

Automorphisms of Automorphism Group of Dihedral Groups

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ABSTRACT. The automorphism group of a Dihedral group of order $2n$ is isomorphic to the holomorph of a cyclic group of order n . The holomorph of a cyclic group of order n is a complete group when n is odd. Hence automorphism groups of Dihedral groups of order $2n$ are its own automorphism groups whenever n is odd. In this paper, we prove that the result is also true for those Dihedral groups of order $2n$ where n is twice a prime number.

1. INTRODUCTION

An automorphism on a group G is a bijection $f : G \rightarrow G$ which preserves the binary operation on G . The set of all automorphisms on a group G under the composition of mappings forms a group, which is denoted by $Aut(G)$. The topic of automorphism group of a group has been of interest to many researchers for a long time. The automorphism group of abelian groups has been analyzed fairly well [2, 13], but the case of non-abelian groups is more complicated and is still an active research area. Finite groups whose automorphism group is abelian were first considered by G. A. Miller [10], who studied a group of order 64 with an abelian automorphism group of order 128. In general, the problem of classification of non-abelian groups with abelian automorphism group still remains an open problem, though solutions exist for a few special cases [1, 4, 6, 12].

The automorphism group of D_{2n} , the dihedral group of order $2n$, is isomorphic to the holomorph of \mathbb{Z}_n , the cyclic group of order n [14]. It is known that the holomorph of a cyclic group of order n is a complete group only when n is odd [9]. Since the automorphism group of a complete group is the group itself, it follows that $Aut(Aut D_{2n})$ is isomorphic to $Aut D_{2n}$ whenever n is odd. In this paper, we prove the result is also true for those Dihedral group of order $2n$ where n is twice a prime number.

Most of the notations, definitions and results we mention in this paper are as in [7] and [5]. For a group G , let $|G|$ the order of G and $o(g)$ denote the order of the element g in G . For integers m and n , the greatest common divisor of m and n is denoted by (m, n) .

For any given natural number n let:

$$\varphi(n) = \begin{array}{l} \text{the number of non-negative integers less than } n \text{ and relatively} \\ \text{prime to } n. \end{array}$$

Also, for $n \geq 1$, \mathbb{Z}_n denotes the group of integers modulo n and \mathbb{Z}_n^* denotes the multiplicative group of integers group modulo n .

Definition 1.1. [7] A subgroup H of a group G is said to be a characteristic subgroup of G if $\phi(H) = H$ for all automorphisms ϕ on G .

Theorem 1.1. [5] *The group $Aut(S_n) \cong S_n$ for all $n \geq 3$ and $n \neq 6$.*

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Theorem 1.2. [5] *Let G be a group and H be a unique subgroup(cyclic) of given order. Then H is a characteristic subgroup.*

Theorem 1.3. [8] *The group \mathbb{Z}_n^* is cyclic if and only if $n = 1, 2, 4, p^k$ or $2p^k$ where p is an odd prime.*

For each natural number $n \geq 3$, define

$$G_n = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \in \mathbb{Z}_n^*, b \in \mathbb{Z}_n \right\}$$

Then G_n is a group of order $n\varphi(n)$ with respect to matrix multiplication. the identity element of G_n is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the inverse of $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{bmatrix}$.

Theorem 1.4. [3] *The group G_n is isomorphic to $\text{Aut}(D_{2n})$ for all positive integer $n \geq 3$.*

2. AUTOMORPHISM GROUPS OF AUTOMORPHISM GROUPS D_{2n}

Now we characterize automorphism groups of $\text{Aut}(D_{2n})$.

Theorem 2.5. *Let $n = 1, 2, 4, p^k$ or $2p^k$ where p is an odd prime. Then*

$$1 + t + t^2 + \dots + t^{\varphi(n)-1} \equiv 0 \pmod{n}$$

for all $t \in \mathbb{Z}_n^*$ and $o(t) = \varphi(n)$.

Proof. The case $n = 1, 2$ and 4 are trivial. So assume that $n = p^k$ or $2p^k$, $t \in \mathbb{Z}_n^*$ and $o(t) = \varphi(n)$. Therefore

$$(1 + t + t^2 + \dots + t^{\varphi(n)-1})(t - 1) = t^{\varphi(n)} - 1 \equiv 0 \pmod{n} \quad (2.1)$$

Claim that $t - 1$ is not congruent to $0 \pmod{p}$. Suppose $t - 1 \equiv 0 \pmod{p}$. Then

$$\begin{aligned} t &= 1 + rp \text{ for some } r \in \mathbb{Z} \\ \implies t^{p^{k-1}} &= (1 + rp)^{p^{k-1}} = 1 + \left\{ p^{k-1} C_1(rp) + p^{k-1} C_2(rp)^2 + \dots + (rp)^{p^{k-1}} \right\} \end{aligned}$$

Each term in the bracket is congruent to zero $\pmod{p^k}$. Hence

$$t^{p^{k-1}} \equiv 1 \pmod{p^k} \quad (2.2)$$

If $n = 2p^k$, then t is odd and hence

$$t^{p^{k-1}} \equiv 1 \pmod{2} \quad (2.3)$$

and

$$1 + t + t^2 + \dots + t^{\varphi(n)-1} \equiv 0 \pmod{2} \quad (2.4)$$

From (2.2) and (2.3), we get

$$\begin{aligned} t^{p^{k-1}} &\equiv 1 \pmod{n}, \text{ when } n = p^k \text{ or } n = 2p^k \\ \implies o(t) \text{ in } \mathbb{Z}_n^* &\leq p^{k-1} < p^{k-1}(p-1) = \varphi(n), \end{aligned}$$

a contradiction to the choice of t . Hence

$$t - 1 \equiv 0 \pmod{p} \quad (2.5)$$

From (2.1), (2.3) and (2.5), we have

$$1 + t + t^2 + \dots + t^{\varphi(n)-1} \equiv 0 \pmod{n}$$

when $n = p^k$ or $2p^k$. □

Theorem 2.6. Let $n = p$ or $2p$ where p is an odd prime. Then

$$1 + z + z^2 + \dots + z^{p-2} \equiv 0 \pmod{n}$$

for all $z \in \mathbb{Z}_n^*$ and $z \neq 1$.

Proof. Let $z \in \mathbb{Z}_n^*$ and $z \neq 1$. Then

$$z - 1 \text{ is not congruent to } 0 \pmod{p} \quad (2.6)$$

Now,

$$(1 + z + z^2 + \dots + z^{p-2})(z - 1) = z^{p-1} - 1 \equiv 0 \pmod{n} \quad (2.7)$$

Hence by (2.6),

$$1 + z + z^2 + \dots + z^{p-2} \equiv 0 \pmod{p} \quad (2.8)$$

If $n = 2p$, then z is odd and hence

$$1 + z + z^2 + \dots + z^{p-2} \equiv 0 \pmod{2} \quad (2.9)$$

From (2.8) and (2.9) we get

$$1 + z + z^2 + \dots + z^{p-2} \equiv 0 \pmod{n} \quad (2.10)$$

for all $z \in \mathbb{Z}_n^*$ and $z \neq 1$ when $n = p$ or $2p$. \square

Theorem 2.7. Let $n = 2, 4, p^k$ or $2p^k$ where p is an odd prime. Let $a = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}$

where $x, t \in \mathbb{Z}_n^*$, $y \in \mathbb{Z}_n$ and $o(t) = \varphi(n)$. Then

- (i) $o(a) = n$
- (ii) $o(b) = \varphi(n)$
- (iii) $b^{-1}a^ib = a^{it^{-1}}$ for all $i \in \mathbb{N}$
- (iv) $b^{-k}a^ib^k = a^{it^{-k}}$ for all $i, k \in \mathbb{N}$
- (v) $\langle a \rangle$ is normal in G_n and $\langle a \rangle \cap \langle b \rangle = \{I\}$
- (vi) $G = \langle a, b \rangle = \{b^ia^j : 0 \leq i \leq \varphi(n) - 1, 0 \leq j \leq n - 1\}$.

Proof. (i) For any $k \in \mathbb{N}$,

$$a^k = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & kx \\ 0 & 1 \end{bmatrix}$$

Therefore $o(a)$ in $G_n = o(x)$ in $\mathbb{Z}_n = n$.

(ii) For any $k \in \mathbb{N}$,

$$b^k = \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} t^k & (1 + t + t^2 + \dots + t^{k-1})y \\ 0 & 1 \end{bmatrix}$$

Now, $b^k = I \implies t^k = 1 \implies k \geq \varphi(n)$.

Also,

$$\begin{aligned} b^{\varphi(n)} &= \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}^{\varphi(n)} = \begin{bmatrix} t^{\varphi(n)} & (1 + t + t^2 + \dots + t^{\varphi(n)-1})y \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \text{ by theorem 2.5} \end{aligned}$$

Therefore $o(b)$ in $G_n = \varphi(n)$.

(iii)

$$\begin{aligned}
 b^{-1}a^ib &= \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^i \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t^{-1} & -yt^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & ix \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} t^{-1} & t^{-1}ix - yt^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t^{-1}ix \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{it^{-1}} = a^{it^{-1}}
 \end{aligned}$$

(iv) Let $i \in \mathbb{N}$. Then $b^{-1}a^ib = a^{it^{-1}}$. Hence the result is true for $k = 1$. Suppose the result is true for $k = n$. Then

$$\begin{aligned}
 b^{-(n+1)}a^ib^{n+1} &= b^{-1}(b^{-n}a^ib^n)b = b^{-1}a^{it^{-n}}b = a^{it^{-n}t^{-1}} \quad ; \text{ by (iii)} \\
 &= a^{it^{-(n+1)}}
 \end{aligned}$$

Hence the result is true for all $i, k \in \mathbb{N}$.

(v) Let $g = \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix} \in G_n$ and $a^i \in \langle a \rangle$. Then

$$\begin{aligned}
 ga^ig^{-1} &= \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^i \begin{bmatrix} z^{-1} & -dz^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & ix \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & -dz^{-1} \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} zz^{-1} & -zdz^{-1} + izx + d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & izx \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{iz} = a^{iz} \in \langle a \rangle
 \end{aligned}$$

Hence $\langle a \rangle$ is normal in G_n .

Let $z \in \langle a \rangle \cap \langle b \rangle$. Then

$$z = a^i = b^j \text{ for some } 0 \leq i \leq n-1, \text{ and } 0 \leq j \leq \varphi(n) - 1.$$

$$\implies z = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^i = \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}^j$$

$$\implies t^j = 1 \text{ for some } 0 \leq j \leq \varphi(n) - 1.$$

Since $o(t) = \varphi(n)$, we have $j = 0$. Therefore $z = b^0 = I$. Hence $\langle a \rangle \cap \langle b \rangle = \{I\}$.

(vi) By (v) we have,

$$G = \langle b \rangle \langle a \rangle = \{b^i a^j : 0 \leq i \leq \varphi(n) - 1, 0 \leq j \leq n - 1\} = \langle a, b \rangle$$

□

Theorem 2.8. Let $n = p$ or $2p$ where p is an odd prime. Then $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$ is a characteristic subgroup of G_n .

Proof. Let $g = \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix} \in G_n$. If $z \neq 1$, then

$$g^{p-1} = \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix}^{p-1} = \begin{bmatrix} z^{p-1} & (1 + z + z^2 + \dots + z^{p-2})d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \text{ by theorem 2.6}$$

Therefore $o(g)$ in $G_n \leq p - 1 < n$.

Let $z = 1$. Then

$$g^k = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & kd \\ 0 & 1 \end{bmatrix} \implies o(g) \text{ in } G_n = o(d) \text{ in } \mathbb{Z}_n$$

Hence the elements of order n in G_n are $\left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : 0 \leq x \leq n-1, (x, n) = 1 \right\}$. Therefore, there are $\varphi(n)$ elements of order n . Since $o\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = n$, we have $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$ is the unique cyclic subgroup of G_n of order n . Hence $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$ is the characteristic subgroup of G_n . \square

Theorem 2.9. *Let $n = p$ or $2p$ where p is an odd prime. Then $|Aut(G_n)| = n\varphi(n)$.*

Proof. Take $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$ where $t \in \mathbb{Z}^*$ such that $o(t) = \varphi(n) = p-1$. Let $\phi : G_n \rightarrow G_n$ be an automorphism. Since $\langle a \rangle$ is a characteristic subgroup of G_n and a has order n , we have $\phi(\langle a \rangle) = \langle a \rangle$ and $\phi(a) = a^i$ for some $0 \leq i \leq n-1$ and $(i, n) = 1$. Assume $\phi(b) = b^l a^m$ for some $0 \leq l \leq \varphi(n)-1$ and $0 \leq m \leq n-1$. By theorem 2.7, we have $b^{-1}ab = a^{t^{-1}}$.

Therefore

$$\begin{aligned} \phi(b^{-1}ab) &= \phi(a^{t^{-1}}) \implies (\phi(b))^{-1}\phi(a)\phi(b) = (\phi(a))^{t^{-1}} \\ \implies (b^l a^m)^{-1}(a^i)(b^l a^m) &= (a^i)^{t^{-1}} \implies a^{-m}b^{-l}a^i b^l a^m = a^{it^{-1}} \\ \implies a^{-m}(b^{-l}a^i b^l)a^m &= a^{it^{-1}} \implies a^{-m}(a^{i(t^{-1})^l})a^m = a^{it^{-1}} \\ \implies a^{i(t^{-1})^l} &= a^{it^{-1}} \implies a^{i(t^{-1})((t^{-1})^{l-1}-1)} = I \\ \implies i(t^{-1})((t^{-1})^{l-1}-1) &\equiv 0 \pmod{n} \quad ; \text{ since } o(a) = n \\ \implies (t^{-1})^{l-1}-1 &\equiv 0 \pmod{n} \implies (t^{-1})^{l-1} \equiv 1 \pmod{n} \\ \implies l-1 &= 0 \quad ; \text{ since } o(t^{-1}) \text{ in } \mathbb{Z}_n^* = \varphi(n) \text{ and } l-1 < \varphi(n) \end{aligned}$$

Hence $\phi(b) = ba^j$ for some $0 \leq j \leq n-1$. Consequently, there are at most $n\varphi(n)$ automorphisms on G_n and hence

$$|Aut(G_n)| \leq n\varphi(n)$$

Conversely, suppose for each $0 \leq i \leq n-1$ and $0 \leq j \leq \varphi(n)-1$, define a map $\phi_{i,j} : G_n \rightarrow G_n$ by

$$\phi_{i,j}(b^l a^m) = \hat{b}^l \hat{a}^m$$

where $\hat{b} = ba^j$, $\hat{a} = a^i$ and $0 \leq l \leq \varphi(n)-1$ and $0 \leq m \leq n-1$. We show that $\phi_{i,j}$ is an automorphism.

Let $b^l a^m, b^k a^s \in G_n$. Then

$$\begin{aligned} b^l a^m b^k a^s &= b^l (a^m b^k) a^s = b^l (b^k a^{m(t^{-1})^k}) a^s \quad ; \text{ by Theorem 2.7} \\ &= b^{l+k} a^{m(t^{-1})^k+s} \\ \therefore \phi_{i,j}(b^l a^m b^k a^s) &= \phi_{i,j}(b^{l+k} a^{m(t^{-1})^k+s}) = (\hat{b})^{l+k} (\hat{a})^{m(t^{-1})^k+s} = (\hat{b})^l (\hat{b})^k (\hat{a})^{m(1/t)^k} (\hat{a})^s \\ &= (\hat{b})^l (\hat{a})^m (\hat{b})^k (\hat{a})^s = \phi_{i,j}(b^l a^m) \phi_{i,j}(b^k a^s) \end{aligned}$$

Hence $\phi_{i,j}$ is a homomorphism. By Theorem 2.7, we have $\langle \hat{a}\hat{b} \rangle = G_n$. Hence $\phi_{i,j}$ is onto. Since G_n is finite, $\phi_{i,j}$ is one-one also. Therefore $\phi_{i,j}$ is an automorphism.

Next we will prove that $\phi_{i,j}$ are different. Suppose $\phi_{i,j} = \phi_{k,s}$ where $0 \leq i, k \leq n-1$ and $0 \leq j, s \leq \varphi(n)-1$. Then

$$\phi_{i,j}(a) = \phi_{k,s}(a) \implies a^i = a^k \implies i = k$$

Again,

$$\phi_{i,j}(b) = \phi_{k,s}(b) \implies ba^j = ba^s \implies a^j = a^s \implies j = s$$

Consequently there are at least $n\varphi(n)$ automorphisms on G_n . Hence

$$|Aut(G_n)| = n\varphi(n)$$

□

Take $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$ where $t \in \mathbb{Z}_n^*(n = p \text{ or } 2p)$ and $o(t) = \varphi(n) = p-1$.

There is a unique automorphism on G_n which map $a \rightarrow a^i$ and $b \rightarrow ba^j$ where $0 \leq i, j \leq n-1$ and $(i, n) = 1$. Denote this automorphism by $\phi_{i,j}$, called automorphism induced by the map $a \rightarrow a^i$ and $b \rightarrow ba^j$. Hence

$$Aut(G_n) = \{\phi_{i,j} : 0 \leq i, j \leq n-1, (i, n) = 1\}$$

□

Theorem 2.10. $Aut(G_n)$ is isomorphic to G_n for $n = p$ or $2p$ where p is an odd prime.

Proof. Define $\psi : Aut(G_n) \rightarrow G_n$ by

$$\psi(\phi_{i,j}) = \begin{bmatrix} i & j \\ 0 & 1 \end{bmatrix}, \quad 0 \leq i, j \leq n-1, (i, n) = 1$$

Now,

$$\phi_{i,j} \circ \phi_{k,s}(a) = \phi_{i,j}(a^k) = a^{ik}$$

and

$$\phi_{i,j} \circ \phi_{k,s}(b) = \phi_{i,j}(ba^k) = \phi_{i,j}(b)\phi_{i,j}(a^k) = ba^j a^{ik} = ba^{ik+j}$$

Hence $\phi_{i,j} \circ \phi_{k,s} = \phi_{l,m}$ where $l \equiv ik \pmod{n}$ and $m \equiv (ik+j) \pmod{n}$.

So,

$$\begin{aligned} \psi(\phi_{i,j} \circ \phi_{k,s}) &= \psi(\phi_{l,m}) = \begin{bmatrix} l & m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ik \pmod{n} & (j+is) \pmod{n} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} i & j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k & s \\ 0 & 1 \end{bmatrix} = \psi(\phi_{i,j})\psi(\phi_{k,s}) \end{aligned}$$

Clearly ψ is one-one and onto. Hence ψ is an isomorphism from $Aut(G_n)$ onto G_n . □

It well known that D_4 is the only abelian Dihedral group and that its group of automorphisms is the symmetric group of order 6 (S_3). Thus $Aut(AutD_4) \cong AutS_3 \cong S_3 \cong AutD_4$. $Aut(AutD_8)$ is isomorphic to G_4 which is a non-abelian group of order 8. D_8 has 4 inner automorphisms in which every element has order 2 except trivial automorphism. Hence $AutD_8 \cong D_8$. So $Aut(AutD_8) \cong AutD_8$.

Hence we have the following.

Theorem 2.11. Let $n = p$ or $2p$ where p is prime. Then $Aut(Aut(D_{2n}))$ is isomorphic to $Aut(D_{2n})$.

3. CONCLUSION

In this paper, it is proved that $\text{Aut}(\text{Aut}(D_{2n}))$ is isomorphic to $\text{Aut}(D_{2n})$ whenever n is twice a prime number. The case when n is even and not twice a prime number will be considered in future work.

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