CREAT. MATH. INFORM.
Volume **32** (2023), No. 2,
Pages 247 - 253

Online version at `https://semnul.com/creative-mathematics/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2023.02.12

Statements and open problems on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that contain informal notions and refer to the current knowledge on $\mathcal{X}$

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ABSTRACT. For a set $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven, we define which elements of $\mathcal{X}$ are classified as known. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions `(1)`–`(4)` and is widely known in number theory or naturally defined, where this term has only informal meaning. `(1)` *A known algorithm with no input returns an integer n satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. `(2)` *A known algorithm for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$. `(3)` *No known algorithm with no input returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$. `(4)` *There are many elements of* $\mathcal{X}$ *and it is conjectured, though so far unproven, that* $\mathcal{X}$ *is infinite.* `(5)` $\mathcal{X}$ *is naturally defined. The infiniteness of* $\mathcal{X}$ *is false or unproven.* $\mathcal{X}$ *has the simplest definition among known sets* $\mathcal{Y} \subseteq \mathbb{N}$ *with the same set of known elements.* The set $\mathcal{X} = \{n \in \mathbb{N} : \textit{the interval } [-1, n] \textit{ contains more than } 29.5 + \frac{11!}{3n+1} \cdot \sin(n) \textit{ primes of the form } k! + 1\}$ satisfies Conditions `(1)`–`(5)` except the requirement that $\mathcal{X}$ is naturally defined. $501893 \in \mathcal{X}$. Condition `(1)` holds with $n = 501893$. $\mathrm{card}(\mathcal{X} \cap [0, 501893]) = 159827$. $\mathcal{X} \cap [501894, \infty) = \{n \in \mathbb{N} : \textit{the interval } [-1, n] \textit{ contains at least } 30 \textit{ primes of the form } k!+1\}$. We present a table that shows satisfiable conjunctions of the form #(Condition 1)∧(Condition 2)∧ #(Condition 3) ∧ (Condition 4) ∧ #(Condition 5), where # denotes the negation ¬ or the absence of any symbol. No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions `(1)`–`(4)` forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.

This article is a continuation of the article [15]. The results of this article and the article [15] were presented at the 25th Conference Applications of Logic in Philosophy and the Foundations of Mathematics, see `http://www.applications-of-logic.uni.wroc.pl/Program-1`. Nicolas D. Goodman observed that epistemic notions increase the scope of mathematics, see [4]. The article [4] does not discuss the notion of the current mathematical knowledge.

1. BASIC DEFINITIONS

Algorithms always terminate. Semi-algorithms may not terminate. There is the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in ZFC) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known), see [2], [10], [12, p. 9], [15]. A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n. Definition 1.1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

Definition 1.1. We say that a non-negative integer k is a known element of $\mathcal{X}$, if $k \in \mathcal{X}$ and we know an algebraic expression that defines k and consists of the following signs: 1 (one), + (addition), − (subtraction), · (multiplication), ^ (exponentiation with exponent in $\mathbb{N}$), ! (factorial of a non-negative integer), ((left parenthesis),) (right parenthesis).

The set of known elements of $\mathcal{X}$ is finite and time-dependent, so cannot be defined in the formal language of classical mathematics. Let t denote the largest twin prime that is

Received: 24.10.2022. In revised form: 24.05.2023. Accepted: 31.05.2023
2020 *Mathematics Subject Classification.* 03F55, 11A41.
Key words and phrases. *constructive algorithms, current knowledge, decidable sets, epistemic notions, informal notions, known algorithms, known elements, special primes.*

smaller than $((((((((9!)!)!)!)!)!)!)!)!$. The number t is an unknown element of the set of twin primes.

Definition 1.2. Conditions `(1)`-`(5)` concern sets $\mathcal{X} \subseteq \mathbb{N}$.
`(1)` A known algorithm with no input returns an integer n satisfying $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
`(2)` A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.
`(3)` No known algorithm with no input returns the logical value of the statement $\mathrm{card}(\mathcal{X}) = \omega$.
`(4)` There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is infinite.
`(5)` $\mathcal{X}$ is naturally defined. The infiniteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.

Condition `(3)` implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions `(1)`-`(4)` and is widely known in number theory or naturally defined, where this term has only informal meaning.

2. Main results

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite, see [13], [14], [16].

Statement 1. *The statement*

$$\exists n \in \mathbb{N}\ (\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, n+3])$$

remains unproven in ZFC and classical logic without the law of excluded middle.

Let $f(1) = 10^6$, and let $f(n+1) = f(n)^{f(n)}$ for every positive integer n.

Statement 2. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies Conditions `(1)`-`(4)`. *Condition* `(5)` *fails for* $\mathcal{X}$.

Proof. Condition `(4)` holds as $\mathcal{X} \supseteq \{0, \ldots, 10^6\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. Due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(10^6)$, see [8]. Thus Condition `(3)` holds. Condition `(2)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (10^6 < k) \wedge (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, Condition `(1)` holds with $n = 10^6$. Condition `(5)` fails as the set of known elements of $\mathcal{X}$ equals $\{0, \ldots, 10^6\}$. □

Statements 3 and 4 provide stronger examples.

Conjecture 2.1. *(*[1, p. 443], [5]*). The are infinitely many primes of the form* $k!+1$.

For a non-negative integer n, let $\rho(n)$ denote $29.5 + \frac{11!}{3n+1} \cdot \sin(n)$.

Statement 3. *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : \textit{the interval } [-1, n] \textit{ contains more than } \rho(n) \textit{ primes of the form } k!+1\}$$

satisfies Conditions `(1)`-`(5)` *except the requirement that* $\mathcal{X}$ *is naturally defined.* $501893 \in \mathcal{X}$. *Condition* `(1)` *holds with* $n = 501893$. $\mathrm{card}(\mathcal{X} \cap [0, 501893]) = 159827$. $\mathcal{X} \cap [501894, \infty) = \{n \in \mathbb{N} : \textit{the interval } [-1, n] \textit{ contains at least } 30 \textit{ primes of the form } k!+1\}$.

Proof. For every integer $n \geqslant 11!$, 30 is the smallest integer greater than $\rho(n)$. By this, if $n \in \mathcal{X} \cap [11!, \infty)$, then $n+1$, $n+2$, $n+3$, $\ldots \in \mathcal{X}$. Hence, Condition (1) holds with $n = 11! - 1$. We explicitly know 24 positive integers k such that $k!+1$ is prime, see [3]. The inequality $\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k!+1 \textit{ is prime}\}) > 24$ remains unproven. Since $24 < 30$, Condition (3) holds. The interval $[-1, 11!-1]$ contains exactly three primes of the form $k!+1$: $1!+1$, $2!+1$, $3!+1$. For every integer $n > 503000$, the inequality $\rho(n) > 3$ holds. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 503000.0 do
if n<1!+1 then r:=0 end_if:
if n>=1!+1 and n<2!+1 then r:=1 end_if:
if n>=2!+1 and n<3!+1 then r:=2 end_if:
if n>=3!+1 then r:=3 end_if:
if r>29.5+(11!/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line [501893.0, 159827], which proves Condition (4). □

To formulate Statement 4 and its proof, we need some lemmas. For a non-negative integer n, let $\theta(n)$ denote the largest integer divisor of $10^{10^{10}}$ smaller than n. For a non-negative integer n, let $\theta_1(n)$ denote the largest integer divisor of 10^{10} smaller than n.

Lemma 2.1. *For every integer* $j > 10^{10^{10}}$, $\theta(j) = 10^{10^{10}}$. *For every integer* $j > 10^{10}$, $\theta_1(j) = 10^{10}$.

Lemma 2.2. *For every integer* $j \in (6553600, 7812500]$, $\theta(j) = 6553600$.

Proof. 6553600 equals $2^{18} \cdot 5^2$ and divides $10^{10^{10}}$. $7812500 < 2^{24}$. $7812500 < 5^{10}$. We need to prove that every integer $j \in (6553600, 7812500)$ does not divide $10^{10^{10}}$. It holds as the set

$$\left\{2^u \cdot 5^v : (u \in \{0, \ldots, 23\}) \wedge (v \in \{0, \ldots, 9\})\right\}$$

contains 6553600 and 7812500 as consecutive elements. □

Lemma 2.3. *The number* $6553600^2 + 1$ *is prime.*

Proof. The following PARI/GP ([9]) command

```
isprime(6553600^2+1,{flag=2})
```

returns 1. This command performs the APRCL primality test, the best deterministic primality test algorithm ([17, p. 226]). It rigorously shows that the number $6553600^2 + 1$ is prime. □

In the next lemmas, the execution of the command `isprime(n,{flag=2})` proves the primality of n. Let κ denote the function

$$\mathbb{N} \ni n \xrightarrow{\kappa} the\_exponent\_of\_2\_in\_the\_prime\_factorization\_of\_\underbrace{n+1} \in \mathbb{N}$$

Lemma 2.4. *The set* $\mathcal{X}_1 = \{n \in \mathbb{N} : (\theta_1(n) + \kappa(n))^2 + 1 \textit{ is prime}\}$ *is infinite.*

Proof. Let $i = 142101504$. By the inequality $2^i \geqslant 2 + 10^{10}$ and Lemma 2.1, for every non-negative integer m, the number

$$\left(\theta_1\left(2^i \cdot (2m+1) - 1\right) + \kappa\left(2^i \cdot (2m+1) - 1\right)\right)^2 + 1 = \left(10^{10} + i\right)^2 + 1$$

is prime. □

Before Open Problem 1, $\mathcal{X}$ denotes the set $\{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1 \textit{ is prime}\}$.

Lemma 2.5. *For every* $n \in \mathcal{X} \cap \left(10^{10^{10}}, \infty\right)$ *and for every non-negative integer* j, $3^j \cdot (n+1) - 1 \in \mathcal{X} \cap \left(10^{10^{10}}, \infty\right)$.

Proof. By the inequality $3^j \cdot (n+1) - 1 \geqslant n$ and Lemma 2.1,

$$\theta\left(3^j \cdot (n+1) - 1\right) + \kappa\left(3^j \cdot (n+1) - 1\right) = 10^{10^{10}} + \kappa(n) = \theta(n) + \kappa(n)$$

□

Lemma 2.6. $\operatorname{card}(\mathcal{X}) \geqslant 629450$.

Proof. By Lemmas 2.2 and 2.3, for every even integer $j \in (6553600, 7812500]$, the number $(\theta(j) + \kappa(j))^2 + 1 = (6553600 + 0)^2 + 1$ is prime. Hence,

$$\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500] \subseteq \mathcal{X}$$

Consequently,

$$\operatorname{card}(\mathcal{X}) \geqslant \operatorname{card}(\{2k : k \in \mathbb{N}\} \cap (6553600, 7812500]) = \frac{7812500 - 6553600}{2} = 629450$$

□

Lemma 2.7. $10242 \in \mathcal{X}$ *and* $10242 \notin \mathcal{X}_1$.

Proof. The number $10240 = 2^{11} \cdot 5$ divides $10^{10^{10}}$. Hence, $\theta(10242) = 10240$. The number $(\theta(10242) + \kappa(10242))^2 + 1 = (10240 + 0)^2 + 1$ is prime. The set

$$\left\{2^u \cdot 5^v : (u \in \{0, \ldots, 10\}) \wedge (v \in \{0, \ldots, 10\})\right\}$$

contains 10000 and 12500 as consecutive elements. Hence, $\theta_1(10242) = 10000$. The number $(\theta_1(10242) + \kappa(10242))^2 + 1 = (10000 + 0)^2 + 1 = 17 \cdot 5882353$ is composite. □

Statement 4. *The set* $\mathcal{X}$ *satisfies Conditions* `(1)` – `(5)` *except the requirement that* $\mathcal{X}$ *is naturally defined.*

Proof. Condition `(2)` holds trivially. Let δ denote $10^{10^{10}}$. By Lemma 2.5, Condition `(1)` holds for $n = \delta$. Lemma 2.5 and the unproven statement $\mathcal{P}_{n^2+1} \cap \left[\delta^2 + 1, \infty\right) \neq \emptyset$ show Condition `(3)`. The same argument and Lemma 2.6 yield Condition `(4)`. By Lemma 2.4, the set $\mathcal{X}_1$ is infinite. Since Definition 1.1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven, Condition `(5)` holds except the requirement that $\mathcal{X}$ is naturally defined. □

The set $\mathcal{X}$ satisfies Condition `(5)` except the requirement that $\mathcal{X}$ is naturally defined. It is true because $\mathcal{X}_1$ is infinite by Lemma 2.4 and Definition 1.1 applies only to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven. Ignoring this restriction, $\mathcal{X}$ still satisfies the same identical condition due to Lemma 2.7.

Proposition 2.1. *No set* $\mathcal{X} \subseteq \mathbb{N}$ *will satisfy Conditions* `(1)` – `(4)` *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies Condition `(1)`. Since Conditions `(1)`–`(3)` will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \ldots \qquad \texttt{(T)}$$

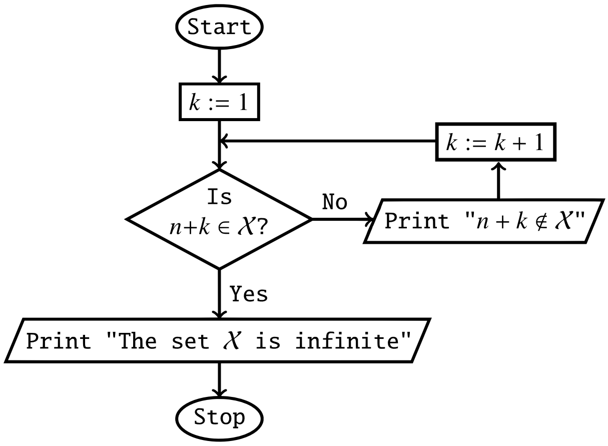

Figure 1 Semi-algorithm that terminates if and only if $\mathcal{X}$ is infinite

The sentences from the sequence `(T)` and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in Condition `(4)`. □

The physical limits of computation ([8]) disprove the assumption of Proposition 2.1.

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions* `(1)`–`(5)`*?*

Open Problem 1 asks about the existence of a year $t \geqslant 2022$ in which the conjunction

$$(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge (\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge (\textit{Condition } 5)$$

will hold for some $\mathcal{X} \subseteq \mathbb{N}$. For every year $t \geqslant 2022$ and for every $i \in \{1, 2, 3\}$, a positive solution to Open Problem i in the year t may change in the future. Currently, the answers to Open Problems 1–5 are negative.

3. Satisfiable conjunctions which consist of Conditions `(1)`–`(5)` and their negations

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge (\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge (\textit{Condition } 5)$$

The set $\mathcal{X} = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\textit{Condition } 1) \wedge (\textit{Condition } 2) \wedge (\textit{Condition } 3) \wedge (\textit{Condition } 4) \wedge \neg(\textit{Condition } 5)$$

The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [7, p. 158] and [11, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [7, p. 159] and [11, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geqslant 5$, see [6, p. 23].

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \; if \; 2^{2^{f(9^9)}} + 1 \; is \; composite \\ \{0, \ldots, 10^6\}, \; otherwise \end{cases}$$

satisfies the conjunction

$$(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)?$$

The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \; if \; 2^{2^{f(9^9)}} + 1 \; is \; composite \\ \{0, \ldots, 10^6\} \cup \\ \{n \in \mathbb{N} : n \; is \; the \; sixth \; prime \; number \; of \; the \; form \; 2^{2^k} + 1\}, \; otherwise \end{cases}$$

satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1 \; is \; composite\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

Table 1 shows satisfiable conjunctions of the form

$$\#(\text{Condition } 1) \land (\text{Condition } 2) \land \#(\text{Condition } 3) \land (\text{Condition } 4) \land \#(\text{Condition } 5)$$

where # denotes the negation ¬ or the absence of any symbol. Table 1 differs from Table 1 in [15] for three sets $\mathcal{X}$. These sets $\mathcal{X}$ have the index *new*.

	(Cond. 2) ∧ (Cond. 3) ∧ (Cond. 4)	(Cond. 2) ∧ ¬(Cond. 3) ∧ (Cond. 4)
(Cond. 1) ∧ (Cond. 5)	Open Problem 1	Open Problem 2
(Cond. 1) ∧ ¬(Cond. 5)	$\mathcal{X}_{new} = \{n \in \mathbb{N} : \; the \; interval \; [-1, n] \; contains \; more \; than \; 29.5 + \frac{11!}{3n+1} \cdot \sin(n) \; primes \; of \; the \; form \; k! + 1\}$	$\mathcal{X}_{new} = \begin{cases} \mathbb{N}, \; if \; 2^{2^{f(9^9)}} + 1 \; is \; composite \\ \{0, \ldots, 10^6\}, \; otherwise \end{cases}$
¬(Cond. 1) ∧ (Cond. 5)	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
¬(Cond. 1) ∧ ¬(Cond. 5)	$\mathcal{X} = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X}_{new} = \begin{cases} \mathbb{N}, \; if \; 2^{2^{f(9^9)}} + 1 \; is \; composite \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n \; is \\ the \; sixth \; prime \; number \; of \\ the \; form \; 2^{2^k} + 1\}, \; otherwise \end{cases}$

Table 1 Five satisfiable conjunctions

Definition 3.3. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Open Problem 4. *Is there a known threshold number of $\mathcal{P}_{n^2+1}$?*

Open Problem 4 asks about the existence of a year $t \geqslant 2022$ in which the implication $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, n]$ will hold for some known integer n.

Let $\mathcal{T}$ denote the set of twin primes.

Open Problem 5. *Is there a known threshold number of $\mathcal{T}$?*

Open Problem 5 asks about the existence of a year $t \geqslant 2022$ in which the implication $\mathrm{card}(\mathcal{T}) < \omega \Rightarrow \mathcal{T} \subseteq (-\infty, n]$ will hold for some known integer n.

References

[1] Caldwell, C. K.; Gallot, Y. On the primality of $n! \pm 1$ and $2 \times 3 \times 5 \times \cdots \times p \pm 1$, *Math. Comp.* **71** (2002), no. 237, 441–448. http://doi.org/10.1090/S0025-5718-01-01315-1.

[2] Case, J.; Ralston, M. Beyond Rogers' non-constructively computable function. in: *The nature of computation*, Lecture Notes in Comput. Sci., 7921, 45–54, Springer, Heidelberg, 2013. http://link.springer.com/chapter/10.1007/978-3-642-39053-1_6.

[3] *Factorial prime* http://en.wikipedia.org/wiki/Factorial_prime.

[4] Goodman, N. D. The knowing mathematician. *Synthese* **60** (1984), no. 1, 21–38. http://link.springer.com/article/10.1007/BF00485616; reprinted in: Leblanc, H., Mendelson, E., Orenstein, A., *Foundations: Logic, Language, and Mathematics,* Springer, 1984, 21–38, http://link.springer.com/book/10.1007/978-94-017-1592-8.

[5] *Is $n! + 1$ often a prime?* http://math.stackexchange.com/questions/853085/is-n-1-often-a-prime.

[6] De Koninck, J.-M.; Luca, F. *Analytic number theory: Exploring the anatomy of integers.* American Mathematical Society, Providence, RI, 2012.

[7] Křížek, M.; Luca, F.; Somer, L. *17 lectures on Fermat numbers: from number theory to geometry.* Springer, New York, 2001.

[8] Lloyd, S. Ultimate physical limits to computation. *Nature* **406** (2000), 1047–1054. http://doi.org/10.1038/35023282.

[9] PARI/GP *online documentation.* http://pari.math.u-bordeaux.fr/dochtml/html/Arithmetic_functions.html.

[10] Reitzig, R. *How can it be decidable whether π has some sequence of digits?* http://cs.stackexchange.com/questions/367/how-can-it-be-decidable-whether-pi-has-some-sequence-of-digits.

[11] Ribenboim, P. *The little book of bigger primes*. 2nd ed., Springer, New York, 2004.

[12] Rogers, H. Jr. *Theory of recursive functions and effective computability*. 2nd ed., MIT Press, Cambridge, MA, 1987.

[13] Sloane, N. J. A. *The On-Line Encyclopedia of Integer Sequences A002496, Primes of the form $n^2 + 1$.* http://oeis.org/A002496.

[14] Sloane, N. J. A., *The On-Line Encyclopedia of Integer Sequences. A083844, Number of primes of the form $x^2 + 1 < 10^n$.* http://oeis.org/A083844.

[15] Tyszka, A. Statements and open problems on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that refer to the current knowledge on $\mathcal{X}$. *J. Appl. Math. Comput.* **16** (2022), no. 2, 31–35. http://doi.org/10.4316/JACSM.202202005.

[16] Wolfram MathWorld *Landau's Problems.* http://mathworld.wolfram.com/LandausProblems.html.

[17] Yan, S. Y. *Number theory for computing.* 2nd ed., Springer, Berlin, 2002.

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