leq r \leq n-2}, a, \underbrace{b, \ldots, b}_{n-r-1})$ with $M \geq a \geq b$ and an unique point $\omega \in A_S$ of the form $(\underbrace{a, \ldots, a}_{n-t-1}, b, \underbrace{m, \ldots, m}_{0 \leq t \leq n-2})$ with $a \geq b \geq m$.

(b) If $f : I_S \to \mathbb{R}$ is strictly 3–convex with respect to e then

$$\forall x \in A_S \Rightarrow E_f(\omega) \leq E_f(x) \leq E_f(\Omega)$$

The equality occurs if and only if $x = \omega$ (on left) or $x = \Omega$ (on right).

Proof. We will prove the theorem first for the case of a 2–convex system.

(a) For the *existence* part we observe first that there exists at least a function $f_0 : I_S \to \mathbb{R}$ strictly 3–convex with respect to e. Indeed, it's easy to see that, for example, $f_0(t) = \int_{t_0}^{t} (e'(s))^2 ds$ is such a function. For this particular function f_0 we consider $E_{f_0} : A_S \to \mathbb{R}$ (defined as in 1.1) and, because E_{f_0} is continuous on the compact set A_S, we get a point $c \in A_S$ for which $E_{f_0}(c) = \sup_{A_S} E_{f_0}$. **The ideea is to show that c is exactly of the desired form** $(\underbrace{M, \ldots, M}_{0 \leq r \leq n-2}, a, \underbrace{b, \ldots, b}_{n-r-1})$ with $M \geq a \geq b$ and for this is enough to prove that for every $1 \leq i < j < k \leq n$ the triple (c_i, c_j, c_k) has $c_i = M$ or $c_j = c_k$. We consider the 3 variable system $S'(e, s', k', 3)$ given by

$$\begin{cases} x_1' + x_2' + x_3' = c_i + c_j + c_k = 3s' \\ e(x_1') + e(x_2') + e(x_3') = e(c_i) + e(c_j) + e(c_k) = 3k' \\ x_1' \geq x_2' \geq x_3' \end{cases}$$

and we observe that $(c_i, c_j, c_k) \in A_{S'}$ must also maximize the sum $f_0(x_1') + f_0(x_2') + f_0(x_3')$ over $A_{S'}$ because, assuming the contrary, we get an $(x_1', x_2', x_3') \in A_{S'}$ such that

$$f_0(x_1') + f_0(x_2') + f_0(x_3') > f_0(c_i) + f_0(c_j) + f_0(c_k)$$

and if we consider the n-tuple c' constructed from c by replacing (c_i, c_j, c_k) with (x_1', x_2', x_3') (and, if necessary, reordering it) it follows that $E_{f_0}(c') > E_{f_0}(c)$, impossible. Thus, we can apply Corollary 3.1 to $(c_i, c_j, c_k) \in A_{S'}$ and conclude that $c_i = M$ or $c_j = c_k$, as desired.

Now, for the *uniqueness* part, let $c, c' \in A_S$ of the same form

$$c = (\underbrace{M, \ldots, M}_{0 \leq r \leq n-2}, a, \underbrace{b, \ldots, b}_{n-r-1}), \ c' = (\underbrace{M, \ldots, M}_{0 \leq r' \leq n-2}, a', \underbrace{b', \ldots, b'}_{n-r'-1})$$

Assuming $r \geq r'$, we consider first the case $r = r'$, hence $c = (a, b \ldots, b)$, $c' = (a', b' \ldots, b)$. If, for example, $a \geq a'$ then $b \leq b'$ and is clear that $\begin{cases} T_1(c) \geq T_1(c') \\ B_i(c) \leq B_i(c') \quad \forall 2 \leq i \leq n \end{cases}$. Thus, by Remark 1.5, $c \succcurlyeq c'$. If $c \neq c'$ then $c \succ c'$ and, applying HLP theorem to the strictly convex function e, we get the contradiction $kn > kn$.

Consider now the case $r > r'$ and write the equality

$$rM + a + (n - r - 1)b = r'M + a' + (n - r' - 1)b'$$

$$\text{as } (r - r' - 1)(M - b') + (M - a') + (a - b) = (n - r)(b' - b)$$

Since the left side is clearly positive, we get $b \leq b'$ and so $\begin{cases} T_i(c) \geq T_i(c') & \forall 1 \leq i \leq r \\ B_i(c) \leq B_i(c') & \forall r+2 \leq i \leq n \end{cases}$ hence, by Remark 1.5, $c \succcurlyeq c'$. If $c \neq c'$ then $c \succ c'$ and, applying HLP theorem to the strictly convex function e, we get again the contradiction $kn > kn$.

Therefore, there is a unique point $\Omega = c$ of de desired form and the ω case is similar.

(b) For this, there is practically nothing left to prove. Let $f : I_S \to \mathbb{R}$ be an arbitrarily strictly 3–convex with respect to e. Because $E_f : A_S \to \mathbb{R}$ is continuous on the compact set A_S, we get a point $c \in A_S$ for which $E_f(c) = \sup_{A_S} E_f$. And, exactly as above for f_0, we find that c must be of the form $(M, \ldots, M, a, b, \ldots, b)$. On the other hand, according to (a), there is an unique point Ω of that form so we must have $c = \Omega$. For the minimum case the proof is similar.

Thus, we have proved (a) and (b) for the case of a 2–convex system. If S is 2–concave, then we consider the dual 2–convex system $S'(h, s, k', n)$ where $h = -e$, $k' = -k$ and, clearly, $A_S = A_{S'}$. On the other hand, according to Remark 1.4, f is also 3–convex with respect to h and so, by the 2–convex case, we get the unique points $\omega, \Omega \in A_{S'} = A_S$ of the desired form, for which $E_f(\omega) \leq E_f(x) \leq E_f(\Omega)\ \forall x \in A_S$ and the conclusion follows. □

Remark 3.7. If $M \notin I_S$ then $r = 0$ and Ω is of the simpler form $\Omega = (a, b \ldots, b)$. Similarly, if $m \notin I_S$ then $t = 0$ and ω gets the simpler form $\omega = (a, \ldots, a, b)$. We can see that, in general, to get the exact value of Ω (for example) we have to solve a two equations system with a, b as unknowns but also with that extra parameter r. But, as we will next see, this r can be estimated in advance and this fact, obviously, simplify solving the above system.

From now on we will assume I_S compact, hence $I_S = [m, M]$.

Lemma 3.3. *Let $I = [m, M]$ a compact interval, $s \in \mathring{I}$ and $C = \{x \in I^n | x_1 + x_2 + \ldots x_n = ns\}$. Then $\exists! \tilde{u} \in C$ of the form $\tilde{u} = (\underbrace{M, \ldots M}_{l_0}, \theta, \underbrace{m, \ldots m}_{n-l_0-1})$ where $0 \leq l_0 \leq n-1$ and $\theta \in [m, M)$.*

Proof. Let $\lambda = \frac{s-m}{M-m} \in (0, 1)$, $l_0 = [n\lambda] \in \{0, \ldots n-1\}$ and $\theta = ns - l_0 M - (n - l_0 - 1)m$. A straightforward calculation give us $\theta = m + \{n\lambda\}(M - m) \in [m, M)$ and, finally, we define $\tilde{u} \stackrel{def}{=} (\underbrace{M, \ldots M}_{l_0}, \theta, \underbrace{m, \ldots m}_{n-l_0-1}) \in C$. Next, if $u' = (\underbrace{M, \ldots M}_{l_0'}, \theta', \underbrace{m, \ldots m}_{n-l_0'-1}) \in C$ with $0 \leq l_0' \leq n-1$ and $\theta' \in [m, M)$ then $\theta' = ns - l_0' M - (n - l_0' - 1)m$ and we immediately get $n\lambda - l_0' = \frac{\theta' - m}{M - m} \in [0, 1)$ so $l_0' = [n\lambda] = l_0$, hence $\tilde{u}$ is unique. □

Remark 3.8. If $A_S \neq \emptyset$ then $k \in [e(s), \tilde{k}]$, where $\tilde{k} \stackrel{def}{=} E(\tilde{u})$ and $E(x) = \frac{1}{n}\sum_{i=1}^n e(x_i)$. Indeed, by Jensen inequality, $E(\bar{x}) \geq E(\bar{s})$ and since $\tilde{u} \succcurlyeq \bar{x} \Rightarrow E(\bar{x}) \leq E(\tilde{u})$ (by HLP).

Moreover, if $k = \tilde{k}$ then $A_S = \{\tilde{u}\}$. Indeed, we get $l_0 e(M) + e(\theta) + (n - l_0 - 1)e(m) = nk$ so $E(\tilde{u}) = k$ and $\tilde{u} \in A_S$. Now, for an arbitrary $\bar{x} = (x_1, \ldots, x_n) \in A_S$ we see that $\bar{x} \preccurlyeq \tilde{u}$ and since $E(\bar{x}) = k = \tilde{k} = E(\tilde{u})$ we deduce from HLP inequality applied to the strictly convex function e that $\bar{x} = \tilde{u}$. Thus $A_S = \{\tilde{u}\}$. Similarly, if $k = e(s)$ then $A_S = \{\bar{s}\}$.

Next, for every $1 \leq p \leq n-1$ we define $k_p = \begin{cases} \frac{pe(M)+(n-p)e(\delta_p)}{n} \text{ if } p \leq l_0 \\ \frac{pe(\gamma_p)+(n-p)e(m)}{n} \text{ if } p > l_0 \end{cases}$ where δ_p, γ_p are given by $pM + (n-p)\delta_p = p\gamma_p + (n-p)m = ns$. By a straightforward calculation we get $\gamma_1 > \gamma_2 > \ldots > \gamma_{n-1} > s > \delta_1 > \delta_2 > \ldots > \delta_{n-1}$ and is also easy to verify that $\delta_p \in [m, s)$ (if $p \leq l_0$), respectively $\gamma_p \in (s, M]$ (if $p > l_0$), hence k_p is well defined.

Lemma 3.4. *Under the above notations we have*

$$\begin{cases} \text{(a) } e(s) < k_1 < \ldots < k_{l_0} \leq \tilde{k} & \text{if } l_0 \geq 1 \\ \text{(b) } \tilde{k} \geq k_{l_0+1} > \ldots > k_{n-1} > e(s) & \text{if } l_0 + 1 \leq n-1 \end{cases}$$

Proof. (a) For $1 \leq p < p+1 \leq l_0$ we have the chain of majorization inequalities

$$(s,\ldots,s) \prec (\underbrace{M,\ldots M}_{p}, \underbrace{\delta_p,\ldots \delta_p}_{n-p}) \prec (\underbrace{M,\ldots M}_{p+1}, \underbrace{\delta_{p+1},\ldots \delta_{p+1}}_{n-p-1}) \preccurlyeq (\underbrace{M,\ldots M}_{l_0}, \theta, m, \ldots m) = \tilde{u}$$

and applying HLP theorem to the strictly convex function e we get $e(s) < k_p < k_{p+1} \leq \tilde{k}$

(b) For $l_0 + 1 \leq p < p+1 \leq n-1$ the conclusion follows similarly using the chain

$$\tilde{u} = (\underbrace{M,\ldots M}_{l_0}, \theta, m, \ldots m) \succcurlyeq (\underbrace{\gamma_p,\ldots \gamma_p}_{p}, \underbrace{m,\ldots m}_{n-p}) \succ (\underbrace{\gamma_{p+1},\ldots \gamma_{p+1}}_{p+1}, \underbrace{m,\ldots m}_{n-p-1}) \succ (s,\ldots,s)$$

□

In the following, we will exemplify only the Ω case (the other being similar). We start with some observations, grouped in the following remark.

Remark 3.9. Fix $p \leq l_0$ and let $\Omega = (\underbrace{M,\ldots,M}_{r}, a, b\ldots, b)$, $Z = (\underbrace{M,\ldots,M}_{p}, \delta_p, \ldots, \delta_p)$.

(a) $rM + a + (n-r-1)b = pM + (n-p)\delta_p = kn$. This is obvious.

(b) We have $r \leq l_0$. Indeed, assuming $r > l_0 = \left[n\frac{s-m}{M-m}\right] \Rightarrow r > n\frac{s-m}{M-m} \Rightarrow (n-r)m > a + (n-r-1)b$ and this is impossible because $a, b \geq m$.

(c) If $k < k_p$ $(p \leq l_0)$ then $r < p$. Indeed, if $r \geq p$ then we observe by *(a)* that $b \leq \delta_p$ and so (by Remark 1.5) $\Omega \succcurlyeq Z$ and, applying HLP theorem to e, we get $k \geq k_p$, a contradiction.

(d) If $k > k_p$ $(p \leq l_0)$ then $r \geq p$. Indeed, if $r < p$ then we infer using *(a)* that $\delta_p \leq b$. Thus, by Remark 1.5, $\Omega \preccurlyeq Z$ and so (by HLP theorem) we get $k \leq k_p$, a contradiction.

Now, we can evaluate r using the *position of* k in the sequence $e(s) < k_1 < \ldots < k_{l_0} < \tilde{k}$.

If $k = e(s)$ or $k = \tilde{k}$ then $A_S = \{\bar{s}\}$, respectively $A_S = \{\tilde{u}\}$ and everything is clear.

If $k = k_p$ for some $1 \leq p \leq l_0$ if follows that $Z \in A_S$. But Z and Ω are of the same form hence, by Theorem 3.5a, we infer that $\Omega = Z$ etc.

If $k \in (k_{l_0}, \tilde{k})$ then , by Remark 3.9b and 3.9d, we get $r = l_0$.

If $k_{p-1} < k < k_p$ for some $2 \leq p \leq l_0$ then, by Remark 3.9c and 3.9d we get $r = p-1$.

Finally, if $e(s) < k < k_1$ then, by Remark 3.9c we get $r = 0$.

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