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On Tensor Product of c-Spaces

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ABSTRACT. This paper is an extension of the research on (cartesian) product of c-spaces. This paper demonstrates that the finite (tensor)product of quotients of c-spaces can be represented as a quotient of its (tensor)product. Some properties of the tensor product of c-spaces have been investigated in this context. Properties of the space of c-continuous functions have been probed and the relevance of the standard c-structure on it has been established.

1. INTRODUCTION

The concept of connectedness was familiar to us through various branches like topology and graph theory. In image analysis, topological connectivity is much useful for studying images defined over a continuous space, whereas graph theoretic connectivity is more useful for studying images defined over a discrete space. There are, however, topological spaces whose connectivity does not result from a graph [10] and vice versa [2]. As discrete images can be viewed as a discretization of the continuous images, compatibility is essential for both type of approaches. Therefore, it is fairly obvious that topological or graph-theoretic connectivity alone is insufficient for practical purposes. Combined strategies are essential. Reinhard Bӧrger's [9] theory of connected sets eliminates the drawbacks of graph theoretical and topological connectivity. He proposed the Theory of Connectivity Classes, an axiomatic approach to connectivity. He conducted a categorical study of these spaces. This space has enormous applications in Pattern Recognition, Signal Processing, Mathematical Morphology and Image Analysis [4, 5, 10, 14, 15].

Dugowson S., Muscat J. and Ronce C. et al. [3, 7, 9] enhanced the structural analysis of this space. Unexplored are structural properties, and this paper is an attempt in that direction. It is hoped that research in this area will stimulate application-based research.

2. PRELIMINARIES

All concepts in this section are taken from [3, 9, 11, 12, 15]. A c-space or a connectivity space is a set X together with a collection $\mathcal{C}$ of subsets such that the following properties hold.

(1) $\phi \in \mathcal{C}$ and $\{x\} \in \mathcal{C}$ for every $x \in X$.
(ii) If $\{C_i : i \in I\}$ be a non empty collection of subsets in $\mathcal{C}$ with $\underset{i\in I}{\cap} C_i \neq \phi$, then $\underset{i\in I}{\cup} C_i \in \mathcal{C}$.

The collection $\mathcal{C}$ of subsets X which satisfy the above axioms is called a *c-structure* or a *connectivity class* of X. Elements of a c-structure are called *connected sets*. Some examples of c-spaces are

(1) Discrete c-space $(X, \mathcal{D}_X)$, where $\mathcal{D}_X = \{\phi\} \cup \{\{x\} : x \in X\}$.
(2) Indiscrete c-space $(X, \mathcal{I}_X)$, where $\mathcal{I}_X = \mathcal{P}(X)$ is the power set of X.

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(3) The real line $\mathbb{R}$ with all intervals.
(4) Co-finite c-spce $(X, \mathcal{C}) = \mathcal{D}_X \cup \{A \subseteq X | A^c \text{ is finite}\}$, where X is infinite.
(5) If X is a topological space, then the collection of all connected sets in X form a c-structure on X and the corresponding c-space is called the associated c-space of X.
(6) Let G be a finite simple graph. Then the collection of all edge connected sub graphs of G form a c-structure on G and the corresponding c-space is called the associated c-space of G.

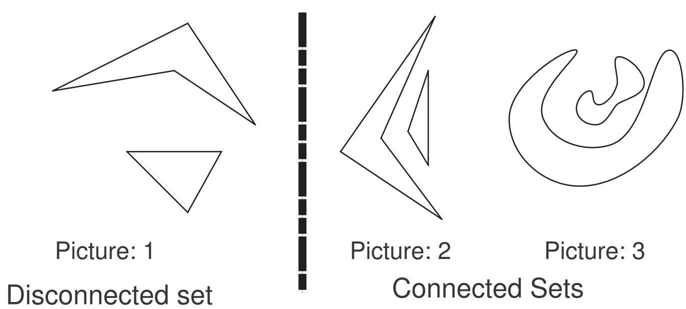
The c-space (X, C_X) is denoted by X unless otherwise stated. Let X and Y be two c-spaces and $f : X \to Y$ be a function. The function f is called *c-continuous* or *catenuous* if it maps connected sets of X to connected sets of Y. Further, a bijection $f : X \to Y$ is said to be a *c-isomorphism* or *catenomorphism* if both f and f^{-1} are c-continuous.

(1) Any continuous function from a topological space X to another topological space Y is clearly c-continuous.
(2) Consider the c-spaces X and Y where $X = \{1, 2, 3\}$, $Y = \{a, b, c, d\}$, $\mathcal{C}_X = \mathcal{D}_X \cup \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\mathcal{C}_Y = \mathcal{D}_Y \cup \{\{a, b\}, \{c, d\}\}$. Define $f : X \to Y$ by $1 \to a$, $2 \to a$ and $3 \to b$. Then f is c-continuous.
(3) If X and Y are as above and if $f : X \to Y$ be defined by $1 \to a$, $2 \to a$ and $3 \to c$, then f is not c-continuous.

We may note that there are c-continuous functions from X to Y that are not continuous. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ This function maps connected sets to connected sets and is not continuous at 0.

Let $\{X_i : i \in I\}$ be a family of c-spaces and $\{f_i : X \to X_i : i \in I\}$ be a family of functions defined on a set X. Let $\mathcal{C} = \{A \subset X : f_i(A) \in \mathcal{C}_{X_i} \text{ for every } i\}$. Then $\mathcal{C}$ is a c-structure on X and is called the strong c-structure generated by the given family of functions. The c-structure on the product space $\prod_{i \in I} X_i$ is the strong c-structure generated by the family of projection functions $\{\pi_i : i \in I\}$.

To visualize the connectedness in the product space, some examples of connected and disconnected sets from $\mathbb{R}^2$ are given below. For figure $P1$, we can note that $\pi_1(P1)$ is an



interval(and hence a connected set in $\mathbb{R}$) where as $\pi_2(P1)$ is the union of two intervals in $\mathbb{R}$ (and hence a disconnected subset of $\mathbb{R}$), so that figure $P1$ is a disconnected subset of $\mathbb{R}^2$. For figures $P2$ and $P3$, it is clear that both projections are intervals in $\mathbb{R}$(and thereby connected in $\mathbb{R}$) and hence are connected in $\mathbb{R}^2$.

Let X be any nonempty set and $\mathcal{B}$ be a collection of subsets of a set X. Then the smallest c-structure on X containing $\mathcal{B}$ is called the c-structure generated by $\mathcal{B}$, and is denoted by

$< \mathcal{B} >$. Elements of $\mathcal{B}$ are called basic connected sets. Any two points in a connected set B in $< \mathcal{B} >$ can be connected by a finite chain of basic connected sets that are contained in B. Let $\{X_i : i \in I\}$ be a family of c-spaces and $\{f_i : X_i \to X : i \in I\}$ be a family of functions. Then the weak c-structure generated by $\{f_i\}_{i\in I}$ is the smallest c-structure on X which make each function f_i c-continuous and is denoted by $< \{f_i : i \in I\} >_W$. In particular, let X and Y be any two c-spaces. Let $f : X \to Y$ be an onto function. Then f is said to be a quotient map or Y is said to be a quotient space of X with respect to f if $\mathcal{C}_Y$ is the smallest c-structure on Y which make f c-continuous. In other words, $\mathcal{C}_Y$ is the weak c-structure on Y generated by $\{f\}$. The following Theorem[1] is true for any category and in particular for the category of c-spaces.

Theorem 2.1. *Let X and Y be two c-spaces such that $f : X \to Y$ is a quotient map. Then for any c-space Z, a function $g : Y \to Z$ is c-continuous if and only if the composite function $g \circ f : X \to Z$ is c-continuous.*

We may note the following proposition.

Proposition 2.1. [3] *Let X be a set, $\mathcal{A}$ a set of subsets of X, $(Y, \mathcal{C}_Y)$ a c-space and $f : X \to Y$ be a function. Then f is c-continuous from $(X, < \mathcal{A} >)$ to $(Y, \mathcal{C}_Y)$ if and only if $f(A) \in \mathcal{C}_Y$ for every $A \in \mathcal{A}$.*

3. On Tensor Product of c-spaces

This section examines the conditions under which the finite product of quotients of c-spaces becomes a quotient of its product. Before proceeding with the primary investigations, it is important to note the following premise.

Proposition 3.2. [13]

(1) *Finite product(cartesian) of quotients of c-spaces need not be the quotients of its product.*
(2) *Let $f : X \to Y$ be a quotient map. Then $I_X \times f : X \times X \to X \times Y$ need not be a quotient map, where I_X is the identity map on X.*

This prompts us to employ the concept of Tensor Product of c-spaces introduced by S. Dugowson[3] in order to solve the problem.

Definition 3.1. [3] Tensor Product
The connectivity tensor product $X_1 \boxtimes X_2$ of two c-spaces X_i, $(i = 1, 2)$ is the set $X_1 \times X_2$ with the generated c-structure $< \{C_1 \times C_2 : C_1 \in \mathcal{C}_{X_1}, C_2 \in \mathcal{C}_{X_2}\} >$ on it.

We may note that [3], for any two c-spaces X_1 and X_2, the c-structure $\mathcal{C}_{X_1 \boxtimes X_2}$ is a smaller c-structure on the set $X_1 \times X_2$ than the c-structure given by the cartesian product, since $C_1 \times C_2 \in \mathcal{C}_{X_1 \times X_2}$ for every $C_1 \in \mathcal{C}_{X_1}$ and $C_2 \in \mathcal{C}_{X_2}$.

Remark 3.1. Without any loss of generality, we can extend the same definition to arbitrary product. That is, given a family of c-spaces, $\{X_i : i \in I\}$, it's tensor product $\underset{i\in I}{\boxtimes} X_i$ is the set $\prod_{i\in I} X_i$ with the generated c-structure $< \{\prod_{i\in I} C_i : C_i \in \mathcal{C}_{X_i}$ for each $i\} >$ on it.

Proposition 3.3. *For $i \in I$, let X_i, Y_i be c-spaces and $\{f_i : X_i \to Y_i : i \in I\}$ be family of functions . Then,*

(1) *The projection functions $\pi_i : \underset{i\in I}{\boxtimes} X_i \to X_i$, for $i \in I$ are c-continuous.*
(2) *The function $h : \underset{i\in I}{\boxtimes} X_i \to \underset{i\in I}{\boxtimes} Y_i$ defined by $h(x) = (f_i(x_i))_{i\in I}$, where $x = (x_i)_{i\in I}$ is c-continuous if and only if each f_i is c-continuous.*

(3) *Let X be a c-space and $\{g_i : X \to X_i : i \in I\}$ be a family of functions. Define $f : X \to \underset{i \in I}{\boxtimes} X_i$ by $f(x) = (g_i(x))_{i \in I}$, $x \in X$. If f is c-continuous, then each g_i is c-continuous. Converse is not true.*

Proof of above statements directly follows from the Proposition 2.1. For the counter example for statement (3), consider the following example.

Consider the c-spaces $X = \{1, 2, 3\}$, $X_1 = \{a, b, c\}$, $X_2 = \{d, e, f\}$ with $\mathcal{C}_X = \mathcal{D}_X \cup \{X\}$, $\mathcal{C}_{X_1} = \mathcal{D}_{X_1} \cup \{X_1\}$ and $\mathcal{C}_{X_2} = \mathcal{D}_{X_2} \cup \{\{d, e\}\}$.

Then the c-structure generated by the collection $\{C_1 \times C_2 : C_1 \in \mathcal{C}_{X_1}, C_2 \in \mathcal{C}_{X_2}\}$ is given by

$\mathcal{D}_{X_1 \times X_2} \cup \{\{(a,d),(a,e)\}, \{(b,d),(b,e)\}, \{(c,d),(c,e)\}, \{(a,d),(b,d),(c,d)\}, \{(a,e),(b,e),(c,e)\}, \{(a,f),(b,f),(c,f)\}, \{(a,d),(a,e),(b,d),(c,d)\}, \{(a,d),(a,e),(b,e),(c,e)\}, \{(a,d),(b,d),(b,e),(c,d)\}, \{(a,e),(b,d),(b,e),(c,e)\}, \{(a,d),(b,d),(c,d),(c,e)\}, \{(a,e),(b,e),(c,d),(c,e)\}, \{(a,d),(a,e),(b,d),(b,e),(c,d),(c,e)\}\}$.

Define $f_1 : X \to X_1$ by $1 \mapsto a$, $2 \mapsto b$ and $3 \mapsto c$ and $f_2 : X \to X_2$ by $1 \mapsto d$, $2 \mapsto e$ and $3 \mapsto e$.

We can verify that both f_1 and f_2 are c-continuous.

Define $f : X \to X_1 \boxtimes X_2$ by $f(x) = (f_1(x), f_2(x))$, $x \in X$.

Now, $f(X) = \{(a,d),(b,e),(c,e)\}$, which is not a connected set in $X_1 \boxtimes X_2$. Hence f is not c-continuous. Thus the converse fails. We may note that the above set $f(X) = \{(a,d),(b,e),(c,e)\}$ is connected in the cartesian product $X_1 \times X_2$.

The next theorem gives a partial settlement to our desired goal.

Theorem 3.2. *Let X_i and Y_i, $i = 1, 2$ be c-spaces such that $f_i : X_i \to Y_i$, $i = 1, 2$ be two quotient maps. Let $Y = Y_1 \boxtimes Y_2$. Then $\mathcal{C}_Y =< \{f_1(C) \times f_2(D) : C \in \mathcal{C}_{X_1}, D \in \mathcal{C}_{X_2}\} >$.*

Proof. By definition, we have $\mathcal{C}_Y =< \{A \times B : A \in \mathcal{C}_{Y_1}, B \in \mathcal{C}_{Y_2}\} >$. Let $\mathcal{C}_{Y_Q} =< \{f_1(C) \times f_2(D) : C \in \mathcal{C}_{X_1}, D \in \mathcal{C}_{X_2}\} >$. Since $f_1(C) \times f_2(D) \in \{A \times B : A \in \mathcal{C}_{Y_1}, B \in \mathcal{C}_{Y_2}\}$ for all $C \in \mathcal{C}_{X_1}$, $D \in \mathcal{C}_{X_2}$, we have

$$\mathcal{C}_{Y_Q} \subseteq \mathcal{C}_Y \tag{3.1}$$

On the other hand, let $K \in \mathcal{C}_Y$. Let $x = (x_1, y_1)$ and $y = (x_2, y_2)$ be two elements of K. Then, there exist basic connected sets $A_i \times B_i$ for $i = 1$ to n such that $A_i \in \mathcal{C}_{Y_1}$, $B_i \in \mathcal{C}_{Y_2}$ for each i, $x \in A_1 \times B_1$, $y \in A_n \times B_n$, $(A_i \times B_i) \cap (A_{i+1} \times B_{i+1}) \neq \phi$ for $i = 1$ to $(n-1)$ and $A_i \times B_i \subseteq K$ for every i.

Let $(a_i, b_i) \in (A_i \times B_i) \cap (A_{i+1} \times B_{i+1})$ for i=1 to (n-1).

Then $a_i, a_{i+1} \in A_{i+1}$ and $b_i, b_{i+1} \in B_{i+1}$ for $i = 0$ to $(n-1)$, where $a_0 = x_1$, $a_n = x_2$, $b_0 = y_1$ and $b_n = y_2$. Since $a_0, a_1 \in A_1$ and since $A_1 \in \mathcal{C}_{Y_1} =< \{f_1(C) : C \in \mathcal{C}_{X_1}\} >$, there exists a finite sequence of basic connected sets $\{f_1(D_i) : D_i \in \mathcal{C}_{X_1}, i = 1, 2, \ldots, m_1\}$ such that $a_0 \in f_1(D_1)$, $a_1 \in f_1(D_{m_1})$, $f_1(D_i) \cap f_1(D_{i+1}) \neq \phi$ for $= 1$ to $(m_1 - 1)$ and $f_1(D_i) \subseteq A_1$ for every i.

Similarly, there exists a finite sequence of basic connected sets $\{f_2(E_i) : E_i \in \mathcal{C}_{X_2}, i = 1, 2, \ldots, n_1\}$ such that $b_0 \in f_2(E_1)$, $b_1 \in f_2(E_{n_1})$, $f_2(E_i) \cap f_2(E_{i+1}) \neq \phi$ for $i = 1$ to $(n_1 - 1)$ and $f_2(E_i) \subseteq B_1$ for every i.

Let $m_1 \leq n_1$. Let $f_1(D_i) = f_1(D_{m_1})$ for $(m_1 + 1) \leq i \leq n_1$.

Now consider the finite sequence $S_1 = \{f_1(D_i) \times f_2(E_i) : i = 1 \text{ to } n_1\}$ of connected sets in $(Y, \mathcal{C}_{Y_Q})$.

Clearly $(a_0, b_0) \in f_1(D_1) \times f_2(E_1)$ and $(a_1, b_1) \in f_1(D_{n_1}) \times f_2(E_{n_1})$. Also,

$$\begin{aligned}[f_1(D_i) \times f_2(E_i)] \bigcap [f_1(D_{i+1}) \times f_2(E_{i+1})] \\ &= [f_1(D_i) \cap f_1(D_{i+1})] \times [f_2(E_i) \bigcap f_2(E_{i+1})] \\ &\neq \phi \text{ for each i=1 to } (n_1\text{-}1)\end{aligned}$$

Further, $f_1(D_i) \times f_2(E_i) \subseteq A_1 \times B_1 \subseteq K$ for every i. Thus S_1 is a finite chain of connected sets in $(Y, \mathcal{C}_{Y_Q})$ containing (a_0, b_0) and (a_1, b_1) and contained in K.

Similarly there exists a finite chain S_2 of connected sets $f_1(D_i) \times f_2(E_i)$, $n_1 + 1 \leq i \leq n_2$ (by renaming D_i's and E_i's accordingly) such that $(a_1, b_1) \in f_1(D_{n_1+1}) \times f_2(E_{n_1+1})$, $(a_2, b_2) \in f_1(D_{n_2}) \times f_2(E_{n_2})$. Further, $f_1(D_i) \times f_2(E_i) \subseteq A_2 \times B_2 \subseteq K$ for every $n_1 + 1 \leq i \leq n_2$.

Since $[f_1(D_{n_1}) \times f_2(E_{n_1})] \bigcap [f_1(D_{n_1+1}) \times f_2(E_{n_1+1})] \neq \phi$, concatenation of the finite chains S_1 and S_2, that is, $S_1 + S_2$ is a finite chain of connected sets in K such that $(a_0, b_0) \in f_1(D_1) \times f_2(E_1)$ and $(a_2, b_2) \in f_1(D_{n_2}) \times f_2(E_{n_2})$.

Proceeding similarly, there is a finite chain $S_1 + S_2 + \ldots + S_n$ of connected sets in K such that $x = (a_0, b_0) \in f_1(D_1) \times f_2(E_1)$ and $y = (a_n, b_n) \in f_1(D_{n_n}) \times f_2(E_{n_n})$.

Thus any two elements of K can be joined by a finite sequence of basic connected sets in $\{f_1(C) \times f_2(D) : C \in \mathcal{C}_{X_1}, D \in \mathcal{C}_{X_2}\}$ and hence $C \in < \{f_1(C) \times f_2(D) : C \in \mathcal{C}_{X_1}, D \in \mathcal{C}_{X_2}\} >$. That is, $K \in \mathcal{C}_{Y_Q}$. Thus

$$\mathcal{C}_Y \subseteq \mathcal{C}_{Y_Q} \tag{3.2}$$

From the equations (3.1) and (3.2), theorem follows. □

The following theorem solves our problem regarding the finite product of quotients. Unresolved is the problem of arbitrary product of quotients.

Theorem 3.3. *Let $\{X_i : i = 1 \text{ to } n\}$ and $\{Y_i : i = 1 \text{ to } n\}$ be two family of c-spaces such that for each i, $f_i : X_i \to Y_i$ be a quotient map. Let $X = \boxtimes_{i=1}^{n} X_i$ and $Y = \boxtimes_{i=1}^{n} Y_i$. Define $f : X \to Y$ by $f = \prod_{i=1}^{n} f_i$. Then Y is a quotient space of X with respect to f.*

That is, in the case of tensor product of c-spaces, finite product of quotients of c-spaces is the quotient of its product.

Proof. Given that $f : X \to Y$ be defined by $f = \prod_{i=1}^{n} f_i$. Then by Proposition 3.3, f is c-continuous.

Also by Theorem 3.2, $\mathcal{C}_Y = < \{\prod_{i=1}^{n} f_i(C) : C \in \mathcal{C}_{X_i}\} >$.

Let $\mathscr{C}$ be any other c-structure on Y with respect to which f is c-continuous. By Proposition 2.1, we have

$$\begin{aligned} f : X \to (Y, \mathscr{C}) \text{ is c-continuous} & \\ \iff \quad & f(C_1 \times C_2 \times \ldots \times C_n) \in \mathscr{C},\ C_i \in \mathcal{C}_{X_i} \text{ for } i = 1 \text{ to } n, \\ \iff \quad & \prod_{i=1}^{n} f_i(C) \in \mathscr{C},\ C \in \mathcal{C}_{X_i}, \\ \iff \quad & < \{\prod_{i=1}^{n} f_i(C) : C \in \mathcal{C}_{X_i}\} > \subseteq \mathscr{C}, \\ \iff \quad & \mathcal{C}_Y \subseteq \mathscr{C}. \end{aligned}$$

Thus $\mathcal{C}_Y$ is the smallest c-structure on Y with respect to which $f : X \to Y$ is c-continuous and hence Y is a quotient space of X with respect to f. □

4. More on the Space of c-continuous Functions

Let X and Y be two c-spaces and $\mathcal{C}(X, Y)$ denotes the set of all c-continuous functions from X to Y. In [3], a c-structure on $\mathcal{C}(X, Y)$ is defined to be as follows. A subset M of $\mathcal{C}(X, Y)$ is said to be connected if for every $K \in \mathcal{C}_X$, $< M, K > \in \mathcal{C}_Y$, where $< M, K > = \bigcup_{f \in M} f(K)$. Let us call this c-structure as the standard c-structure on $\mathcal{C}(X, Y)$.

Unless otherwise specified, from here onwards, $\mathcal{C}(X,Y)$ is considered as a c-space with the standard c-structure.

In [3], it is also proved that, M is connected in $\mathcal{C}(X,Y)$ if and only if for all $x \in X$, $< M, \{x\} > \in \mathcal{C}_Y$.

Proposition 4.4. *Let X and Y be two c-spaces. Then the evaluation map $e : X \boxtimes \mathcal{C}(X,Y) \to Y$ defined by $e(x,f) = f(x)$, for $x \in X$ and $f \in \mathcal{C}(X,Y)$ is c-continuous.*

Proof. To prove the c-continuity of e, by Proposition 2.1, it is enough to prove that $e(C_1 \times C_2)$ is connected in Y for $C_1 \in \mathcal{C}_X$ and $C_2 \in \mathcal{C}_{\mathcal{C}(X,Y)}$.

Since C_2 is connected in $\mathcal{C}(X,Y)$, in particular we have, $\bigcup\limits_{f \in C_2} f(C_1)$ is connected in Y. That is,

$$\{f(x) : x \in C_1, f \in C_2\} \text{ is connected in } Y.$$

Hence $\{e(x,f) : x \in C_1, f \in C_2\}$ is connected in Y. Thus $e(C_1 \times C_2)$ is connected in Y. □

Remark 4.2. The above proposition will not be true if we replace tensor product with cartesian product. That is, the evaluation map $e : X \times \mathcal{C}(X,Y) \to Y$ defined by $e(x,f) = f(x)$, for $x \in X$ and $f \in \mathcal{C}(X,Y)$ need not be c-continuous.

For example, consider the c-spaces $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$, where X=$\{a,b,c\}$, $Y = \{1,2,3\}$, $\mathcal{C}_X = \mathcal{D}_X \cup \{\{a,b\}, X\}$ and $\mathcal{C}_Y = \mathcal{D}_Y \cup \{\{1,2\},\{2,3\},Y\}$.

Define two functions f_1 and f_2 from X to Y as

$$f_1(x) = \begin{cases} 1 & if\ x = a \\ 2 & if\ x = b \\ 3 & if\ x = c \end{cases}$$

and

$$f_2(x) = \begin{cases} 2 & if\ x = a \\ 3 & if\ x = b \\ 3 & if\ x = c \end{cases}$$

It can be easily verified that both f_1 and f_2 are c-continuous.

Since $< \{f_1, f_2\}, \{a\} >= \{1,2\}$, $< \{f_1, f_2\}, \{b\} >= \{2,3\}$ and $< \{f_1, f_2\}, \{c\} >= \{3\}$, $\{f_1, f_2\}$ is connected in $\mathcal{C}(X,Y)$.

Consider the connected set $C = \{\{a, f_1\}, \{b, f_2\}\}$ in product space $X \times \mathcal{C}(X,Y)$. Now,

$$\begin{aligned} e(C) &= \{f_1(a), f_2(b)\} \\ &= \{1,3\} \end{aligned}$$

is not connected in Y. Hence $e : X \times \mathcal{C}(X,Y) \to Y$ is not c-continuous

Theorem 4.4. *Let X, Y and Z be three c-spaces. Then a map $f : X \boxtimes Z \to Y$ is c-continuous if and only the induced map $\hat{f} : Z \to \mathcal{C}(X,Y)$ is c-continuous, where $\hat{f}(z)(x) = f(x,z)$.*

Proof. Let $\hat{f} : Z \to \mathcal{C}(X,Y)$ be c-continuous.

We know that by Proposition 4.4, the evaluation map $e : X \boxtimes \mathcal{C}(X,Y) \to Y$ defined by $e(x,f) = f(x)$, for $x \in X$ and $f \in \mathcal{C}(X,Y)$ is c-continuous. Now consider the diagram

$$X \boxtimes Z \overset{id_X \times \hat{f}}{\to} X \boxtimes \mathcal{C}(X,Y) \overset{e}{\to} Y.$$

Since $f = e \circ (id_X \times \hat{f})$, f is c-continuous.

Conversely let $f : X \boxtimes Z \to Y$ is c-continuous. Let C be a connected set in Z. To show $\hat{f}(C)$ is connected in $\mathcal{C}(X,Y)$, let K be any connected set in X.

$$\begin{aligned} < \hat{f}(C), K > &= \bigcup_{x\in C} \hat{f}(x)(K) \\ &= \bigcup_{x\in C}\bigcup_{y\in K} \hat{f}(x)(y) \\ &= \bigcup_{x\in C}\bigcup_{y\in K} \{f(y,x)\} \\ &= f(K\times C) \end{aligned}$$

Since f is c-continuous, $f(K \times C)$ is connected in Y. Thus $\hat{f}(C)$ is connected in Y. Since $C \in \mathcal{C}_Z$ is arbitrary, $\hat{f}$ is c-continuous. □

Theorem 4.5. *Let $\mathcal{C}'(X,Y)$ denote the set $\mathcal{C}(X,Y)$ with some c-structure $\mathfrak{C}$. If the evaluation map $e : X \boxtimes \mathcal{C}'(X,Y) \to Y$ is c-continuous, then $\mathfrak{C}$ is contained in the standard c-structure on $\mathcal{C}(X,Y)$.*

Proof. Let $e : X \boxtimes \mathcal{C}'(X,Y) \to Y$ is c-continuous. Then by Theorem 4.4, the induced map $\hat{e} : \mathcal{C}'(X,Y) \to \mathcal{C}(X,Y)$ is c-continuous, where $\hat{e}(f)(x) = e(x,f)$, for $x \in X$ and $f \in \mathcal{C}'(X,Y)$.

Let C be a connected set in $\mathcal{C}'(X,Y)$. Then $\hat{e}(C)$ is connected in $\mathcal{C}(X,Y)$. Thus for every $x \in X$, $< \hat{e}(C), \{x\} >$ is connected in Y. But,

$$\begin{aligned} < \hat{e}(C), \{x\} > &= \bigcup_{f\in C} \hat{e}(f)(x) \\ &= \bigcup_{f\in C} \{e(x,f)\} \\ &= \bigcup_{f\in C} \{f(x)\} \\ &= < C, \{x\} > \end{aligned}$$

Thus $< C, \{x\} >$ is connected in Y for $x \in X$ and hence C is connected in the standard c-structure on $\mathcal{C}(X,Y)$. Consequently, $\mathfrak{C}$ is contained in the standard c-structure on $\mathcal{C}(X,Y)$. □

Even though the following theorem is a special case of the Theorem 3.3 which we proved earlier, it fortifies the relevance of the standard c-structure on the function space $\mathcal{C}(X,Y)$.

Theorem 4.6. *Let A, B and X be three c-spaces. Let $p : A \to B$ be a quotient map. Then $id_X \times p : X \boxtimes A \to X \boxtimes B$ is a quotient map, where id_X is the identity map on X.*

Proof. Let $p : A \to B$ be a quotient map. Let $(X \boxtimes B)_q$ denotes the quotient space of $X \times A$ with respect to the map $\pi = id_X \times p$. Let $g : X \boxtimes B \to (X \boxtimes B)_q$ be the identity map. We claim g is a c-isomorphism.

Since $\pi : X \boxtimes A \to (X \boxtimes B)_q$ is c-continuous, by Theorem 4.4, the induced map $\hat{\pi} : A \to \mathcal{C}(X, (X \boxtimes B)_q)$ defined by $\hat{\pi}(a)(x) = \pi(x,a)$ $a \in A$, $x \in X$ is c-continuous.

Define $\hat{g} : B \to \mathcal{C}(X, (X \boxtimes B)_q)$ by $\hat{g}(b)(x) = g(x,b)$. We claim $\hat{g}$ is c-continuous.

For $a \in A$, $x \in X$ we have $\hat{\pi}(a)(x) = \pi(x,a) = (id_X \times p)(x,a) = (x, p(a))$. Thus,

$$\begin{aligned} (\hat{g} \circ p)(a)(x) &= \hat{g}(p(a))(x) \\ &= (x, p(a)) \\ &= \hat{\pi}(a)(x) \end{aligned}$$

Hence $\hat{g} \circ p = \hat{\pi}$. Since p is a quotient map, by Theorem 2.1, $\hat{g}$ is c-continuous. Hence by Theorem 4.4, $g : X \boxtimes B \to (X \boxtimes B)_q$ is c-continuous.

It can be easily verified that $g^{-1} : (X \boxtimes B)_q \to X \boxtimes B$ is c-continuous. Since g is bijective, g is a c-isomorphism. Hence the proof. □

Remark 4.3. In TOP, the category of topological spaces, the corresponding results holds if X is a locally compact Hausdorff topological space [6].

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