CREAT. MATH. INFORM.
Volume **33** (2024), No. 1,
Pages 47 - 56

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2024.01.05

Hardy Spaces and Integral Means of Certain Integral Operators on Analytic Functions

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ABSTRACT. In this paper, we determine the Hardy spaces of certain integral operators on normalised analytic functions defined in the open unit disk in the complex plane with the prior knowledge of the Hardy spaces of the functions or their derivatives in the integral.

1. INTRODUCTION

Given an analytic function $f : \Delta \to \mathbb{C}$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, the integral means of f are defined as

$$M_p(r,f) = \begin{cases} \{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\}^{1/p}, & \text{if } 0 < p < \infty \\ max_{0\leq\theta<2\pi}|f(re^{i\theta})|, & \text{if } p = \infty. \end{cases}$$

For $0 < p \leq \infty$, a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in Δ is said to belong to the Hardy space H^p if the integral mean $M_p(r,f)$ is bounded as $r \to 1^-$. i.e.,

$$\lim_{r\to1^-} M_p(r,f) \leq K,$$

where K is a constant depending on f. When $p = \infty$, the class H^∞ consists of all bounded analytic functions in Δ. In particular, when $p = 2$, H^2 consists of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in the open unit disk with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. If $0 < p < q \leq \infty$, then $H^p \supset H^q \supset H^\infty$ [2].
Let $\mathcal{A}$ denote the class of analytic functions f defined on Δ with the normalization [4]

$$f(0) = f'(0) - 1 = 0,$$

having the Taylor's series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Several integral operators on the subclasses of analytic functions on the unit disk were studied in the past [1, 3, 6, 8, 9, 10, 11, 14].

In this article, we construct the integral operators F_i's considering the Hornich operators on functions in Class $\mathcal{A}$, we determine the inclusion theorems involving Hardy spaces of F_i's provided the Hardy space in which the functions f's or its derivatives used in constructing F_i's are known. We also analyse a bound for their integral means by establishing a relation between the integral means of these integral operators and that of the integrands. In addition, we examine the limiting behavior of the Taylor's coefficients of these integral operators.

Received: 20.01.2023. In revised form: 17.07.2023. Accepted: 24.07.2023
2000 *Mathematics Subject Classification.* 30H10, 47G10, 30C45, 30C50.
Key words and phrases. *Analytic Functions, Hardy Spaces, Integral Operators, Integral Means.*
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2. Preliminaries

In this section we recall a subclass of analytic normalised functions introduced and studied by Mac Gregor in the year 1962.

Definition 2.1. [12] A function $f \in \mathcal{A}$ is said to be in class $\mathcal{R}$ if it satisfies the inequality

$$Re(f'(z)) > 0, \text{ for all } z \in \Delta.$$

We require the following lemmas to substantiate our results.

Lemma 2.1 ([13], p. 40). *If $f \in H^p$ and $g \in H^q$ then $fg \in H^{\frac{pq}{p+q}}$.*

Lemma 2.2 ([2], p. 88). *If $f' \in H^p$ then $f \in H^{\frac{p}{1-p}}$, if $p < 1$.*

Lemma 2.3 ([2], p. 88). *If $f' \in H^p$ then $f \in H^\infty$, if $p \geq 1$.*

Lemma 2.4 ([10], p. 04). *If $f \in \mathcal{A}$ satisfies $z^\gamma f(z) \in H^p, (0 < p < \infty)$ for some real γ, then $f \in H^p$.*

Lemma 2.5 ([5], p. 408). *If $f(z) \in H^p, (0 < p < 1)$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$ then $a_n = o(n^{\frac{1}{p}-1})$.*

Lemma 2.6 ([2], p. 34). *If $f \in \mathcal{R} \implies f' \in H^p$ for all $p < 1 \implies f \in H^{\frac{p}{1-p}}$, for all $0 < p < 1$.*

3. Main results

In this section we state and prove the main findings of our research work.

Theorem 3.1. *Let $f_i \in H^{p_i}$ for $i = 1, 2, \cdots, n$ where $0 < p_i < \infty$ and*

$$F_1(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(\zeta)}{\zeta} \right)^{m_i} d\zeta, \quad m_i \in \mathbb{N}. \tag{3.1}$$

(i) If $\prod_{i=1}^n p_i < \sum_{i=1}^n m_i \hat{p}_i$, where $\hat{p}_i = p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n$, then

$$F_1 \in H^\mu, \text{ where } \mu = \frac{\prod_{i=1}^n p_i}{\sum_{i=1}^n m_i \hat{p}_i - \prod_{i=1}^n p_i}.$$

(ii) If $\prod_{i=1}^n p_i \geq \sum_{i=1}^n m_i \hat{p}_i$, where $\hat{p}_i = p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n$, then $F_1 \in H^\infty$.

Proof. Let $f_i \in H^{p_i}$ for $i = 1, 2, \cdots, n$.
Using Lemma 2.1, we obtain

$$(f_i)^{m_i} \in H^{\frac{p_i}{m_i}}, m_i \in \mathbb{N}, i = 1, 2, \cdots, n$$

and

$$\prod_{i=1}^{n} (f_i)^{m_i} \in H^\lambda, \lambda = \frac{\prod_{i=1}^n p_i}{\sum_{i=1}^n m_i \hat{p}_i}, \text{ where } \hat{p}_i = p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n.$$

On differentiating (3.1), we have,

$$z^{(\sum_{i=1}^n m_i)} F_1'(z) = \prod_{i=1}^{n} (f_i(z))^{m_i}.$$

Using Lemma 2.4, we get,

$$F_1'(z) \in H^{\lambda}, \lambda = \frac{\prod_{i=1}^{n} p_i}{\sum_{i=1}^{n} m_i \hat{p_i}}, \hat{p_i} = p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n.$$

We have the following 2 cases:

(i) When $\prod_{i=1}^{n} p_i < \sum_{i=1}^{n} m_i \hat{p_i}$, we have $\lambda < 1$.
Using Lemma 2.2,

$$F_1 \in H^{\mu}, \text{ where } \mu = \frac{\prod_{i=1}^{n} p_i}{\sum_{i=1}^{n} m_i \hat{p_i} - \prod_{i=1}^{n} p_i} \text{ and } \hat{p_i} = p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n.$$

(ii) When $\prod_{i=1}^{n} p_i \geq \sum_{i=1}^{n} m_i \hat{p_i}$, we have $\lambda \geq 1$.
Using Lemma 2.3,

$$F_1 \in H^{\infty}.$$

□

Theorem 3.2. *Let $f_i \in \mathcal{R}, i = 1, 2, \cdots, k$, $g_j \in H^{q_j}$ for $j = 1, 2, \cdots, l$, where $0 < q_j < \infty$ and*

$$F_2(z) = \int_0^z \prod_{i=1}^{k} (f_i'(\zeta))^{m_i} \prod_{j=1}^{l} \left(\frac{g_j(\zeta)}{\zeta} \right)^{n_j} dt, \quad m_i,\ n_j \in \mathbb{N}, \tag{3.2}$$

then

$$F_2 \in H^{\mu}, \text{ where } \mu = \frac{p(\prod_{j=1}^{l} q_j)}{(\sum_{i=1}^{k} m_i)(\prod_{j=1}^{l} q_j) + p(\sum_{j=1}^{l} n_j \hat{q_j} - \prod_{j=1}^{l} q_j)},$$

for all $p < 1$, $\hat{q_j} = q_1 q_2 \cdots q_{j-1} q_{j+1} \cdots q_l$.

Theorem 3.3. *Let $f_i \in \mathcal{R}$ for $i = 1, 2, \cdots k$ and*

$$F_3(z) = \int_0^z \prod_{i=1}^{k} (f_i'(\zeta))^{m_i} d\zeta, \quad m_i \in \mathbb{N}, \tag{3.3}$$

then

$$F_3 \in H^{\mu} \text{ where } \mu = \frac{p}{(\sum_{i=1}^{k} m_i) - p} \text{ for all } p < 1.$$

Proof. Let $f_i \in \mathcal{R}$ for $i = 1, 2, \cdots, k$.
$f_i' \in H^p$, for $i = 1, 2, \cdots, k$ and for all $p < 1.$, using Lemma 2.6.
Using Lemma 2.1, we obtain,

$$(f_i')^{m_i} \in H^{\frac{p}{m_i}}, \text{ for all } p < 1 \text{ and } m_i \in \mathbb{N} \text{ for } i = 1, 2, \cdots, k$$

and

$$\prod_{i=1}^{k} (f_i')^{m_i} \in H^{\frac{p}{\sum_{i=1}^{k} m_i}}. \tag{3.4}$$

On differentiating (3.3), we get,

$$F_3'(z) = \prod_{i=1}^{k} (f_i'(z))^{m_i}, m_i \in \mathbb{N} \text{ for } i = 1, 2, \cdots, k.$$

We have,

$$F_3' \in H^{\frac{p}{\sum_{i=1}^{k} m_i}}, \text{ by } (3.4).$$

Since $p<1$, it follows that $p<\sum_{i=1}^{k} m_i$.
Therefore by using Lemma 2.2, we have,

$$F_3 \in H^{\mu} \text{ where } \mu = \frac{p}{(\sum_{i=1}^{k} m_i) - p} \text{ for all } p<1.$$

□

Theorem 3.4. *Let* $f_i \in \mathcal{R}$ *for* $i=1,2,\cdots k$ *and*

$$F_4(z) = \int_0^z \prod_{i=1}^{k}\Big(\frac{f_i(\zeta)}{\zeta}\Big)^{m_i} (f_i^{'}(\zeta))^{n_i} d\zeta, \quad m_i, n_i \in \mathbb{N}, \tag{3.5}$$

then

$$F_4 \in H^{\mu} \textit{ where } \mu = \frac{p}{\big(\sum_{i=1}^{k} n_i\big) + \big(\sum_{i=1}^{k} m_i\big) - p\big(\sum_{i=1}^{k} m_i - 1\big)} \textit{ for all } p<1.$$

Proof. Let $f_i \in \mathcal{R}$ for $i=1,2,\cdots,k$.
By Lemma 2.6, we have,

$$f_i^{'} \in H^p, \text{ for } i=1,2,\cdots k \text{ and for all } p<1. \tag{3.6}$$

Now by Lemma 2.1,

$$(f_i^{'})^{n_i} \in H^{\frac{p}{n_i}}, \text{for all } p<1 \text{ and } n_i \in \mathbb{N} \text{ for } i=1,2,\cdots,k$$

and

$$\prod_{i=1}^{k}(f_i^{'})^{n_i} \in H^{\frac{p}{\sum_{i=1}^{k} n_i}}.$$

From (3.6) and Lemma 2.2, we get,

$$f_i \in H^{\frac{p}{1-p}}, \text{ for all } p<1. \tag{3.7}$$

Using (3.7) and Lemma 2.1, we have,

$$\prod_{i=1}^{k}(f_i)^{m_i} \in H^{\frac{p}{(1-p)\left(\sum_{i=1}^{k} m_i\right)}}.$$

$$\implies \prod_{i=1}^{k}(f_i)^{m_i}(f_i^{'})^{n_i} \in H^{\frac{p}{\left(\sum_{i=1}^{k} n_i\right)+(1-p)\left(\sum_{i=1}^{k} m_i\right)}}.$$

On differentiating (3.5), we get,

$$z^{(\sum_{i=1}^{k} m_i)} F_4^{'}(z) = \prod_{i=1}^{k}(f_i(z))^{m_i}\big(f_i^{'}(z)\big)^{n_i} \in H^{\frac{p}{\left(\sum_{i=1}^{k} n_i\right)+(1-p)\left(\sum_{i=1}^{k} m_i\right)}}.$$

Using Lemma 2.4, we have,

$$F_4^{'} \in H^{\frac{p}{\left(\sum_{i=1}^{k} n_i\right)+(1-p)\left(\sum_{i=1}^{k} m_i\right)}}.$$

Since $p<1$, it follows that $p<(\sum_{i=1}^{k} n_i)+(1-p)(\sum_{i=1}^{k} m_i)$.
Therefore by Lemma 2.2,

$$F_4 \in H^{\mu} \text{ where } \mu = \frac{p}{(\sum_{i=1}^{k} n_i) + (\sum_{i=1}^{k} m_i) - p(\sum_{i=1}^{k} m_i - 1)}$$

for all $p<1$ and $m_i, n_i \in \mathbb{N}$, $i=1,2,\cdots,k$. □

Corollary 3.1. [7] *Let* $f_1 \in H^{p_1}$ *and* $f_2 \in H^{p_2}$ *where* $0 < p_1, p_2 < \infty$ *and*

$$F_5(z) = \int_0^z \left(\frac{f_1(\zeta)}{\zeta}\right)^{m_1} \left(\frac{f_2(\zeta)}{\zeta}\right)^{m_2} d\zeta, m_1, m_2 \in \mathbb{N}.$$

(i) If $p_1p_2 < p_1m_2 + p_2m_1$, *then*

$$F_5 \in H^{\frac{p_1p_2}{p_1m_2+p_2m_1-p_1p_2}}.$$

(ii) If $p_1p_2 \geq p_1m_2 + p_2m_1$, *then* $F_5 \in H^\infty$.

Corollary 3.2. [7] *Let* $f_1 \in \mathcal{R}$ *and* $g_1 \in H^Q$, *where* $Q = \frac{r}{r-1}, r > 1$ *and*

$$F_6(z) = \int_0^z (f_1'(\zeta))^m \left(\frac{g_1(\zeta)}{\zeta}\right)^n d\zeta, m, n \in \mathbb{N},$$

then

$$F_6 \in H^{\frac{pQ}{mQ+p(n-Q)}} \text{ for all } p < 1.$$

Corollary 3.3. [7] *Let* $f_1, f_2 \in \mathcal{R}$ *and*

$$F_7(z) = \int_0^z (f_1'(\zeta))^{m_1}(f_2'(\zeta))^{m_2} d\zeta, m_1, m_2 \in \mathbb{N},$$

then

$$F_7 \in H^{\frac{p}{(m_1+m_2)-p}} \text{ for all } p < 1.$$

Corollary 3.4. [7] *Let* $f_1 \in H^{p_1}$ *where* $0 < p_1 < \infty$. *and*

$$F_8(z) = \int_0^z \left(\frac{f_1(\zeta)}{\zeta}\right)^{m_1} d\zeta, m_1 \in \mathbb{N}.$$

(i) If $p_1 < m_1$, *then* $F_8 \in H^{\frac{p_1}{m_1-p_1}}$.
(ii) If $p_1 \geq m_1$, *then* $F_8 \in H^\infty$.

Corollary 3.5. [7] *Let* $f_1 \in \mathcal{R}$ *and* $F_9(z) = \int_0^z (f_1'(\zeta))^{m_1} d\zeta$, $m_1 \in \mathbb{N}$, *then* $F_9 \in H^{\frac{p}{m_1-p}}$ *for all* $p < 1$.

In the upcoming results, we determine the Hardy spaces of these integral operators by varying the integrands in class $\mathcal{A}$ and their exponentials over $\mathbb{R}^+$.

Theorem 3.5. *Let* $f \in H^p$ *and* $g \in H^q$, $0 < p, q < \infty$ *and*

$$F_{10}(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\alpha \left(\frac{g(\zeta)}{\zeta}\right)^\beta d\zeta, \quad \alpha, \beta > 0. \tag{3.8}$$

(i) If $pq < \alpha q + \beta p$ *then* $F_{10} \in H^\mu$ *where* $\mu = \frac{pq}{\alpha q + \beta p - pq}$.
(ii) If $pq \geq \alpha q + \beta p$ *then* $F_{10} \in H^\infty$.
(iii) $M_\lambda(r, F_{10}') \leq M^\alpha_{\alpha\lambda m}(r, \frac{f}{z}) M^\beta_{\beta\lambda n}(r, \frac{g}{z})$ *where* $\lambda \in (0, \infty]$ *and* $\frac{1}{m} + \frac{1}{n} = 1, m, n > 1$.
(iv) If $F_{10}(z) = z + \sum_{n=2}^\infty a_n z^n$ *and* $\frac{\alpha}{p} + \frac{\beta}{q} > 2$ *then* $a_n = o(n^{\frac{1}{\mu}-1})$.

Proof. Let $f \in H^p$, then it implies that,

$$\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

is bounded.
If $h(z) = (f(z))^\alpha$, for some $\alpha > 0$,

$$\lim_{r\to 1} M_x^x(r,h) = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^{\alpha x} d\theta$$

is bounded if $\alpha x < p$ which implies $x < \frac{p}{\alpha}$.
Thus, $h = f^\alpha \in H^{\frac{p}{\alpha}}$.
Similarly, if $g \in H^q$, then $g^\beta \in H^{\frac{q}{\beta}}$.
By using Lemma 2.1, we get,

$$(f)^\alpha (g)^\beta \in H^{\frac{pq}{\alpha q + \beta p}}, \text{ where } \alpha, \beta > 0 \text{ and } 0 < p, q < \infty. \tag{3.9}$$

Differentiating (3.8), we get,

$$z^{\alpha+\beta} F'_{10}(z) = (f(z))^\alpha (g(z))^\beta.$$

Also from equation (3.9), we have,

$$z^{\alpha+\beta} F'_{10}(z) \in H^{\frac{pq}{\alpha q + \beta p}}$$

and hence from Lemma 2.4, it follows that

$$F'_{10}(z) \in H^{\frac{pq}{\alpha q+\beta p}}.$$

(i) If $pq < \alpha q + \beta p$, then by Lemma 2.2, we have,

$$F_{10} \in H^\mu, \quad \mu = \frac{pq}{\alpha q + \beta p - pq}, \alpha, \beta > 0 \text{ and } 0 < p, q < \infty.$$

(ii) If $pq \geq \alpha q + \beta p$, then by Lemma 2.3, we have,

$$F_{10} \in H^\infty.$$

(iii) Differentiating (3.8) we get,

$$F'_{10}(z) = \left(\frac{f(z)}{z}\right)^\alpha \left(\frac{g(z)}{z}\right)^\beta, \quad \alpha, \beta > 0.$$

Therefore,

$$\begin{aligned} M_\lambda^\lambda(r, F'_{10}) &= \frac{1}{2\pi}\int_0^{2\pi} |F'_{10}(re^{i\theta})|^\lambda d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|\left(\frac{f(z)}{z}\right)^\alpha \left(\frac{g(z)}{z}\right)^\beta\right|^\lambda d\theta \\ &\leq \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f(z)}{z}\right|^{\alpha\lambda m} d\theta\right)^{\frac{1}{m}} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{g(z)}{z}\right|^{\beta\lambda n} d\theta\right)^{\frac{1}{n}} \\ &\text{where } \frac{1}{m} + \frac{1}{n} = 1,\ m, n > 1. \\ &\implies M_\lambda(r, F'_{10}) \leq M^\alpha_{\alpha\lambda m}\big(r, \frac{f}{z}\big) M^\beta_{\beta\lambda n}\big(r, \frac{g}{z}\big). \end{aligned}$$

(iv) If $\frac{\alpha}{p} + \frac{\beta}{q} > 2$, then

$$a_n = o\big(n^{\frac{1}{\mu}-1}\big), \mu = \frac{pq}{\alpha q + \beta p - pq}, \ \alpha, \beta > 0 \text{ and } 0 < p, q < \infty, \text{by Lemma 2.5.}$$

□

We now state the results for different integral operators, the proof of which are similar to above and hence omitted.

Theorem 3.6. *Let $f \in H^p$, g be analytic in Δ, $g' \in H^q$, $0 < p, q < \infty$ and*

$$F_{11}(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^{\alpha} (g'(\zeta))^{\beta} d\zeta, \ \alpha, \beta > 0.$$

(i) *If $pq < \alpha q + \beta p$ then $F_{11} \in H^{\mu}$, $\mu = \frac{pq}{\alpha q + \beta p - pq}$.*

(ii) *If $pq \geq \alpha q + \beta p$ then $F_{11} \in H^{\infty}$.*

(iii) $M_{\lambda}(r, F_{11}^{'}(z)) \leq M^{\alpha}_{\alpha\lambda m}\big(r, \frac{f(z)}{z}\big) M^{\beta}_{\beta\lambda n}(r, g'(z))$, *$\lambda \in (0, \infty]$ and $\frac{1}{m} + \frac{1}{n} = 1$, $m, n > 1$.*

(iv) *If $F_{11}(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\frac{\alpha}{p} + \frac{\beta}{q} > 2$ then $a_n = o\big(n^{(\frac{1}{\mu})-1}\big)$.*

Theorem 3.7. *Let f, g be analytic in Δ and $f \in H^p$, $g' \in H^q$, $0 < p, q < \infty$ and*

$$F_{12}(z) = \int_0^z (f'(\zeta))^{\alpha} (g'(\zeta))^{\beta} d\zeta, \ \alpha, \beta > 0.$$

(i) *If $pq < \alpha q + \beta p$ then $F_{12} \in H^{\mu}$, $\mu = \frac{pq}{\alpha q + \beta p - pq}$.*

(ii) *If $pq \geq \alpha q + \beta p$ then $F_{12} \in H^{\infty}$.*

(iii) $M_{\lambda}(r, F_{12}^{'}(z)) \leq M^{\alpha}_{\alpha\lambda m}(r, f'(z)) M^{\beta}_{\beta\lambda n}(r, g'(z))$, *$\lambda \in (0, \infty]$ and $\frac{1}{m} + \frac{1}{n} = 1$, $m, n > 1$.*

(iv) *If $F_{12}(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\frac{\alpha}{p} + \frac{\beta}{q} > 2$ then $a_n = o\big(n^{(\frac{1}{\mu})-1}\big)$.*

Theorem 3.8. *Let $f \in H^p$, $0 < p < \infty$ and*

$$F_{13}(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^{\alpha} d\zeta, \ \alpha > 0.$$

(i) *If $p < \alpha$ then $F_{13} \in H^{\mu}$, $\mu = \frac{p}{\alpha - p}$.*

(ii) *If $p \geq \alpha$ then $F_{13} \in H^{\infty}$.*

(iii) $M_{\lambda}(r, F_{13}^{'}(z)) = M^{\alpha}_{\alpha\lambda m}\big(r, \frac{f(z)}{z}\big)$, *$\lambda \in (0, \infty]$.*

(iv) *If $F_{13}(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\frac{\alpha}{p} > 2$ then $a_n = o\big(n^{(\frac{1}{\mu})-1}\big)$.*

Theorem 3.9. *Let f be analytic in Δ with $f' \in H^p$, $0 < p < \infty$ and*

$$F_{14}(z) = \int_0^z (f'(\zeta))^{\alpha} d\zeta, \ \alpha > 0.$$

(i) *If $p < \alpha$ then $F_{14} \in H^{\mu}$, $\mu = \frac{p}{\alpha - p}$.*

(ii) *If $p \geq \alpha$ then $F_{14} \in H^{\infty}$.*

(iii) $M_{\lambda}(r, F_{14}^{'}(z)) = M^{\alpha}_{\alpha\lambda m}(r, f'(z))$, *$\lambda \in (0, \infty]$.*

(iv) *If $F_{14}(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\frac{\alpha}{p} > 2$ then $a_n = o(n^{(\frac{1}{\mu})} - 1)$.*

Theorem 3.10. *Let f be analytic in Δ with $f' \in H^p$, $0 < p < \infty$ and*

$$F_{15}(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\alpha (f'(\zeta))^\beta d\zeta, \alpha, \beta > 0. \tag{3.10}$$

(i) If $p < 1$ and $p < \frac{\alpha+\beta}{1+\alpha}$ then $F_{15} \in H^{\mu_1}$, $\mu_1 = \frac{p}{\alpha(1-p)+\beta-p}$.

(ii) If $p < 1$ and $p \geq \frac{\alpha+\beta}{1+\alpha}$ then $F_{15} \in H^\infty$.

(iii) If $p \geq 1$ and $p < \beta$ then $F_{15} \in H^{\mu_2}$, $\mu_2 = \frac{p}{\beta-p}$.

(iv) If $p \geq 1$ and $p \geq \beta$ then $F_{15} \in H^\infty$.

(v) $M_\lambda(r, F_{15}^{'}(z)) \leq M^\alpha_{\alpha\lambda m}(r, \frac{f(z)}{z}) M^\beta_{\beta\lambda n}(r, f'(z))$, $\lambda \in (0, \infty]$ and $\frac{1}{m} + \frac{1}{n} = 1$, $m, n > 1$.

(vi) If $F_{15}(z) = z + \sum_{n=2}^\infty a_n z^n$ and $p < min\{1, \frac{\alpha+\beta}{2+\alpha}\}$ then $a_n = o(n^{(\frac{1}{\mu_1})-1})$.

(vii) If $F_{15}(z) = z + \sum_{n=2}^\infty a_n z^n$ with $p \leq 1$ and $p < \frac{\beta}{2}$ then $a_n = o(n^{(\frac{1}{\mu_2})-1})$.

Proof. Let $f' \in H^p, p < 1$, then by Lemma 2.2, we obtain,

$$f \in H^{\frac{p}{1-p}}.$$

Using Lemma 2.1, we have,

$$\left(\frac{f(z)}{z}\right)^\alpha \in H^{\frac{p}{\alpha(1-p)}},$$

$$(f'(z))^\beta \in H^{\frac{p}{\beta}}.$$

and

$$\left(\frac{f(z)}{z}\right)^\alpha (f'(z))^\beta \in H^m, m = \frac{p}{\alpha(1-p)+\beta}.$$

On differentiating (3.10), we get,

$$z^\alpha F_{15}^{'} = (f(z))^\alpha (f'(z))^\beta, \alpha, \beta > 0.$$

Using Lemma 2.4, we obtain,

$$F_{15}^{'} \in H^m, m = \frac{p}{\alpha(1-p)+\beta}, \ p < 1 \text{ and } \alpha, \beta > 0.$$

(i) When $p < \alpha(1-p) + \beta$:
Using Lemma 2.2, we have,

$$F_{15} \in H^{\mu_1}, \mu_1 = \frac{p}{\alpha(1-p)+\beta-p}.$$

(ii) When $p \geq \alpha(1-p) + \beta$:
Using Lemma 2.3, we have,

$$F_{15} \in H^\infty.$$

If $f' \in H^p$, where $p \geq 1$, by Lemma 2.3 $f \in H^\infty$.
Using Lemma 2.1, we have,

$$\left(\frac{f(z)}{z}\right)^\alpha \in H^\infty,$$

$$(f'(z))^\beta \in H^{\frac{p}{\beta}}$$

and

$$\left(\frac{f(z)}{z}\right)^{\alpha}(f'(z))^{\beta} \in H^n, n = \frac{p}{\beta}.$$

Therefore,

$$F_{15}^{'} \in H^n \text{ where } n = \frac{p}{\beta} \text{ with } p \geq 1 \text{ and } \beta > 0.$$

(iii) When $p < \beta$,
Using Lemma 2.2, we get,

$$F_{15} \in H^{\mu_2}, \mu_2 = \frac{p}{\beta - p}.$$

(iv) When $p \geq \beta$,
Using Lemma 2.3, we get,

$$F_{15} \in H^{\infty}.$$

The remainder of the proof is similar in lines with the proof of Theorem 3.5. □

4. CONCLUSION

In this paper, we have determined the Hardy space in which the integral operators lie, provided one knows the same of the functions or it's derivatives in the integrand of the integral operator. We have estimated an inequality of the integral means connecting the integral operators and the functions in the integrand. A bound for the Taylor coefficients of the integral operator, depending on the Hardy space in which the integral operator lie is also determined in this paper.

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