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On a useful lemma that relates quasi-nonexpansive and demicontractive mappings in Hilbert spaces

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ABSTRACT. We give a brief account on a basic result (Lemma 3.2) which is a very useful tool in proving various convergence theorems in the framework of the iterative approximation of fixed points of demicontractive mappings in Hilbert spaces. This Lemma relates the class of quasi-nonexpansive mappings, by one hand, and the class of k-demicontractive mappings (quasi k-strict pseudocontractions), on the other hand and essentially states that the class of demicontractive mappings, which strictly includes the class of quasi-nonexpansive mappings, can be embedded in the later by means of an averaged perturbation. From the point of view of the fixed point problem, this means that any convergence result for Krasnoselskij-Mann iterative algorithms in the class of k-demicontractive mappings can be derived from its counterpart from quasi-nonexpansive mappings.

1. INTRODUCTION

Nonexpansive type operators are extremely important in the metric fixed point theory, both from the theoretical point of view and especially for their large areas of applications, see [21] for a very recent survey. In this note we shall refer mainly to the following classes of mappings: nonexpansive, quasi-nonexpansive, k-strictly pseudocontractive (in the sense of Browder and Petryshyn) and quasi k-strictly pseudocontractive (commonly called demicontractive), which, although largely well known, are defined in the following for the sake of completeness.

Let H be a real Hilbert space with norm and inner product denoted as usually by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $C\subset H$ be a closed and convex set and $T:C\to C$ be a self mapping. Denote by

$$Fix\,(T)=\{x\in C: Tx=x\}$$

the set of fixed points of T.

Definition 1.1. The mapping T is said to be:

1) *nonexpansive* if

$$\|Tx-Ty\|\leq\|x-y\|,\ \text{for all } x,y\in C. \tag{1.1}$$

2) *quasi-nonexpansive* if $Fix\,(T)\neq\emptyset$ and

$$\|Tx-y\|\leq\|x-y\|,\ \text{for all } x\in C \text{ and } y\in Fix\,(T). \tag{1.2}$$

3) *k-strictly pseudocontractive* of the Browder-Petryshyn type ([26]) if there exists $k<1$ such that

$$\|Tx-Ty\|^2\leq\|x-y\|^2+k\|x-y-Tx+Ty\|^2,\forall x,y\in C. \tag{1.3}$$

4) *k-demicontractive* ([63]) or *quasi k-strictly pseudocontractive* (see [24]) if $Fix\,(T)\neq\emptyset$ and there exists a positive number $k<1$ such that

$$\|Tx-y\|^2\leq\|x-y\|^2+k\|x-Tx\|^2, \tag{1.4}$$

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for all $x \in C$ and $y \in Fix\,(T)$.

It is known, see the remarks following Definition 3.2, that Definition 1.1 4) is equivalent, in the setting of a Hilbert space, with Definition 3.2, that is, (1.4) is equivalent to (3.7).

Let us denote by $\mathcal{NE}$, $\mathcal{QNE}$, $\mathcal{SPC}$ and $\mathcal{DC}$ the classes of nonexpansive, quasi-nonexpansive, k-strictly pseudocontractive (in the sense of Browder and Petryshyn) and quasi k-strictly pseudocontractive (demicontractive), respectively.

In Metrical Fixed Point Theory there was a long standing and there still exists a steadily increasing interest for studying the existence and approximation of fixed points of mappings in all of the above four classes of mappings and in many related ones like asymptotically nonexpansive, firmly nonexpansive etc.

Most of the literature is devoted to mappings in the classes $\mathcal{NE}$, $\mathcal{QNE}$, $\mathcal{SPC}$ but, starting with the year 2008, there was also an increasing interest for studying the mappings in the class $\mathcal{DC}$, see the very recent survey [21] and especially the consistent list of references therein, of which most are also included here, for the sake of completeness, see [2]-[13], [20]-[25], [28]-[109], [111]-[178].

In order to establish convergence theorems for fixed point iteration schemes, some authors ([88], [63], [98],...) have used implicitly or explicitly ([91], [20]) a lemma that relates the classes $\mathcal{QNE}$ and $\mathcal{DC}$.

The aim of this note is to review some of the most important moments in the process of discovering and use of this Lemma in order to prove convergence theorems in the class of demicontractive operators.

2. THE COMPLETE INCLUSION DIAGRAM OF THE CLASSES $\mathcal{NE}$, $\mathcal{QNE}$, $\mathcal{SPC}$ AND $\mathcal{DC}$

To our best knowledge, there is no any paper that includes together a diagram of the four classes of nonexpansive type mappings $\mathcal{NE}$, $\mathcal{QNE}$, $\mathcal{SPC}$ and $\mathcal{DC}$, which should clearly show by appropriate examples the complete map of the relationships existing between all of them.

So, we are doing this is the present section, mainly for its use in this note but also for the importance itself of such a diagram.

The next two simple examples show that $\mathcal{NE}$ and $\mathcal{QNE}$ are independent sets, i.e., $\mathcal{NE} \cap \mathcal{QNE} \neq \emptyset$, $\mathcal{NE}$ is not included in $\mathcal{QNE}$ and $\mathcal{QNE}$ is not included in $\mathcal{NE}$.

Example 2.1. Let H be the real line with the usual norm, $C = [0, 1]$ and $T_1x = 1 + x$, $x \in [0, 1]$. Then: 1) $T_1 \in \mathcal{NE}$; 2) $Fix\,(T_1) = \emptyset$; 3) $T_1 \notin \mathcal{QNE}$.

Example 2.2. Let H be the real line with the usual norm, $C = [0, 2]$ and $T_2x = 2 - x$, $x \in [0, 2]$. Then: 1) $T_2 \in \mathcal{NE}$; 2) $T_2 \in \mathcal{QNE}$; 3) $Fix\,(T_2) = \{1\}$.

The following lemma follows immediately from Definition 1.1.

Lemma 2.1.
$$\mathcal{NE} \subseteq \mathcal{SPC}; \tag{2.5}$$
$$\mathcal{QNE} \subseteq \mathcal{DC}. \tag{2.6}$$

By means of the next example we show that inclusion (2.5) is strict, i.e., $\mathcal{NE} \subsetneq \mathcal{SPC}$.

Example 2.3. Let H be the real line with the usual norm, $C = \left[\dfrac{1}{2}, 2\right]$ and $T_3 : C \to C$ defined by $T_3(x) = \dfrac{1}{x}, \forall x \in C$. Then: 1) $Fix\,(T_3) \neq \emptyset$; 2) $T_3 \in \mathcal{SPC}$ (and hence $T_3 \in \mathcal{DC}$); 3) $T_3 \notin \mathcal{NE}$; 4) $T_3 \notin \mathcal{QNE}$.

Proof. 1) $Fix\,(T_3) = \{1\}$;

2) By (1.3), $T_3 \in \mathcal{SPC}$ if there exists $k \in (0,1)$ such that, for all $x, y \in C$,

$$\|T_3x - T_3y\|^2 \leq \|x-y\|^2 + k\|x - y - T_3x + T_3y\|^2,$$

which in our case reduces to

$$\left|\frac{1}{x} - \frac{1}{y}\right|^2 \leq |x-y|^2 + k\left|x - \frac{1}{x} - y + \frac{1}{y}\right|^2 \Leftrightarrow 1 \leq x^2y^2 + k(1+xy)^2, x, y \in \left[\frac{1}{2}, 2\right].$$

By denoting $t := xy$, it follows that $t \in \left[\frac{1}{4}, 4\right]$ and hence we have to prove that there exists $k > 0$ such that

$\dfrac{1-t^2}{(1+t)^2} \leq k < 1$, for all $t \in \left[\frac{1}{4}, 4\right]$. Consider the function $f(t) := \dfrac{1-t^2}{(1+t)^2}$, $t \in \left[\frac{1}{4}, 4\right]$.

Since $f'(t) = -\dfrac{2}{(1+t)^2} < 0$, it follows that f is strictly decreasing on $\left[\frac{1}{4}, 4\right]$, which implies

$f(t) \leq f\left(\frac{1}{4}\right) = \frac{3}{5}$, for all $t \in \left[\frac{1}{4}, 4\right]$.

This shows that one can choose $k = \frac{3}{5}$ and so, T_3 is $\frac{3}{5}$-strictly pseudocontractive.

3) Assume $T_3 \in \mathcal{NE}$, i.e., $|T_3x - T_3y| \leq |x-y|$, $\forall x, y \in C = \left[\frac{1}{2}, 2\right]$ and take $x = \frac{1}{2}$ and $y = 1$ to get $|2-1| \leq \left|\frac{1}{2} - 1\right| \Leftrightarrow 1 \leq \frac{1}{2}$, a contradiction. Hence $T_3 \notin \mathcal{NE}$.

4) Assume $T_3 \in \mathcal{QNE}$, i.e., $|T_3x - 1| \leq |x-1|$, $\forall x \in C$ and take $x = \frac{1}{2}$ and $y = 1$ as in the previous case to get $|2-1| \leq \left|\frac{1}{2} - 1\right| \Leftrightarrow 1 \leq \frac{1}{2}$, a contradiction. So, $T_3 \notin \mathcal{QNE}$.

□

Example 2.4. Let H be the real line with the usual norm and $C = [0,2]$. Define $T_4 : [0,2] \to [0,2]$ by $T_4x = \dfrac{x^2+2}{x+1}$, for all $x \in [0,2]$. Then: 1) $Fix\,(T_4) \neq \emptyset$; 2) $T_4 \in \mathcal{QNE}$; 3) $T_4 \notin \mathcal{NE}$; 4) $T_4 \in \mathcal{SPC}$.

Proof. 1) $Fix\,(T_4) = \{2\}$;

2) For $y = 2$ and $x \in [0,2]$, by (1.2) we have

$$|T_4x - 2| = \left|\frac{x^2+2}{x+1} - 2\right| = \frac{x}{x+1} \cdot |x-2| \leq |x-2|,\ x \in [0,2],$$

and so $T_4 \in \mathcal{QNE}$.

3) Just consider $x = 0$ and $y = \frac{1}{3}$ in (1.1) to get

$$\frac{5}{12} = \left|T_4 0 - T_4\frac{1}{3}\right| \leq \left|0 - \frac{1}{3}\right| = \frac{1}{3},$$

a contradiction since $\frac{5}{12} > \frac{1}{3}$. So, $T_4 \notin \mathcal{NE}$.

4) For any $x \neq y$, we divide by $(x-y)^2$ the strict pseudocontractive condition (1.3)

$$\left|\frac{x^2+2}{x+1} - \frac{y^2+2}{y+1}\right|^2 \leq |x-y|^2 + k\left|x - y - \frac{x^2+2}{x+1} + \frac{y^2+2}{y+1}\right|^2$$

to get the inequality

$$t^2 \leq 1 + k(1-t)^2 \Leftrightarrow \frac{t+1}{t-1} \leq k,$$

where we denoted

$$t = \frac{xy + x + y - 2}{(x+1)(y+1)}.$$

Since $-2 \leq t < 1$, it follows that the inequality

$$\frac{t+1}{t-1} \leq k$$

is satisfied for any $k \geq \frac{1}{3}$ (the maximum value of $\frac{t+1}{t-1}$ on the interval $[-2, 1)$).

This proves that $T_4 \in \mathcal{SPC}$.

□

Example 2.5. Let H be the real line with the usual norm and $C = [0, 1]$. Define T_5 on C by $T_5 x = \frac{1}{3}$, if $0 \leq x < \frac{2}{3}$ and $T_5 x = \frac{1}{2}$, if $\frac{2}{3} \leq x \leq 1$. Then:

1) $Fix\,(T_5) \neq \emptyset$; 2) $T_5 \in \mathcal{QNE}$; 3) $T_5 \notin \mathcal{NE}$; 4) $T_5 \notin \mathcal{SPC}$.

Proof. 1) $Fix\,(T_5) = \left\{\frac{1}{3}\right\}$;

2) Hence, the quasi-nonexpansiveness condition for T_5 becomes

$$\left|Tx - \frac{1}{3}\right| \leq \left|x - \frac{1}{3}\right|, \forall x \in [0, 1]$$

which, for $x \in \left[0, \frac{2}{3}\right)$ is obvious, while, for $x \in \left[\frac{2}{3}, 1\right]$ reduces to the obvious inequality $x \geq \frac{1}{2}$.

3) and 4): since T_5 is not continuous, we have $T_5 \notin \mathcal{NE}$ and $T_5 \notin \mathcal{SPC}$. □

Remark 2.1. *In order to illustrate how the discontinuity affects the strict pseudocontractivity, we give a direct proof of item 4) in the previous Example.*

Assume $T_5 \in \mathcal{SPC}$. Then, there exists a positive $k < 1$ such that

$$\|T_5 x - T_5 y\|^2 \leq \|x - y\|^2 + k\|x - y - T_5 x + T_5 y\|^2, \forall x, y \in [0, 1].$$

For $x \in \left[0, \frac{2}{3}\right)$ and $y \in \left[\frac{2}{3}, 1\right]$ this reduces to

$$\left|\frac{1}{3} - \frac{1}{2}\right|^2 \leq |x - y|^2 + k\left|x - y - \left(\frac{1}{3} - \frac{1}{2}\right)\right|^2.$$

Now take $y = \frac{2}{3}$ and let $x \to \frac{2}{3}$ in the previous inequality to get

$$\left(\frac{1}{6}\right)^2 \leq k \cdot \left(\frac{1}{6}\right)^2 \Leftrightarrow 1 \leq k$$

which contradicts the fact that $k < 1$.

The next example shows that the inclusion (2.6) is strict, i.e., $\mathcal{QNE} \subsetneq \mathcal{DC}$.

Example 2.6. Let H be the real line with the usual norm and $C = [0, 1]$. Define T_6 on C by $T_6 x = \frac{7}{8}$, if $0 \leq x < 1$ and $T_6 1 = \frac{1}{4}$. Then:

1) $Fix\,(T_6) \neq \emptyset$; 2) $T_6 \in \mathcal{DC}$; 3) $T_6 \notin \mathcal{NE}$; 4) $T_6 \notin \mathcal{QNE}$; 5) $T_6 \notin \mathcal{SPC}$.

Proof. 1) $Fix\,(T_6) = \left\{\frac{7}{8}\right\}$;

2) By taking $y = \frac{7}{8}$ and $x \in [0, 1)$, inequality (1.4) becomes:

$$|T_6 x - y|^2 = 0 \le |x - y|^2 + k|x - T_6 x|^2,$$

which obviously holds, for any $k > 0$.

It remains to check (1.4) for the case $x = 1$, which yields

$$\left|\frac{1}{4} - \frac{7}{8}\right|^2 \le \left|1 - \frac{7}{8}\right|^2 + k\left|1 - \frac{1}{4}\right|^2$$

and which holds true for any $k \ge \frac{2}{3}$. Hence T_6 is $\frac{2}{3}$-demicontractive.

3) To show that T_6 is not quasi-nonexpansive, take $x = 1$ and $y = \frac{7}{8}$ in (1.2), to get $\frac{5}{8} \le \frac{1}{8}$, a contradiction. Hence T_6 is not quasi-nonexpansive.

4) To prove that T_6 is not nonexpansive take $x = 1$ and $y = \frac{7}{8}$ in (1.1) to get the same contradiction as above.

5) Assume T_6 is k-strictly pseudocontractive, that is, there exists $k < 1$ such that (1.3) holds for any $x, y \in [0, 1]$. By taking $x \in [0, 1)$ and $y = 1$ in (1.3) we have

$$\left(\frac{5}{8}\right)^2 \le (x-1)^2 + k\left(x - 1 - \frac{5}{8}\right)^2, \; x \in [0, 1),$$

from which, by letting $x \to 1$ we obtain $1 \le k < 1$, a contradiction.

Hence T_6 is not strictly pseudocontractive. □

We also note that if $T \in \mathcal{SPC}$ and $Fix\,(T) \neq \emptyset$, then $T \in \mathcal{DC}$. Now, based on Lemma 2.1 and Examples 2.1-2.6, we have the following complete map of the relationships between the four classes of nonexpansive type mappings in Definition 1.1.

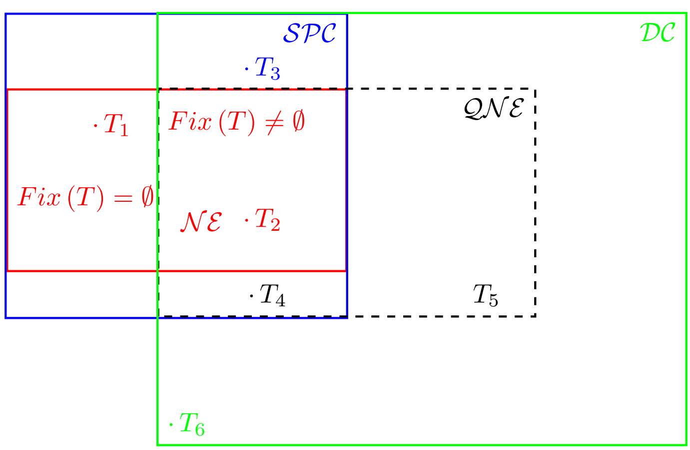

Figure 1. Diagram of the relationships between the classes $\mathcal{NE}$, $\mathcal{QNE}$, $\mathcal{SPC}$ and $\mathcal{DC}$

3. A Lemma that relates quasi-nonexpansive and demicontractive mappings

The main aim of this section is to present some historical facts about the use and formulation of an important lemma that relates quasi-nonexpansive and demicontractive mappings.

This result is of particular importance in proving convergence theorems for some fixed point iterative schemes like Krasnoselskij, Krasnoselskij-Mann etc. in the class of demicontractive mappings, by reducing the arguments to the same algorithms but in the class of quasi-nonexpansive mappings.

We state it in the form it has been presented and used in the paper [20] and, for the sake of completeness, we also give its proof.

Lemma 3.2 ([20], Lemma 3.2)**.** *Let H be a real Hilbert space, $C \subset H$ be a closed and convex set. If $T : C \to C$ is k-demicontractive, then for any $\lambda \in (0, 1-k)$, T_λ is quasi-nonexpansive.*

Proof. By hypothesis, we have $Fix\,(T) \neq \emptyset$ and there exists $k < 1$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2,\ x \in C \text{ and } y \in Fix\,(T)$$

which is equivalent to

$$\langle Tx - x, x - y\rangle \leq \frac{k-1}{2} \cdot \|x - Tx\|^2, x \in C,\ y \in Fix\,(T).$$

Then, for all $x \in C$ and $y \in Fix\,(T)$, we have

$$\|T_\lambda x - y\|^2 = \|\lambda(Tx - x) + x - y\|^2 = \|x - y\|^2 + 2\lambda\langle Tx - x, x - y\rangle$$
$$+\lambda^2\|Tx - x\|^2 \leq \|x - y\|^2 + (\lambda^2 + \lambda k - \lambda)\|Tx - x\|^2$$
$$= \|x - y\|^2 + \frac{\lambda^2 + \lambda k - \lambda}{\lambda^2} \cdot \|T_\lambda x - x\|^2,\ x \in C,\ y \in Fix\,(T).$$

So, if $\lambda^2 + \lambda k - \lambda < 0$, that is, $\lambda < 1 - k$, then the above inequality implies that

$$\|T_\lambda x - y\|^2 \leq \|x - y\|^2,\ x \in C,\ y \in Fix\,(T),$$

i.e., that T_λ is quasi-nonexpansive. □

We are now interested to trace back on the use of this simple but important Lemma. As it has been shown in the very recent survey paper [21], the demicontractive mappings were introduced independently in 1977 by Măruşter [88] and Hicks and Kubicek [63], respectively, in the setting of a Hilbert space.

The same notion has been introduced in 1973 by Măruşter [87], in the particular case of $\mathbb{R}^n$, but for the case of the nonlinear equation $U(x) = 0$. By simply taking $U = I - T$, one finds the same concept as the one introduced in [88]. This was the reason why in the survey paper [21] we have considered 1973 as the birth date of demicontractive mappings.

In order to present some facts about the early use of Lemma 3.2, we also give here Măruşter's definition [88] of demicontractive mappings. It is important to note that the term "demicontractive" was coined by Hicks and Kubicek [63], who introduced it by means of inequality (1.4).

Definition 3.2 (Măruşter [88])**.** Let H be a real Hilbert space and C a closed convex subset of H. A mapping $T : C \to C$ such that $Fix\,(T) \neq \emptyset$ is said to satisfy *condition (A)* if there exists $\lambda > 0$ such that

$$\langle x - Tx, x - x^*\rangle \geq \lambda\|Tx - x\|^2, \forall x \in C, x^* \in Fix\,(T). \tag{3.7}$$

Despite the fact that the two definitions were introduced in the same year and in very visible magazines, it was not apparent for a rather long time that the two inequalities (1.4) and (3.7), which involve different formulas, are actually equivalent in the setting of a Hilbert space.

This fact was observed more than two decades later, by Moore [97] and is based on the following identity, valid in a real Hilbert space:

$$\|x-x^*\|^2+k\|x-Tx\|^2-\|Tx-x^*\|^2=2\langle x-x^*,x-Tx\rangle-(1-k)\|x-Tx\|^2,$$

see [97]) for more details.

In our recent paper [20], based on Lemma 3.2, we have explicitly proven that, in Hilbert spaces, any convergence result for a Krasnoselkij type fixed point iterative algorithm in the class of demicontractive mappings can be deduced from its counterpart in the class of quasi-nonexpansive mappings.

But this fact was known and used implicitly long before by a few researchers that were working in this area. Our aim is to survey all those attempts that precede the more recent papers [136], [148] and [20], where Lemma 3.2 was explicitly stated.

1) In the proof of Theorem 1 in Măruşter [88], the author used the same arguments like the ones in the proof of Lemma 3.2.

Indeed, if we adapt the notations in [88] to our current ones, i.e., we denote the fixed point of T by x^* instead of ξ and the parameter t_k involved in the Mann iteration by t, what Măruşter [88] did, see the first 4 rows on page 70, is the following

$$\|T_tx-x^*\|^2=\|x-x^*-t(x-Tx)\|^2=\|x-x^*\|^2-2t\langle x-Tx,x-x^*\rangle$$
$$+t^2\|x-Tx\|^2\leq\|x-x^*\|^2+t(2\lambda-t)\|x-Tx\|^2$$

and since $2\lambda-t>0$, it follows that

$$\|T_tx-x^*\|\leq\|x-x^*\|,\ x\in C,\ x^*\in Fix\,(T),$$

which means that T_t is quasi-nonexpansive for $0<t<2\lambda$.

On the other hand, if we keep in mind the relationship between λ in (3.7) and k in (1.4), that is, $\lambda=\dfrac{1-k}{2}$, then we get exactly the condition on the parameter in Lemma 3.2 that ensures that the averaged operator T_t is quasi-nonexpansive.

As a matter of fact, in [88] all the above calculations were performed directly for the sequence $x_{k+1}=T_tx_k$ and not for the mapping T_t.

2) In the proof of Théoréme in [87], the same arguments were used, but for the case of the nonlinear equation $U(x)=0$. By simply taking $U=I-T$, the proof actually shows that the mapping T_μ is quasi-nonexpansive for $\mu<2\eta$, where η corresponds to λ in (3.7).

Similarly to [88], the author did all the calculations in [87] for the sequence $x_{p+1}=T_\mu x_p$ and not for the mapping T_μ.

3) In the proof of Theorem 1 in [63], the authors performed similar calculations to those in [88] but for the sequence $v_{n+1}=T_{d_n}v_n$ and not for the averaged mapping T_{d_n}.

4) In a series of papers from the period 2003-2009, see [89], [86], [90]-[92], Măruşter used Lemma 3.2 and even presented a complete proof of it, but in the framework of the proof of the main result established there. For example, in [89], this is done in the proof of Theorem 2. Lemma 3.2 is also explicitly stated and proved and then used to apply Theorem 1 in [89] (about quasi-nonexpansive mappings) to get the desired conclusion. Similar formulations of Lemma 3.2 do appear under various forms in the subsequent papers by Măruşter [86], [90]-[92].

5) In Remark 2.1 from Moudafi [98], Lemma 3.2 is explicitly stated and proven, as follows.

"Let T be a k-demicontractive self-mapping on $\mathcal{H}$ with $Fix\,(T) \neq \emptyset$ and set $T_w := (1-w)I+wT$ for $w \in (0,1]$. It is obviously checked that $Fix\,(T) = Fix\,(T_w)$. Moreover, T_w is quasi-nonexpansive for w small enough. Indeed, given an arbitrary $(x,q) \in \mathcal{H}\times Fix\,(T)$, we have

$$|T_w x - q|^2 = |(x-q) + w(Tx-x)|^2$$
$$= |x-q|^2 - 2w\langle x-q, x-Tx\rangle + w^2|Tx-x|^2$$

which by (1.5) (i.e., the demicontractive condition in Măruşter's form) yields

$$|T_w x - q|^2 \leq |x-q|^2 - w(1-k-w)|Tx-x|^2.$$

Consequently, if $w \in (0, 1-k]$, then T_w is quasi-nonexpansive..."

This explicit statement and its proof are reproduced in Maingé and Moudafi [84] (Remark 2.1), in Maingé [79] (Remark 4.2) and in some other papers by the same authors.

6) It appears that Tang et al. [136] were the first ones to state explicitly Lemma 3.2, by referring to Remark 2.1 from Moudafi [98].

7) The present author, who was not aware of the implicit or explicit statements of Lemma 3.2 reviewed previously, formulated it as an auxiliary result (Lemma 3.2) in [20], and, based on it, presented simpler and unifying proofs for the pioneering papers by Măruşter [88] and Hicks and Kubicek [63].

The title of [20], *Approximating fixed points results for demicontractive mappings could be derived from their quasi-nonexpansive counterparts*, as well as its first conclusions reproduced below should be taken into consideration by all researchers dealing with the study of demicontractive mappings.

"1. In this paper we have shown that the convergence theorems for Mann iteration used for approximating the fixed points of demicontractive mappings in Hilbert spaces could be derived from the corresponding convergence theorems in the class of quasi-nonexpansive mappings.

2. Our derivation is based on an imbedding technique described by Lemma 3.2, which essentially shows that if T is k-demicontractive, then for any $\lambda \in (0, 1-k)$, T_λ is quasi-nonexpansive.

3. In this way we obtained a unifying technique of proof for various well known results in the fixed point theory of demicontractive mappings that has been illustrated for the case of the first two classical convergence results in the class of demicontractive mappings in literature: Măruşter [88] and Hicks and Kubicek [63]."

We note that a similar technique also works for k-strict pseudocontractions, which can be embedded in the class of nonexpansive mappings in Hilbert spaces. This fact was first exploited by Browder and Petryshyn [26], [110], and also used much later by Zhou [174] in the case of nonself mappings.

4. Conclusions

1. I this paper we gave a brief account on a basic result (Lemma 3.2) which is a very useful tool in proving various convergence theorems in the framework of the iterative approximation of fixed points of demicontractive mappings in Hilbert spaces. This lemma relates the class of quasi-nonexpansive mappings, by one hand, and the class of k-demicontractive mappings (or quasi k-strict pseudocontractions), on the other hand and essentially states that the class of demicontractive mappings, which strictly includes the class of quasi-nonexpansive mappings, can be embedded in the later by means of an averaged perturbation.

2. From the point of view of the fixed point problem, this means that any convergence result for Krasnoselskij-Mann iterative algorithms in the class of demicontractive mappings can be derived from its corresponding counterpart established for quasi-nonexpansive mappings.

3. The nonexpansive mappings are important in solving various problems in data science, like image recovery, machine learning, signal processing, neural networks etc. This was the reason why, in Section 2, we presented, by means of appropriate examples, the complete map (Figure 1) of the relationships existing amongst four important such classes: nonexpansive mappings, quasi-nonexpansive mappings, strictly pseudocontractive mappings and demicontractive mappings. To our best knowledge, this is the first time such a diagram is pictured.

4. In this context, we also collected an almost complete list of references related to the study of fixed point problem in the class of demicontractive mappings, mainly taken from [21].

5. The main message of this note for researchers working in that area is to use Lemma 3.2 when dealing with convergence theorems of Krasoselskij-Man type in the class of demicontractive mappings, in order to unify and simplify the proofs.

6. One of the main aims of this note was to trace back on the awareness and use of Lemma 3.2. We thus discovered that its inception started with the pioneering works on demicontractive mappings, due to Măruşter [87], [88] and Hicks and Kubicek [63], and that the first explicit statement and proof of this lemma in due to Măruşter [89], who did it within the proof of Theorem 2 [89].

7. A similar technique works for k-strict pseudocontractions, which can be embedded in the class of nonexpansive mappings in Hilbert spaces, first exploited by Browder and Petryshyn [26], see also [110], and also used much later by Zhou [174] in the case of nonself mappings, but this should be the subject of another paper.

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The author thanks his PhD student Liviu Socaciu who pointed out an error in Example 2.4, regarding the "proof" of the fact that T_4 would not be strictly pseudocontractive. As a result, in the corrected version we have shown that in fact T_4 is strictly pseudocontractive. This also lead to the inclusion of Example 2.5, which presents the function T_5 which is quasi-nonexpansive but is not strictly pseudocontractive.

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