CREAT. MATH. INFORM.
Volume **32** (2023), No. 2,
Pages 229 - 235

Online version at `https://semnul.com/creative-mathematics/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2023.02.10

Automorphisms of Automorphism Group of Dihedral Groups

SADANANDAN SAJIKUMAR, SIVADASAN VINOD and GOPINADHAN SATHIKUMARI BIJU

ABSTRACT. The automorphism group of a Dihedral group of order $2n$ is isomorphic to the holomorph of a cyclic group of order n. The holomorph of a cyclic group of order n is a complete group when n is odd. Hence automorphism groups of Dihedral groups of order $2n$ are its own automorphism groups whenever n is odd. In this paper, we prove that the result is also true for those Dihedral groups of order $2n$ where n is twice a prime number.

1. INTRODUCTION

An automorphism on a group G is a bijection $f : G \to G$ which preserves the binary operation on G. The set of all automorphisms on a group G under the composition of mappings forms a group, which is denoted by $Aut(G)$. The topic of automorphism group of a group has been of interest to many researchers for a long time. The automorphism group of abelian groups has been analyzed fairly well [2, 13], but the case of non-abelian groups is more complicated and is still an active research area. Finite groups whose automorphism group is abelian were first considered by G. A. Miller [10], who studied a group of order 64 with an abelian automorphism group of order 128. In general, the problem of classification of non-abelian groups with abelian automorphism group still remains an open problem, though solutions exist for a few special cases [1, 4, 6, 12].

The automorphism group of D_{2n}, the dihedral group of order $2n$, is isomorphic to the holomorph of $\mathbb{Z}_n$, the cyclic group of order n [14]. It is known that the holomorph of a cyclic group of order n is a complete group only when n is odd [9]. Since the automorphism group of a complete group is the group itself, it follows that $Aut(AutD_{2n})$ is isomorphic to $AutD_{2n}$ whenever n is odd. In this paper, we prove the result is also true for those Dihedral group of order $2n$ where n is twice a prime number.

Most of the notations, definitions and results we mention in this paper are as in [7] and [5]. For a group G, let $|G|$ the order of G and $o(g)$ denote the order of the element g in G. For integers m and n, the greatest common divisor of m and n is denoted by (m, n).

For any given natural number n let:

$$\varphi(n) = \text{ the number of non-negative integers less than } n \text{ and relatively prime to } n.$$

Also, for $n \geq 1$, $\mathbb{Z}_n$ denotes the group of integers modulo n and $\mathbb{Z}_n^*$ denotes the multiplicative group of integers group modulo n.

Definition 1.1. [7] A subgroup H of a group G is said to be a characteristic subgroup of G if $\phi(H) = H$ for all automorphisms ϕ on G.

Theorem 1.1. [5] *The group $Aut(S_n) \cong S_n$ for all $n \geq 3$ and $n \neq 6$.*

Received: 22.09.2022. In revised form: 03.04.2023. Accepted: 10.04.2023
2000 *Mathematics Subject Classification.* 20D45, 20F28.
Key words and phrases. *automorphism group, Dihedral group, characteristic subgroup.*
Corresponding author: Vinod S.; wenod76@gmail.com

Theorem 1.2. [5] *Let G be a group and H be a unique subgroup(cyclic) of given order. Then H is a characteristic subgroup.*

Theorem 1.3. [8] *The group $\mathbb{Z}_n^*$ is cyclic if and only if $n = 1, 2, 4, p^k$ or $2p^k$ where p is an odd prime.*

For each natural number $n \geq 3$, define

$$G_n = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \ : \ a \in \mathbb{Z}_n^*, b \in \mathbb{Z}_n \right\}$$

Then G_n is a group of order $n\varphi(n)$ with respect to matrix multiplication. the identity element of G_n is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the inverse of $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{bmatrix}$.

Theorem 1.4. [3] *The group G_n is isomorphic to $Aut(D_{2n})$ for all positive integer $n \geq 3$.*

2. Automorphism groups of Automorphism groups D_{2n}

Now we characterize automorphism groups of $Aut(D_{2n})$.

Theorem 2.5. *Let $n = 1, 2, 4, p^k$ or $2p^k$ where p is an odd prime. Then*

$$1 + t + t^2 + \ldots + t^{\varphi(n)-1} \equiv 0 \, mod(n)$$

for all $t \in \mathbb{Z}_n^*$ and $o(t) = \varphi(n)$.

Proof. The case $n = 1, 2$ and 4 are trivial. So assume that $n = p^k$ or $2p^k$, $t \in \mathbb{Z}_n^*$ and $o(t) = \varphi(n)$. Therefore

$$(1 + t + t^2 + \ldots + t^{\varphi(n)-1})(t - 1) = t^{\varphi(n)} - 1 \equiv 0 \, mod(n) \tag{2.1}$$

Claim that $t - 1$ is not congruent to $0 \, mod(p)$. Suppose $t - 1 \equiv 0 \, mod(p)$. Then

$$t = 1 + rp \ \text{ for some } r \in \mathbb{Z}$$

$$\implies t^{p^{k-1}} = (1 + rp)^{p^{k-1}} = 1 + \left\{ {}^{p^{k-1}}C_1(rp) + {}^{p^{k-1}}C_2(rp)^2 + \ldots + (rp)^{p^{k-1}} \right\}$$

Each term in the bracket is congruent to zero $mod \, p^k$. Hence

$$t^{p^{k-1}} \equiv 1 \, mod(p^k) \tag{2.2}$$

If $n = 2p^k$, then t is odd and hence

$$t^{p^{k-1}} \equiv 1 \, mod(2) \tag{2.3}$$

and

$$1 + t + t^2 + \ldots + t^{\varphi(n)-1} \equiv 0 \, mod(2) \tag{2.4}$$

From (2.2) and (2.3), we get

$$t^{p^{k-1}} \equiv 1 \, mod(n), \ \text{ when } n = p^k \text{ or } n = 2p^k$$

$$\implies o(t) \text{ in } \mathbb{Z}_n^* \leq p^{k-1} < p^{k-1}(p - 1) = \varphi(n),$$

a contradiction to the choice of t. Hence

$$t - 1 \equiv 0 \, mod(p) \tag{2.5}$$

From (2.1), (2.3) and (2.5), we have

$$1 + t + t^2 + \ldots + t^{\varphi(n)-1} \equiv 0 \, mod(n)$$

when $n = p^k$ or $2p^k$. □

Theorem 2.6. *Let $n = p$ or $2p$ where p is an odd prime. Then*

$$1 + z + z^2 + \ldots + z^{p-2} \equiv 0 \, mod(n)$$

for all $z \in \mathbb{Z}_n^*$ and $z \neq 1$.

Proof. Let $z \in \mathbb{Z}_n^*$ and $z \neq 1$.Then

$$z - 1 \text{ is not congruent to } 0 \, mod(p) \tag{2.6}$$

Now,

$$(1 + z + z^2 + \ldots + z^{p-2})(z - 1) = z^{p-1} \equiv 0 \, mod(n) \tag{2.7}$$

Hence by (2.6),

$$1 + z + z^2 + \ldots + z^{p-2} \equiv 0 \, mod(p) \tag{2.8}$$

If $n = 2p$, then z is odd and hence

$$1 + z + z^2 + \ldots + z^{p-2} \equiv 0 \, mod(2) \tag{2.9}$$

From (2.8) and (2.9) we get

$$1 + z + z^2 + \ldots + z^{p-2} \equiv 0 \, mod(n) \tag{2.10}$$

for all $z \in \mathbb{Z}_n^*$ and $z \neq 1$ when $n = p$ or $2p$. □

Theorem 2.7. *Let $n = 2, 4, p^k$ or $2p^k$ where p is an odd prime. Let $a = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}$ where $x, t \in \mathbb{Z}_n^*, y \in \mathbb{Z}_n$ and $o(t) = \varphi(n)$. Then*

(i) $o(a) = n$
(ii) $o(b) = \varphi(n)$
(iii) $b^{-1}a^i b = a^{it^{-1}}$ *for all $i \in \mathbb{N}$*
(iv) $b^{-k}a^i b^k = a^{it^{-k}}$ *for all $i, k \in \mathbb{N}$*
(v) $\langle a \rangle$ *is normal in G_n and $\langle a \rangle \cap \langle b \rangle = \{I\}$*
(vi) $G = \langle a, b \rangle = \{b^i a^j \ : \ 0 \leq i \leq \varphi(n) - 1, \, 0 \leq j \leq n - 1\}$.

Proof. (i) For any $k \in \mathbb{N}$,

$$a^k = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & kx \\ 0 & 1 \end{bmatrix}$$

Therefore $o(a)$ in $G_n = o(x)$ in $\mathbb{Z}_n = n$.

(ii) For any $k \in \mathbb{N}$,

$$b^k = \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} t^k & (1 + t + t^2 + \ldots + t^{k-1})y \\ 0 & 1 \end{bmatrix}$$

Now, $b^k = I \implies t^k = 1 \implies k \geq \varphi(n)$.
Also,

$$b^{\varphi(n)} = \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}^{\varphi(n)} = \begin{bmatrix} t^{\varphi(n)} & (1 + t + t^2 + \ldots + t^{\varphi(n)-1})y \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \text{ by theorem 2.5}$$

Therefore $o(b)$ in $G_n = \varphi(n)$.

(iii)

$$b^{-1}a^ib = \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^i \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t^{-1} & -yt^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & ix \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} t^{-1} & t^{-1}ix - yt^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t^{-1}ix \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{it^{-1}} = a^{it^{-1}}$$

(iv) Let $i \in \mathbb{N}$. Then $b^{-1}a^ib = a^{it^{-1}}$. Hence the result is true for $k = 1$. Suppose the result is true for $k = n$. Then

$$b^{-(n+1)}a^ib^{n+1} = b^{-1}(b^{-n}a^ib^n)b = b^{-1}a^{it^{-n}}b = a^{it^{-n}t^{-1}} \qquad ; \text{by } (iii)$$
$$= a^{it^{-(n+1)}}$$

Hence the result is true for all $i, k \in \mathbb{N}$.

(v) Let $g = \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix} \in G_n$ and $a^i \in \langle a \rangle$. Then

$$ga^ig^{-1} = \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^i \begin{bmatrix} z^{-1} & -dz^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & ix \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & -dz^{-1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} zz^{-1} & -zdz^{-1} + izx + d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & izx \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{iz} = a^{iz} \in \langle a \rangle$$

Hence $\langle a \rangle$ is normal in G_n.
Let $z \in \langle a \rangle \cap \langle b \rangle$. Then

$$z = a^i = b^j \text{ for some } 0 \le i \le n-1, \quad \text{and } 0 \le j \le \varphi(n) - 1.$$

$$\implies z = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^i = \begin{bmatrix} t & y \\ 0 & 1 \end{bmatrix}^j$$

$$\implies t^j = 1 \text{ for some } 0 \le j \le \varphi(n) - 1.$$

Since $o(t) = \varphi(n)$, we have $j = 0$. Therefore $z = b^0 = I$. Hence $\langle a \rangle \cap \langle b \rangle = \{I\}$.

(vi) By (v) we have,

$$G = \langle b \rangle \langle a \rangle = \{b^ia^j \ : \ 0 \le i \le \varphi(n) - 1, \, 0 \le j \le n-1\} = \langle a, b \rangle$$

□

Theorem 2.8. *Let $n = p$ or $2p$ where p is an odd prime. Then $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$ is a characteristic subgroup of G_n.*

Proof. Let $g = \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix} \in G_n$. If $z \neq 1$, then

$$g^{p-1} = \begin{bmatrix} z & d \\ 0 & 1 \end{bmatrix}^{p-1} = \begin{bmatrix} z^{p-1} & (1 + z + z^2 + \ldots + z^{p-2})d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad ; \text{by theorem 2.6}$$

Therefore $o(g)$ in $G_n \le p - 1 < n$.
Let $z = 1$. Then

$$g^k = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & kd \\ 0 & 1 \end{bmatrix} \implies o(g) \text{ in } G_n = o(d) \text{ in } \mathbb{Z}_n$$

Hence the elements of order n in G_n are $\left\{\begin{bmatrix}1 & x\\0 & 1\end{bmatrix} : 0 \leq x \leq n-1, (x,n)=1\right\}$. Therefore, there are $\varphi(n)$ elements of order n. Since $o\left(\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\right) = n$, we have $\left\langle\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\right\rangle$ is the unique cyclic subgroup of G_n of order n. Hence $\left\langle\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\right\rangle$ is the characteristic subgroup of G_n. □

Theorem 2.9. *Let $n = p$ or $2p$ where p is an odd prime. Then $|Aut(G_n)| = n\varphi(n)$.*

Proof. Take $a = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ and $b = \begin{bmatrix}t & 0\\0 & 1\end{bmatrix}$ where $t \in \mathbb{Z}^*$ such that $o(t) = \varphi(n) = p-1$. Let $\phi : G_n \to G_n$ be an automorphism. Since $\langle a \rangle$ is a characteristic subgroup of G_n and a has order n, we have $\phi(\langle a \rangle) = \langle a \rangle$ and $\phi(a) = a^i$ for some $0 \leq i \leq n-1$ and $(i,n) = 1$. Assume $\phi(b) = b^l a^m$ for some $0 \leq l \leq \varphi(n)-1$ and $0 \leq m \leq n-1$. By theorem 2.7, we have $b^{-1}ab = a^{t^{-1}}$.

Therefore

$$\phi(b^{-1}ab) = \phi(a^{t^{-1}}) \implies (\phi(b))^{-1}\phi(a)\phi(b) = (\phi(a))^{t^{-1}}$$
$$\implies (b^l a^m)^{-1}(a^i)(b^l a^m) = (a^i)^{t^{-1}} \implies a^{-m}b^{-l}a^i b^l a^m = a^{it^{-1}}$$
$$\implies a^{-m}(b^{-l}a^i b^l)a^m = a^{it^{-1}} \implies a^{-m}(a^{i(t^{-1})^l})a^m = a^{it^{-1}}$$
$$\implies a^{i(t^{-1})^l} = a^{it^{-1}} \implies a^{i(t^{-1})((t^{-1})^{l-1}-1)} = I$$

$$\implies i(t^{-1})((t^{-1})^{l-1}-1) \equiv 0 \, mod(n) \qquad ; \text{since } o(a) = n$$
$$\implies (t^{-1})^{l-1} - 1 \equiv 0 \, mod(n) \implies (t^{-1})^{l-1} \equiv 1 \, mod(n)$$
$$\implies l - 1 = 0 \qquad ; \text{since } o(t^{-1}) \text{ in } \mathbb{Z}_n^* = \varphi(n) \text{ and } l-1 < \varphi(n)$$

Hence $\phi(b) = ba^j$ for some $0 \leq j \leq n-1$. Consequently, there are at most $n\varphi(n)$ automorphisms on G_n and hence

$$|Aut(G_n)| \leq n\varphi(n)$$

Conversely, suppose for each $0 \leq i \leq n-1$ and $0 \leq j \leq \varphi(n)-1$, define a map $\phi_{i,j} : G_n \to G_n$ by

$$\phi_{i,j}(b^l a^m) = \hat{b}^l \hat{a}^m$$

where $\hat{b} = ba^j$, $\hat{a} = a^i$ and $0 \leq l \leq \varphi(n)-1$ and $0 \leq m \leq n-1$. We show that $\phi_{i,j}$ is an automorphism.

Let $b^l a^m, b^k a^s \in G_n$. Then

$$b^l a^m b^k a^s = b^l(a^m b^k)a^s = b^l(b^k a^{m(t^{-1})^k})a^s \qquad ; \text{by Theorem 2.7}$$
$$= b^{l+k}a^{m(t^{-1})^k+s}$$
$$\therefore \phi_{i,j}(b^l a^m b^k a^s) = \phi_{i,j}(b^{l+k}a^{m(t^{-1})^k+s}) = (\hat{b})^{l+k}(\hat{a})^{m(t^{-1})^k+s} = (\hat{b})^l(\hat{b})^k(\hat{a})^{m(1/t)^k}(\hat{a})^s$$
$$= (\hat{b})^l(\hat{a})^m(\hat{b})^k(\hat{a})^s = \phi_{i,j}(b^l a^m)\phi_{i,j}(b^k a^s)$$

Hence $\phi_{i,j}$ is a homomorphism. By Theorem 2.7, we have $\langle \hat{a}\hat{b} \rangle = G_n$. Hence $\phi_{i,j}$ is onto. Since G_n is finite, $\phi_{i,j}$ is one-one also. Therefore $\phi_{i,j}$ is an automorphism.

Next we will prove that $\phi_{i,j}$ are different. Suppose $\phi_{i,j} = \phi_{k,s}$ where $0 \leq i, k \leq n-1$ and $0 \leq j, s \leq \varphi(n) - 1$. Then

$$\phi_{i,j}(a) = \phi_{k,s}(a) \implies a^i = a^k \implies i = k$$

Again,

$$\phi_{i,j}(b) = \phi_{k,s}(b) \implies ba^j = ba^s \implies a^j = a^s \implies j = s$$

Consequently there are at least $n\varphi(n)$ automorphisms on G_n. Hence

$$|Aut(G_n)| = n\varphi(n)$$

□

Take $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$ where $t \in \mathbb{Z}_n^*(n = p$ or $2p)$ and $o(t) = \varphi(n) = p - 1$. There is a unique automorphism on G_n which map $a \to a^i$ and $b \to ba^j$ where $0 \leq i, j \leq n-1$ and $(i, n) = 1$. Denote this automorphism by $\phi_{i,j}$, called automorphism induced by the map $a \to a^i$ and $b \to ba^j$. Hence

$$Aut(G_n) = \{\phi_{i,j} \ : \ 0 \leq i, j \leq n-1, (i, n) = 1\}$$

□

Theorem 2.10. *$Aut(G_n)$ is isomorphic to G_n for $n = p$ or $2p$ where p is an odd prime.*

Proof. Define $\psi : Aut(G_n) \to G_n$ by

$$\psi(\phi_{i,j}) = \begin{bmatrix} i & j \\ 0 & 1 \end{bmatrix}, \ 0 \leq i, j \leq n-1, (i, n) = 1$$

Now,

$$\phi_{i,j} \circ \phi_{k,s}(a) = \phi_{i,j}(a^k) = a^{ik}$$

and

$$\phi_{i,j} \circ \phi_{k,s}(b) = \phi_{i,j}(ba^k) = \phi_{i,j}(b)\phi_{i,j}(a^k) = ba^j a^{ik} = ba^{ik+j}$$

Hence $\phi_{i,j} \circ \phi_{k,s} = \phi_{l,m}$ where $l \equiv ik \, mod(n)$ and $m \equiv (ik + j) \, mod(n)$.

So,

$$\psi(\phi_{i,j} \circ \phi_{k,s}) = \psi(\phi_{l,m}) = \begin{bmatrix} l & m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ik \, mod(n) & (j + is) \, mod(n) \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} i & j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k & s \\ 0 & 1 \end{bmatrix} = \psi(\phi_{i,j})\psi(\phi_{k,s})$$

Clearly ψ is one-one and onto. Hence ψ is an isomorphism from $Aut(G_n)$ onto G_n. □

It well known that D_4 is the only abelian Dihedral group and that its group of automorphisms is the symmetric group of order $6(S_3)$. Thus $Aut(AutD_4) \cong AutS_3 \cong S_3 \cong AutD_4$. $Aut(AutD_8)$ is isomorphic to G_4 which is a non-abelian group of order 8. D_8 has 4 inner automorphisms in which every element has order 2 except trivial automorphism. Hence $AutD_8 \cong D_8$. So $Aut(AutD_8) \cong AutD_8$.

Hence we have the following.

Theorem 2.11. *Let $n = p$ or $2p$ where p is prime. Then $Aut(Aut(D_{2n}))$ is isomorphic to $Aut(D_{2n})$.*

3. Conclusion

In this paper, it is proved that $Aut(Aut(D_{2n}))$ is isomorphic to $Aut(D_{2n})$ whenever n is twice a prime number. The case when n is even and not twice a prime number will be considered in future work.

References

[1] Adney, J. E.; Yen, T. Automorphisms of p-group. *Illinois J. Math.* **9** (1965), 137–143.
[2] Christopher, H.; Darren, R. Automorphisms of finite abelian groups. *Amer. Math. Monthly* **114** (2007), no. 10, 917–923.
[3] Conrad, K. Dihedral groups-II, *http://www.math.uconn.edu/kconrad/blurbs/grouptheory/dihedral2. pdf*, 2009.
[4] Curran, M. J. Semidirect product groups with abelian automorphism groups. *J. Austral. Math. Soc. Ser. A.* **42** (1987), no. 1, 84–91.
[5] Dummit, D. S.; Foote, R. M. *Abstract algebra*. Wiley Hoboken, 2003.
[6] Earn ley, B. E. On finite groups whose group of automorphisms is abelian. *PhD thesis*, Wayne State University, Detroit, Michigan, 1975.
[7] Gallian, J. A. *Contemporary Abstract Algebra*. D. C. Heath and Company, 1994.
[8] Guichard, D. R. When is $U(n)$ cyclic? An algebraic approach. *Math. Mag*. **72** (1999), no. 2, 139–142.
[9] Miller, GA. On the holomorph of a cyclic group. *Trans. Amer. Math. Soc.* **4** (1903), no. 2, 153–160.
[10] Miller, G. A. A non-abelian group whose group of isomorphism is abelian. *Messenger Math.* **43** (1913), 124–125.
[11] Miller, G. A. Automorphisms of the dihedral groups. *Proc. Nat. Acad. Sci. U.S.A.* **28** (1942), no. 9, 368–371.
[12] Morigi, M. On p-groups with abelian automorphism group. *Rend. Sem. Mat. Univ. Padova* **92** (1994), 47–58.
[13] Ranum, A. The Group of Classes of Congruent Matrices with Application to the Group of Isomorphisms of any Abelian Group. *Trans. Amer. Math. Soc.* **8** (1907), 71–91.
[14] Walls, G. L. Automorphism groups. *The Amer. Math. Monthly.* **93** (1986), no. 6, 459–462.

Department of Mathematics
College of Engineering Trivandrum
Thiruvananthapuram-695016, Kerala, India
Email address: sajikumar.s@cet.ac.in
Email address: wenod76@gmail.com
Email address: gsbiju@cet.ac.in