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Simpson Type Tensorial Norm Inequalities for Continuous Functions of Selfadjoint Operators in Hilbert Spaces

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ABSTRACT. In this paper several tensorial norm inequalities for continuous functions of selfadjoint operators in Hilbert spaces have been obtained. Multiple inequalities of the form

$$\begin{aligned}&\left\|\frac{1}{3}\left(2\exp(A)\otimes 1-\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)+2\cdot 1\otimes\exp(B)\right)\right.\\&\left.-(1\otimes B-A\otimes 1)^{-1}\times\Big(\exp(1\otimes B)-\exp(A\otimes 1)\Big)\right\|\\&\leqslant\|1\otimes B-A\otimes 1\|\,\frac{5}{12}\exp(M)\end{aligned}$$

are obtained with variations due to the convexity properties of the mapping f.

1. INTRODUCTION

The notion of a tensor has its origin in the 19th century, when it was formulated by Gibbs, though he didn't formally use the word tensor but that of dyadic. In modern terminology, it can be seen as the origin of the tensor definition and its introduction to mathematics. Interplay of inequalities in mathematics is ubiquous, and as such it has applications in tensors as well. Mathematics and other scientific fields are highly influenced by inequalities. Many types of inequalities exist, but those involving Jensen, Ostrowski, Hermite-Hadamard, and Minkowski hold particular significance among them. More about inequalities and its history can be found in the books [22, 24]. Many papers have been published concerning the generalizations of these inequalities, see [1–5, 8–10, 26–31] and the references therein.

Since our paper is about tensorial Simpson type inequalities, we give a brief introduction to the topic of inequalities of that type. In 1938, A. Ostrowski [23], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)dt$ and the value $f(x)$, $x\in[a,b]$.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b) and $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<+\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty(b-a),$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

If we take $x=\frac{a+b}{2}$ we get the midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant\frac{1}{4}\|f'\|_\infty(b-a),$$

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with $\frac{1}{4}$ as best possible constant.
Recent advances concerning the theory of inequalities in Hilbert spaces will be shown to supplement the presentation of this work. Dragomir [18] gave the following Mond-Pecarić type inequality.

Theorem 1.2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subset [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$\frac{f(m)+f(M)}{2} \geq \langle \frac{f(A)+f((m+M)1_H - A)}{2} x, x\rangle$$
$$\geq \frac{f(\langle Ax, x\rangle) + f(m+M-\langle Ax, x\rangle)}{2} \geq f\left(\frac{m+M}{2}\right),$$

for each $x \in H$ with $||x|| = 1$.
In addition, if $x \in H$ with $||x|| = 1$ and $\langle Ax, x\rangle \neq \frac{m+M}{2}$, then also

$$\frac{f(\langle Ax, x\rangle) + f(m+M-\langle Ax, x\rangle)}{2}$$
$$\geq \frac{2}{\frac{m+M}{2} - \langle Ax, x\rangle} \int_{\langle Ax,x\rangle}^{m+M-\langle Ax,x\rangle} f(u)du \geq f\left(\frac{m+M}{2}\right).$$

Another interesting result is the Hermite-Hadamard inequality in the selfadjoint operator sense given by Dragomir [19]

Theorem 1.3. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$f\left(\frac{A+B}{2}\right) \leq \left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right]$$
$$\leq \int_0^1 f((1-t)A + tB)dt$$
$$\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \leq \frac{f(A)+f(B)}{2}.$$

In order to derive similar inequalities of the tensorial type, we need the following introduction and preliminaries.
Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$ by following [6] , we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k) dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k)$$

as bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.
If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction extends the definition of Kornyi [21] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,...,t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

Since we will be using tensorial products, we will define in the following what tensors and tensorial products are in short, for more consult the following book [20].
Let U, V and W be vector spaces over the same field F. A mapping $\Phi : U \times V \to W$ is called a bilinear mapping if it is linear in each variable separately. Namely, for all $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$ and $a, b \in F$,
$\Phi(au_1 + bu_2, v) = a\Phi(u_1, v) + b\Phi(u_2, v)$,
$\Phi(u, av_1 + bv_2) = a\Phi(u, v_1) + b\Phi(u, v_2)$. If $W = F$, a bilinear mapping $\Phi : U \times V \to F$ is called a bilinear function.
Let $\otimes : U \times V \to W$ be a bilinear mapping. The pair $(W, \otimes)$ is called a tensor product space of U and V if it satisfies the following conditions:
1. Generating property $< Im\otimes >= W$;
2. Maximal span property $dim < Im\otimes >= dimU \cdot dimV$.
The member $w \in W$ is called a tensor, but not all tensors in W are products of two vectors of the form $u \otimes v$. The notation $< Im\otimes >$ denotes the span.
Example
Let $u = (x_1, .., x_m) \in \mathbb{R}^m$ and $V = (y_1, ..., y_n) \in \mathbb{R}^n$. We can view u and v as column vectors. Namely,

$$u = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, v = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

are $m \times 1$ and $n \times 1$ matrices respectively.
We define $\otimes : \mathbb{R}^m \times \mathbb{R}^n \to M_{m,n}$,

$$u \otimes v = uv^t = \begin{bmatrix} x_1y_1 \cdots x_1y_n \\ \vdots \\ x_my_1 \cdots x_my_n \end{bmatrix},$$

an $m \times n$ matrix with entries $A_{ij} = x_iy_j$. $(M_{m,n}, \otimes)$ is a tensor product space of $\mathbb{R}^m$ and $\mathbb{R}^n$. Tensors do not need to be matrices. This is just one model given. For more consult the following book [20].

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D)$$

that holds for any $A, B, C, D \in B(H)$.
From the property we can deduce easily the following consequences

$$A^n \otimes B^n = (A \otimes B)^n, n \geqslant 0,$$

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B,$$

which can be extended, for two natural numbers m, n we have

$$(A \otimes 1)^n (1 \otimes B)^m = (1 \otimes B)^m (A \otimes 1)^n = A^n \otimes B^m.$$

The current research concerning tensorial inequalities can be seen in the following papers, [11–15]. The following Lemma which we require can be found in a paper of Dragomir [16]. We provide the proof for the education purposes and for the easier understanding of the paper.

Lemma 1.1. *Assume A and B are selfadjoint operators with $Sp(A) \subset I$ and $Sp(B) \subset J$. Let f, h be continuous on I, g, k continuous on J and ψ continuous on an interval K that contains the sum of the intervals $h(I) + k(J)$, then*

$$\begin{aligned}&(f(A) \otimes 1 + 1 \otimes g(B))\psi(h(A) \otimes 1 + 1 \otimes k(B)) \\ &= \int_I \int_J (f(t) + g(s))\psi(h(t) + k(s))dE_t \otimes dF(s),\end{aligned} \tag{1.1}$$

where A and B have the spectral resolutions,

$$A = \int_I tdE_t \text{ and } B = \int_J sdF_s.$$

Proof. By Stone-Weierstrass, any continuous function can be approximated by a sequence of polynomials, therefore it is enough to prove the equality for the power function $\psi(t) = t^n$ with n any natural number.
For any natural number $n \geqslant 1$ we have

$$\begin{aligned}\mathcal{S} &= \int_I \int_J (f(t) + g(s))(h(t) + k(s))^n dE_t \otimes dF(s) \\ &= \int_I \int_J (f(t) + g(s)) \sum_{k=0}^{n} C_n^k [h(t)]^k [k(s)]^{n-k} dE_t \otimes dF(s) \\ &= \sum_{k=0}^{n} C_n^k \Big[\int_I \int_J f(t)[h(t)]^k [k(s)]^{n-k} dE_t \otimes dF(s) \\ &\quad + \int_I \int_J [h(t)]^k g(s)[k(s)]^{n-k} dE_t \otimes dF(s) \Big].\end{aligned}$$

Rewriting the first and second part respectively as

$$\int_I \int_J f(t)[h(t)]^k [k(s)]^{n-k} dE_t \otimes dF(s) = (f(A) \otimes 1)(h(A) \otimes 1)^k (1 \otimes k(B))^{n-k},$$

$$\int_I \int_J [h(t)]^k g(s)[k(s)]^{n-k} dE_t \otimes dF(s) = (1 \otimes g(B))(h(A) \otimes 1)^k (1 \otimes k(B))^{n-k}.$$

Proving the claims above can be shown using the properties of the tensors given. Therefore, we have

$$\begin{aligned}S &= (f(A) \otimes 1 + 1 \otimes g(B)) \sum_{k=0}^{n} C_n^k (h(A) \otimes 1)^k (1 \otimes k(B))^{n-k} \\ &= (f(A) \otimes 1 + 1 \otimes g(B))(h(A) \otimes 1 + 1 \otimes k(B))^n\end{aligned}$$

for which the commutativity of $h(A) \otimes 1$ and $1 \otimes k(B)$ has been employed. □

Proof of the following Theorem can be found in a paper by Dragomir [16].

Theorem 1.4. *Assume A and B are selfadjoint operators with $Sp(A) \subset I$ and $Sp(B) \subset J$. Let f be continuous on I; g continuous on J and ψ continuous on an interval K that contains the product of the intervals $f(I)g(J)$, then*

$$\psi(f(A) \otimes g(B)) = \int_I \int_J \psi(f(t)g(s))dE_t \otimes dF(s), \tag{1.2}$$

where A and B have the spectral resolutions

$$A = \int_I tdE_t, B = \int_J sdF_s.$$

Proof of the following Theorem can be found in a paper by Dragomir [16].

Theorem 1.5. *Assume A and B are selfadjoint operators with $Sp(A) \subset I$ and $Sp(B) \subset J$. Let h be continuous on I, k continuous on J and ψ continuous on an interval U that contains the sum of the intervals $h(I) + k(J)$, then*

$$\psi(h(A) \otimes 1 + 1 \otimes k(B)) = \int_I \int_J \psi(h(t) + k(s)) dE_t \otimes dF(s), \tag{1.3}$$

where A and B have the spectral resolutions

$$A = \int_I t dE_t, B = \int_J s dF_s.$$

In the paper written by Budak [7] , the authors used Lemma 1 to obtain their results. We will use the following Lemma, which is obtained by setting $\alpha = 1$ in their result and by using simple substitutions on the integrals on the left hand side.

Lemma 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping (a, b) such that $f' \in L_1([a, b])$. Then, the following equality holds:*

$$\begin{aligned} &\frac{1}{3}\left(2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right) \\ &-\frac{1}{2}\left(\int_0^1 f\left((1-k)a + k\frac{a+b}{2}\right) dk + \int_0^1 f\left((1-k)\frac{a+b}{2} + kb\right) dk\right) \\ &= \frac{b-a}{4}\int_0^1 \left(u + \frac{1}{3}\right)\left(f'\left(\left(\frac{1-u}{2}\right)a + \left(\frac{1+u}{2}\right)b\right)\right. \\ &\left. -f'\left(\left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)b\right)\right)du. \end{aligned}$$

2. Main results

Lemma 2.3. *Assume that f is continuously differentiable on I, A and B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\begin{aligned} &\frac{1}{3}\left(2f(A) \otimes 1 - f\left(\frac{A \otimes 1 + 1 \otimes B}{2}\right) + 2 \cdot 1 \otimes f(B)\right) \\ &-\frac{1}{2}\int_0^1 \left(f\left(\left(1 - \frac{k}{2}\right)A \otimes 1 + \frac{k}{2} 1 \otimes B\right) + f\left(\left(\frac{1-k}{2}\right)A \otimes 1 + \frac{1+k}{2} 1 \otimes B\right)\right) dk \\ &= \frac{1 \otimes B - A \otimes 1}{4}\int_0^1 u f'\left(\left(\frac{1-u}{2}\right)A \otimes 1 + \left(\frac{1+u}{2}\right)1 \otimes B\right) du \\ &+\frac{1 \otimes B - A \otimes 1}{12}\int_0^1 f'\left(\left(\frac{1-u}{2}\right)A \otimes 1 + \left(\frac{1+u}{2}\right)1 \otimes B\right) du \\ &-\frac{1 \otimes B - A \otimes 1}{4}\int_0^1 u f'\left(\left(\frac{1+u}{2}\right)A \otimes 1 + \left(\frac{1-u}{2}\right)1 \otimes B\right) du \\ &-\frac{1 \otimes B - A \otimes 1}{12}\int_0^1 f'\left(\left(\frac{1+u}{2}\right)A \otimes 1 + \left(\frac{1-u}{2}\right)1 \otimes B\right) du. \end{aligned} \tag{2.4}$$

Proof. We will start the proof with the Lemma 1.2. Simplifying and factoring arguments in the integral on the left hand side we get

$$\frac{1}{3}\left(2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right)$$

$$-\frac{1}{2}\left(\int_0^1 f\left(\left(1-\frac{k}{2}\right)a+\frac{k}{2}b\right)dk+\int_0^1 f\left(\left(\frac{1-k}{2}\right)a+b\left(\frac{1+k}{2}\right)\right)dk\right)$$
$$=\frac{b-a}{4}\int_0^1\left(u+\frac{1}{3}\right)\left(f'\left(\left(\frac{1-u}{2}\right)a+\left(\frac{1+u}{2}\right)b\right)\right.$$
$$\left.-f'\left(\left(\frac{1+u}{2}\right)a+\left(\frac{1-u}{2}\right)b\right)\right)du.$$

Separating the right hand side and assuming that A and B have the spectral resolutions

$$A=\int tdE(t) \text{ and } B=\int sdF(s).$$

If we take the integral $\int_I\int_I dE_t\otimes dF_s$, then we get

$$\frac{1}{3}\int_I\int_I\left(2f(t)-f\left(\frac{t+s}{2}\right)+2f(s)\right)dE_t\otimes dF_s$$
$$-\frac{1}{2}\int_I\int_I\int_0^1 f\left(\left(1-\frac{k}{2}\right)t+\frac{k}{2}s\right)dkdE_t\otimes dF_s$$
$$-\frac{1}{2}\int_I\int_I\int_0^1 f\left(\left(\frac{1-k}{2}\right)t+\left(\frac{1+k}{2}s\right)\right)dkdE_t\otimes dF_s$$
$$=\int_I\int_I\frac{s-t}{4}\int_0^1 uf'\left(\left(\frac{1-u}{2}\right)t+\left(\frac{1+u}{2}\right)s\right)dudE_t\otimes dF_s$$
$$+\int_I\int_I\frac{s-t}{12}\int_0^1 f'\left(\left(\frac{1-u}{2}\right)t+\left(\frac{1+u}{2}\right)s\right)dudE_t\otimes dF_s$$
$$-\int_I\int_I\frac{s-t}{4}\int_0^1 uf'\left(\left(\frac{1+u}{2}\right)t+\left(\frac{1-u}{2}\right)s\right)dudE_t\otimes dF_s$$
$$-\int_I\int_I\frac{s-t}{12}\int_0^1 f'\left(\left(\frac{1+u}{2}\right)t+\left(\frac{1-u}{2}\right)s\right)dudE_t\otimes dF_s.$$

By utilizing the Fubini's Theorem and Lemma 1 for appropriate choices of the functions involved, we have successively

$$\int_I\int_I\int_0^1 f\left(\left(1-\frac{k}{2}\right)t+\frac{k}{2}s\right)dkdE_t\otimes dF_s$$
$$=\int_0^1\int_I\int_I f\left(\left(1-\frac{k}{2}\right)t+\frac{k}{2}s\right)dE_t\otimes dF_sdk$$
$$=\int_0^1 f\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)dk,$$

$$\int_I\int_I\frac{s-t}{4}\int_0^1 uf'\left(\left(\frac{1-u}{2}\right)t+\left(\frac{1+u}{2}\right)s\right)dudE_t\otimes dF_s$$
$$=\int_0^1 u\int_I\int_I\frac{s-t}{4}f'\left(\frac{1-u}{2}t+\frac{1+u}{2}s\right)dE_t\otimes dF_sdu$$
$$=\frac{1\otimes B-A\otimes 1}{4}\int_0^1 uf'\left(\frac{1-u}{2}A\otimes 1+\frac{1+u}{2}1\otimes B\right)du.$$

By utilizing these relations, we get the equality. □

Now we state our first Simpson type inequality of this paper.

Theorem 2.6. *Assume that f is continuously differentiable on I with* $\|f'\|_{I,+\infty} := \sup_{t\in I}|f'(t)| < +\infty$ *and* A, B *are selfadjoint operators with* $Sp(A), Sp(B) \subset I$*, then*

$$\left\|\frac{1}{3}\left(2f(A)\otimes 1 - f\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) + 2\cdot 1\otimes f(B)\right)\right. \tag{2.5}$$
$$\left.-\frac{1}{2}\int_0^1\left(f\left(\left(1-\frac{k}{2}\right)A\otimes 1 + \frac{k}{2}1\otimes B\right) + f\left(\left(\frac{1-k}{2}\right)A\otimes 1 + \frac{1+k}{2}1\otimes B\right)\right)dk\right\|$$
$$\leqslant \|1\otimes B - A\otimes 1\|\frac{5}{12}\|f'\|_{I,+\infty}.$$

Proof. If we take the operator norm of the previously obtained Lemma and use the triangle inequality, we get

$$\left\|\frac{1}{3}\left(2f(A)\otimes 1 - f\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) + 2\cdot 1\otimes f(B)\right)\right.$$
$$\left.-\frac{1}{2}\int_0^1\left(f\left(\left(1-\frac{k}{2}\right)A\otimes 1 + \frac{k}{2}1\otimes B\right) + f\left(\left(\frac{1-k}{2}\right)A\otimes 1 + \frac{1+k}{2}1\otimes B\right)\right)dk\right\|$$
$$\leqslant \frac{1}{4}\|1\otimes B - A\otimes 1\|\left\|\int_0^1 uf'\left(\left(\frac{1-u}{2}\right)A\otimes 1 + \left(\frac{1+u}{2}\right)1\otimes B\right)du\right\|$$
$$+\frac{1}{12}\|1\otimes B - A\otimes 1\|\left\|\int_0^1 f'\left(\left(\frac{1-u}{2}\right)A\otimes 1 + \left(\frac{1+u}{2}\right)1\otimes B\right)du\right\|$$
$$+\frac{1}{4}\|1\otimes B - A\otimes 1\|\left\|\int_0^1 uf'\left(\left(\frac{1+u}{2}\right)A\otimes 1 + \left(\frac{1-u}{2}\right)1\otimes B\right)du\right\|$$
$$+\frac{1}{12}\|1\otimes B - A\otimes 1\|\left\|\int_0^1 f'\left(\left(\frac{1+u}{2}\right)A\otimes 1 + \left(\frac{1-u}{2}\right)1\otimes B\right)du\right\|.$$

By the properties of the integral and norm, we have

$$\left\|\int_0^1 uf'\left(\left(\frac{1-u}{2}\right)A\otimes 1 + \left(\frac{1+u}{2}\right)1\otimes B\right)du\right\|$$
$$\leqslant \int_0^1 u\left\|f'\left(\left(\frac{1-u}{2}\right)A\otimes 1 + \left(\frac{1+u}{2}\right)1\otimes B\right)\right\|du,$$

and

$$\left\|\int_0^1 uf'\left(\left(\frac{1+u}{2}\right)A\otimes 1 + \left(\frac{1-u}{2}\right)1\otimes B\right)du\right\|$$
$$\leqslant \int_0^1 u\left\|f'\left(\left(\frac{1+u}{2}\right)A\otimes 1 + \left(\frac{1-u}{2}\right)1\otimes B\right)\right\|du.$$

Realize here that by Lemma 1,

$$\left|f'\left(\left(\frac{1-u}{2}\right)A\otimes 1 + \left(\frac{1+u}{2}\right)1\otimes B\right)\right| = \int_I\int_I\left|f'\left(\frac{1-u}{2}t + \frac{1+u}{2}s\right)\right|dE_t\otimes dF_s.$$

Since

$$\left|f'\left(\frac{1-u}{2}t + \frac{1+u}{2}s\right)\right| \leqslant \|f'\|_{I,+\infty}$$

for $u\in[0,1]$ and $t, s\in I$. If we take the integral over $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\left|f'\left(\left(\frac{1-u}{2}\right)A\otimes 1 + \left(\frac{1+u}{2}\right)1\otimes B\right)\right| = \int_I\int_I\left|f'\left(\frac{1-u}{2}t + \frac{1+u}{2}s\right)\right|dE_t\otimes dF_s$$
$$\leqslant \|f'\|_{I,+\infty}\int_I\int_I dE_t\otimes dF_s = \|f'\|_{I,+\infty}.$$

From which we get the following

$$\left\|\int_0^1 uf'\left(\left(\frac{1-u}{2}\right)A\otimes 1+\left(\frac{1+u}{2}\right)1\otimes B\right)du\right\|$$

$$\leqslant \int_0^1 u\left\|f'\left(\left(\frac{1-u}{2}\right)A\otimes 1+\left(\frac{1+u}{2}\right)1\otimes B\right)\right\|du \leqslant \|f'\|_{I,+\infty}\int_0^1 udu=\frac{1}{2}\|f'\|_{I,+\infty}.$$

Similarly, we obtain

$$\int_0^1 u\left\|f'\left(\left(\frac{1+u}{2}\right)A\otimes 1+\left(\frac{1-u}{2}\right)1\otimes B\right)\right\|du \leqslant \|f'\|_{I,+\infty}\int_0^1 udu=\frac{1}{2}\|f'\|_{I,+\infty}.$$

Combining the properties given we obtain the original inequality

$$\left\|\frac{1}{3}\left(2f(A)\otimes 1-f\left(\frac{A\otimes 1+1\otimes B}{2}\right)+2\cdot 1\otimes f(B)\right)\right.$$
$$\left.-\frac{1}{2}\int_0^1\left(f\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)+f\left(\left(\frac{1-k}{2}\right)A\otimes 1+\frac{1+k}{2}1\otimes B\right)\right)dk\right\|$$
$$\leqslant \|1\otimes B-A\otimes 1\|\frac{5}{12}\|f'\|_{I,+\infty}.$$

□

In the next Theorem we utilize the convexity properties to obtain the results.

Theorem 2.7. *Assume that f is continuously differentiable on I and f' is convex and A, B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\left\|\frac{1}{3}\left(2f(A)\otimes 1-f\left(\frac{A\otimes 1+1\otimes B}{2}\right)+2\cdot 1\otimes f(B)\right)\right. \tag{2.6}$$
$$\left.-\frac{1}{2}\int_0^1\left(f\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)+f\left(\left(\frac{1-k}{2}\right)A\otimes 1+\frac{1+k}{2}1\otimes B\right)\right)dk\right\|$$
$$\leqslant \frac{5}{24}\|1\otimes B-A\otimes 1\|\left(\|f'(A)\|+\|f'(B)\|\right)$$

Proof. Since $|f'|$ is convex on I, then we get

$$\left|f'\left(\frac{1-u}{2}t+\frac{1+u}{2}s\right)\right|\leqslant\frac{1-u}{2}|f'(t)|+\frac{1+u}{2}|f'(s)|$$

for all $u\in[0,1]$ and $t,s\in I$. If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\left|f'\left(\frac{1-u}{2}A\otimes 1+\frac{1+u}{2}1\otimes B\right)\right|=\int_I\int_I\left|f'\left(\frac{1-u}{2}t+\frac{1+u}{2}s\right)\right|dE_t\otimes dF_s$$
$$\leqslant\int_I\int_I\left[\frac{1-u}{2}|f'(t)|+\frac{1+u}{2}|f'(s)|\right]dE_t\otimes dF_s$$
$$=\frac{1-u}{2}|f'(A)|\otimes 1+\frac{1+u}{2}1\otimes|f'(B)|.$$

If we take the norm in the inequality, we get the following

$$\left\|f'\left(\frac{1-u}{2}A\otimes 1+\frac{1+u}{2}1\otimes B\right)\right\|\leqslant\left\|\frac{1-u}{2}|f'(A)|\otimes 1+\frac{1+u}{2}1\otimes|f'(B)|\right\|$$
$$\leqslant\frac{1-u}{2}\||f'(A)|\otimes 1\|+\frac{1+u}{2}\|1\otimes|f'(B)|\|$$
$$=\frac{1-u}{2}\|f'(A)\|+\frac{1+u}{2}\|f'(B)\|.$$

Therefore, we obtain

$$\int_0^1 u \left\| f'\left(\frac{1-u}{2} A\otimes 1 + \frac{1+u}{2} 1\otimes B\right)\right\| du$$
$$\leqslant \int_0^1 u\cdot \frac{1-u}{2} du \cdot \|f'(A)\| + \int_0^1 u\cdot \frac{1+u}{2} du\cdot \|f'(B)\|$$
$$= \frac{1}{12}\|f'(A)\| + \frac{5}{12}\|f'(B)\|.$$

Similarly,

$$\int_0^1 \left\| f'\left(\frac{1+u}{2} A\otimes 1 + \frac{1-u}{2} 1\otimes B\right)\right\| du$$
$$\leqslant \int_0^1 u\cdot \frac{1+u}{2} du \cdot \|f'(A)\| + \int_0^1 u\cdot \frac{1-u}{2} du\cdot \|f'(B)\|$$
$$= \frac{5}{12}\|f'(A)\| + \frac{1}{12}\|f'(B)\|.$$

Combining everything, we get the desired result. □

We recall the following definition of a P convex function which will be needed for our next Theorem. A nonnegative function f defined on the segment S is said to be a function of P type if

$$f(\lambda x + (1-\lambda)y) \leqslant f(x) + f(y),\ x, y\in S,\ 0\leqslant \lambda\leqslant 1.$$

Theorem 2.8. *Assume that f is continuously differentiable on I and $|f'|$ is a P convex function and A, B are selfadjoint operators with $Sp(A), Sp(B)\subset I$, then*

$$\left\| \frac{1}{3}\left(2f(A)\otimes 1 - f\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) + 2\cdot 1\otimes f(B)\right)\right. \tag{2.7}$$
$$\left. -\frac{1}{2}\int_0^1 \left(f\left(\left(1-\frac{k}{2}\right)A\otimes 1 + \frac{k}{2} 1\otimes B\right) + f\left(\left(\frac{1-k}{2}\right)A\otimes 1 + \frac{1+k}{2} 1\otimes B\right)\right) dk\right\|$$
$$\leqslant \frac{5}{12}\|1\otimes B - A\otimes 1\|\left(\|f'(A)\| + \|f'(B)\|\right).$$

Proof. Since $|f'|$ is P convex on I, then we get

$$\left| f'\left(\frac{1+u}{2} s + \frac{1-u}{2} t\right)\right| \leqslant |f'(t)| + |f'(s)|.$$

for all $u\in[0,1]$ and $t, s\in I$.
If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\left| f'\left(\frac{1+u}{2} 1\otimes B + \frac{1-u}{2} A\otimes 1\right)\right| = \int_I\int_I \left| f'\left(\frac{1+u}{2} s + \frac{1-u}{2} t\right)\right| dE_t\otimes dF_s$$
$$\leqslant \int_I\int_I \Big[|f'(t)| + |f'(s)|\Big] dE_t\otimes dF_s$$
$$= |f'(A)|\otimes 1 + 1\otimes |f'(B)|.$$

for all $u\in[0,1]$.
If we take the norm in the inequality, we get the following

$$\left\| f'\left(\frac{1+u}{2} s + \frac{1-u}{2} t\right)\right\| \leqslant \||f'(A)|\otimes 1 + 1\otimes |f'(B)|\|$$
$$\leqslant \||f'(A)|\otimes 1\| + \|1\otimes |f'(B)|\| = \|f'(A)\| + \|f'(B)\|.$$

Similarly, we obtain

$$\left|f'\left(\frac{1-u}{2}s+\frac{1+u}{2}t\right)\right| \leqslant |f'(t)|+|f'(s)|.$$

for all $u \in [0,1]$ and $t, s \in I$.
If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\begin{aligned}\left|f'\left(\frac{1-u}{2}1\otimes B+\frac{1+u}{2}A\otimes 1\right)\right| &= \int_I\int_I \left|f'\left(\frac{1-u}{2}s+\frac{1+u}{2}t\right)\right| dE_t\otimes dF_s \\ &\leqslant \int_I\int_I \Big[|f'(t)|+|f'(s)|\Big] dE_t\otimes dF_s \\ &= |f'(A)|\otimes 1+1\otimes |f'(B)|.\end{aligned}$$

for all $u \in [0,1]$.
If we take the norm in the inequality, we get the following

$$\begin{aligned}\left\|f'\left(\frac{1-u}{2}s+\frac{1+u}{2}t\right)\right\| &\leqslant \||f'(A)|\otimes 1+1\otimes |f'(B)|\| \\ &\leqslant \||f'(A)|\otimes 1\|+\|1\otimes |f'(B)|\| = \|f'(A)\|+\|f'(B)\|.\end{aligned}$$

Therefore we obtain,

$$\int_0^1 u\left\|f'\left(\frac{1+u}{2}1\otimes B+\frac{1-u}{2}A\otimes 1\right)\right\| du$$

$$\leqslant \int_0^1 u\left(\|f'(A)\|+\|f'(B)\|\right) du = \frac{1}{2}\left(\|f'(A)\|+\|f'(B)\|\right).$$

Similarly,

$$\int_0^1 \left\|f'\left(\frac{1-u}{2}1\otimes B+\frac{1+u}{2}A\otimes 1\right)\right\| du$$

$$\leqslant \int_0^1 \left(\|f'(A)\|+\|f'(B)\|\right) du = \left(\|f'(A)\|+\|f'(B)\|\right).$$

Using these properties on all the terms , we obtain the following

$$\begin{aligned}&\frac{1}{8}\|1\otimes B-A\otimes 1\|\left(\|f'(A)\|+\|f'(B)\|\right) \\ &+\frac{1}{12}\|1\otimes B-A\otimes 1\|\left(\|f'(A)\|+\|f'(B)\|\right) \\ &+\frac{1}{8}\|1\otimes B-A\otimes 1\|\left(\|f'(A)\|+\|f'(B)\|\right) \\ &+\frac{1}{12}\|1\otimes B-A\otimes 1\|\left(\|f'(A)\|+\|f'(B)\|\right) \\ &=\frac{5}{12}\|1\otimes B-A\otimes 1\|\left(\|f'(A)\|+\|f'(B)\|\right).\end{aligned}$$

and by summing up all gives us the desired inequality. □

3. Some examples and consequences

It is known that if U and V are commuting, that is $UV = VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V) = \exp(V)\exp(U) = \exp(U+V).$$

Also, if U is invertible and $a, b \in \mathbb{R}$ and $a < b$ then

$$\int_a^b \exp(tU)dt = U^{-1}\left[\exp(bU) - \exp(aU)\right].$$

Moreover, if U and V are commuting and $V - U$ is invertible, then

$$\int_0^1 \exp((1-k)U + kV)dk = \int_0^1 \exp(k(V-U))\exp(U)dk$$
$$= \left(\int_0^1 \exp(k(V-U))dk\right)\exp(U)$$
$$= (V-U)^{-1}\left[\exp(V-U) - I\right]\exp(U) = (V-U)^{-1}\left[\exp(V) - \exp(U)\right].$$

Since the operators $U = A \otimes 1$ and $V = 1 \otimes B$ are commutative and if $1 \otimes B - A \otimes 1$ is invertible, then

$$\int_0^1 \exp((1-k)A \otimes 1 + k1 \otimes B)dk$$
$$= (1 \otimes B - A \otimes 1)^{-1}\left[\exp(1 \otimes B) - \exp(A \otimes 1)\right].$$

Corollary 3.1. *If A, B are selfadjoint operators with $Sp(A), Sp(B) \subset [m, M]$ and $1 \otimes B - A \otimes 1$ is invertible, then by Theorem 2.6* (2.5), *we get*

$$\left\|\frac{1}{3}\left(2\exp(A) \otimes 1 - \exp\left(\frac{A \otimes 1 + 1 \otimes B}{2}\right) + 2 \cdot 1 \otimes \exp(B)\right)\right. \tag{3.8}$$
$$\left. -(1 \otimes B - A \otimes 1)^{-1} \times \left(\exp(1 \otimes B) - \exp(A \otimes 1)\right)\right\|$$
$$\leqslant \|1 \otimes B - A \otimes 1\| \frac{5}{12}\exp(M).$$

Corollary 3.2. *Since for $f(t) = \exp(t)$, $t \in \mathbb{R}$, $|f'|$ is convex, then by Theorem 2.7* (2.6) *we get*

$$\left\|\frac{1}{3}\left(2\exp(A) \otimes 1 - \exp\left(\frac{A \otimes 1 + 1 \otimes B}{2}\right) + 2 \cdot 1 \otimes \exp(B)\right)\right. \tag{3.9}$$
$$\left. -(1 \otimes B - A \otimes 1)^{-1} \times \left(\exp(1 \otimes B) - \exp(A \otimes 1)\right)\right\|$$
$$\leqslant \frac{5}{24}\|1 \otimes B - A \otimes 1\| \left(\|\exp(A)\| + \|\exp(B)\|\right).$$

Corollary 3.3. *Choosing $f(t) = \exp(t), t \in \mathbb{R}$, and since $|f'|$ is P convex, then by Theorem 2.8* (2.7) *we get*

$$\left\|\frac{1}{3}\left(2\exp(A) \otimes 1 - \exp\left(\frac{A \otimes 1 + 1 \otimes B}{2}\right) + 2 \cdot 1 \otimes \exp(B)\right)\right. \tag{3.10}$$
$$\left. -(1 \otimes B - A \otimes 1)^{-1} \times \left(\exp(1 \otimes B) - \exp(A \otimes 1)\right)\right\|$$
$$\leqslant \frac{5}{12}\|1 \otimes B - A \otimes 1\| \left(\|\exp(A)\| + \|\exp(B)\|\right).$$

The obtained results concerning the inequalities given and the examples given differ from the ones in the existing literature because they are of Simpson type, and not of the classical Ostrowski type which was given by Dragomir [17]. The other results concerning the tensorial inequalities in Hilbert space are of the Hermite-Hadamard type as given by [16, 31]. Therefore the inequalities and the approach given in this paper are indeed new, and yield new boundaries of the tensorial type in Hilbert space.

4. Conclusion

Tensors have become important in various fields, for example in physics because they provide a concise mathematical framework for formulating and solving physical problems in fields such as mechanics, electromagnetism, quantum mechanics, and many others. As such inequalities are crucial in numerical aspects. Reflected in this work is the Simpson tensorial inequality. Using the Lemma which we derived enabled us to obtain various types of Simpson type inequalities. Examples of specific convex functions and their inequalities using our results are given in the section some examples and consequences. Plans for future research can be reflected in the fact that the obtained inequalities in this work can be sharpened or generalized by using other methods.

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