CREAT. MATH. INFORM.
Volume **33** (2024), No. 1,
Pages 77 - 85

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2024.01.08

On Prime Ideal Space of a Partially Ordered Ternary Semigroup

DATTATRAY SHINDE and MACHCHHINDRA GOPHANE

ABSTRACT. In this paper, we introduced the hull-kernel topology τ on the set $\mathcal{P}$ of prime ideals in a partially ordered ternary semigroup T and investigated various topological properties of the structure space $(\mathcal{P}, \tau)$. We also obtained some useful results about compactness and connectedness of the set of all prime full ideals of T.

1. INTRODUCTION

In 1932, Lehmer [8] studied the literature of a ternary algebraic system. The ternary semigroup is a particular case of n-ary semigroups. So many results on ternary semigroups have an analogous version for n-ary semigroups. The ideal theory in ternary semigroups was introduced by F. M. Sioson in 1965. Shabir and Bashir [11] introduced and studied the notion of prime, semiprime and irreducible ideals in ternary semigroups.

Iampan [2] has introduced the notion of partially ordered ternary semigroups, which is a generalization of an ordered semigroup and a ternary semigroup. In [9], the ideal theory of a partially ordered ternary semigroups is introduced. Siva Rami Reddy et al. [10] defined and studied the notions of complete prime ideals, prime ideals, complete semiprime ideals, semiprime ideals of partially ordered ternary semigroups. Shinde and Gophane [12] introduced and studied the notions of prime, semiprime and irreducible pseudo symmetric ideals in partially ordered ternary semigroups and proved that the set of all strongly irreducible pseudo symmetric ideals is topologized.

Kar [4] introduced and studied the concept of the structure space of ternary semirings. He also studied the various properties of this structure space. The notion of the structure space of Γ- semigroups was introduced by Kar and Chattopadhyay in [5]. Kostaq et al. [7] introduced the some special classes of all proper prime k-ideals, prime ideals and strongly irreducible ideals in Γ- semirings. They also obtained the topological spaces of these ideals of Γ- semirings. Jagtap and Pawar [3] studied the space of prime ideals of a Γ- semiring and properties of the space of prime ideals of a Γ- semiring.

In this article, we introduce and study the concept of the structure space of partially ordered ternary semigroups. We consider the set $\mathcal{P}$ of all prime ideals of a partially ordered ternary semigroup T and build the topology τ on $\mathcal{P}$ using the closure operator defined in terms of intersection and inclusion relations among these ideals of partially ordered ternary semigroup T. We investigate various topological properties of space $(\mathcal{P}, \tau)$. This topological space $(\mathcal{P}, \tau)$ is referred as the structure space of the partially ordered ternary semigroup T. We also studied the compactness, connectedness and separation axioms in this topological space $(\mathcal{P}, \tau)$.

Received: 26.04.2023. In revised form: 16.11.2023. Accepted: 23.11.2023
2020 *Mathematics Subject Classification.* 06F99, 20M12, 20N99.
Key words and phrases. *Partially ordered ternary semigroup, prime ideal, hull-kernel topology, structure space.*
Corresponding author: Dattatray Shinde; shindednmaths@gmail.com

2. Preliminaries

Definition 2.1. [8]A non-empty set T with a ternary operation [ ] : $T \times T \times T \longrightarrow T$ is called a ternary semigroup if [ ] satisfies the associative law, $[[a\ b\ c]\ d\ e] = [a\ [b\ c\ d]\ e] = [a\ b\ [c\ d\ e]]$, for all $a, b, c, d, e \in T$.

Definition 2.2. [2] A ternary semigroup T is said to be a partially ordered ternary semigroup if there exist a partially ordered relation $\leq$ on T such that, $a \leq b \Rightarrow xya \leq xyb, xay \leq xby, axy \leq bxy$ for all $a, b, x, y \in T$. In this section, we write T for a partially ordered ternary semigroup.

Definition 2.3. [9] An element $0 \in T$ is said to be a zero element of T if $0xy = x0y = xy0 = 0$ and $0 \leq t$ for all $x, y, t \in T$.

Definition 2.4. [9]An element $e \in T$ is said to be an identity element of T if $exx = xxe = xex = x$ and $x \leq e$ for all $x \in T$.

Let X be a non-empty subset of T. We denote, $\{t \in T : t \leq x, \text{ for some } x \in X\}$ by $(X]$.

Definition 2.5. [1]A non-empty subset I of T is said to be a left (respectively, right, lateral) ideal of T if $TTI \subseteq I$ (respectively, $ITT \subseteq I$, $TIT \subseteq I$) and $(I] = I$. A non-empty subset I of T is said to be an ideal of T if it is a left ideal, a right ideal and a lateral ideal of T.

Definition 2.6. [9]Let X be the non-empty subset of T. The intersection of all ideals of T containing X is called an ideal of T generated by X and it is denoted by $\langle X \rangle$. The ideal generated by $\{a\}$ for some $a \in T$ is denoted by $\langle a \rangle$.

Definition 2.7. [10]An ideal I of T is said to be a prime ideal of T provided I_1, I_2, I_3 are ideals of T and $I_1 I_2 I_3 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$.

Definition 2.8. [10]An ideal I of T is said to be a semiprime ideal of T provided P is ideal of T and $P^n \subseteq I$ for some odd natural number n implies $P \subseteq I$.

Definition 2.9. [12]An ideal I of T is said to be a pseudo symmetric ideal if $x, y, z \in T$, $xyz \in I$ implies $xsytz \in I\ \forall s, t \in T$.

Definition 2.10. [12]A proper pseudo symmetric ideal I of T is said to be a prime pseudo symmetric ideal of T if $I_1 I_2 I_3 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$ where I_1, I_2, I_3 are the pseudo symmetric ideals of T.

Definition 2.11. [12]A proper pseudo symmetric ideal I of T is said to be a semiprime pseudo symmetric ideal of T if P is a pseudo symmetric ideal of T and $P^n \subseteq I \Rightarrow P \subseteq I$ for some odd natural number n.

Theorem 2.1. [12] *The non-empty intersection of an arbitrary collection of prime pseudo symmetric ideals of T is a semiprime pseudo symmetric ideal of T.*

Definition 2.12. [12]A proper pseudo symmetric ideal I of T is said to be irreducible (respectively strongly irreducible) pseudo symmetric ideal of T if $I_1 \cap I_2 \cap I_3 = I$ (respectively $I_1 \cap I_2 \cap I_3 \subseteq I$) implies $I_1 = I$ or $I_2 = I$ or $I_3 = I$ (respectively $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$) for any pseudo symmetric ideals I_1, I_2, I_3 of T.

3. Prime Ideal Space

In this article, we write T for a partially ordered ternary semigroup with zero, unless otherwise specified.

Let $\mathcal{P}$ be the family of all prime ideals of T. For any subset A of $\mathcal{P}$, we define $\overline{A} = \{I \in \mathcal{P} : \bigcap_{I_\alpha \in A} I_\alpha \subseteq I\}$. It can be seen that $\overline{\emptyset} = \emptyset$.

Theorem 3.2. *Let A, B be any two subsets of $\mathcal{P}$ then*
(i) $A \subseteq \overline{A}$
(ii) $\overline{\overline{A}} = \bar{A}$
(iii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$
(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. (i) By the definition of $\overline{A}$, for every $\alpha, I_\alpha \in A$. Therefore $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I_\alpha$ implies $I_\alpha \in \overline{A}$. Hence $A \subseteq \overline{A}$.

(ii) By (i), we have $\overline{A} \subseteq \overline{\overline{A}}$. Let $I \in \overline{\overline{A}}$. Then $\bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I$. Now, $I_\alpha \in \overline{A}$ implies that $\bigcap_{I_\gamma \in A} I_\gamma \subseteq I_\alpha$ for all $\alpha \in \Delta$, where Δ denotes the indexing set. Thus $\bigcap_{I_\gamma \in A} I_\gamma \subseteq \bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I$. Therefore $\bigcap_{I_\gamma \in A} I_\gamma \subseteq I$. So $I \in \overline{A}$ and hence $\overline{\overline{A}} \subseteq \overline{A}$. Thus $\overline{\overline{A}} = \bar{A}$.

(iii) Suppose that $A \subseteq B$. Let $I \in \overline{A}$. Then $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I$. Since $A \subseteq B$, it follows that $\bigcap_{I_\alpha \in B} I_\alpha \subseteq \bigcap_{I_\alpha \in A} I_\alpha \subseteq I$. This shows that $I \in \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$.

(iv) To prove, $\overline{A \cup B} = \overline{A} \cup \overline{B}$, firstly we prove that $\overline{A \cup B \cup C} = \overline{A} \cup \overline{B} \cup \overline{C}$ for any subset C of $\mathcal{P}$. From (iii), $\overline{A} \subseteq \overline{A \cup B \cup C}, \overline{B} \subseteq \overline{A \cup B \cup C}$ and $\overline{C} \subseteq \overline{A \cup B \cup C}$. This implies that, $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cup B \cup C}$. Now let $I \in \overline{A \cup B \cup C}$. Then $\bigcap_{I_\alpha \in A \cup B \cup C} I_\alpha \subseteq I$. Obviously, $\bigcap_{I_\alpha \in A \cup B \cup C} I_\alpha = \left(\bigcap_{I_\alpha \in A} I_\alpha\right) \cap \left(\bigcap_{I_\alpha \in B} I_\alpha\right) \cap \left(\bigcap_{I_\alpha \in C} I_\alpha\right)$. Since $\bigcap_{I_\alpha \in A} I_\alpha, \bigcap_{I_\alpha \in B} I_\alpha$ and $\bigcap_{I_\alpha \in C} I_\alpha$ are ideals of T, also

$$\left(\bigcap_{I_\alpha \in A} I_\alpha\right)\left(\bigcap_{I_\alpha \in B} I_\alpha\right)\left(\bigcap_{I_\alpha \in C} I_\alpha\right) \subseteq \left(\bigcap_{I_\alpha \in A} I_\alpha\right) \cap \left(\bigcap_{I_\alpha \in B} I_\alpha\right) \cap \left(\bigcap_{I_\alpha \in C} I_\alpha\right) = \bigcap_{I_\alpha \in A \cup B \cup C} I_\alpha \subseteq I.$$

As I is a prime ideal of T, we get $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I$ or $\bigcap_{I_\alpha \in B} I_\alpha \subseteq I$ or $\bigcap_{I_\alpha \in C} I_\alpha \subseteq I$, i.e. either $I \in \overline{A}$ or $I \in \overline{B}$ or $I \in \overline{C}$. Hence $I \in \overline{A} \cup \overline{B} \cup \overline{C}$. This shows that $\overline{A \cup B \cup C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$ and hence $\overline{A \cup B \cup C} = \overline{A} \cup \overline{B} \cup \overline{C}$. Since $\overline{\emptyset} = \emptyset$, putting $C = \emptyset$, we get $\overline{A \cup B} = \overline{A} \cup \overline{B}$. □

Remark 3.1. The mapping $A \longrightarrow \overline{A}$ is a closure operator on $\mathcal{P}$.

Definition 3.13. The closure operator $A \longrightarrow \overline{A}$ induces a topology τ on $\mathcal{P}$. This topology τ is called the hull-kernel topology and the topological space $(\mathcal{P}, \tau)$ is called the structure space of the partially ordered ternary semigroup T.

For any ideal I of T, we define $X(I) = \{J \in \mathcal{P} : I \subseteq J\}$ and $Y(I) = \mathcal{P} \setminus X(I) = \{J \in \mathcal{P} : I \nsubseteq J\}$.

Theorem 3.3. *Any closed set in $\mathcal{P}$ is of the form $X(I)$ where I is a ideal of T.*

Proof. Let $\overline{A}$ be any closed set in $\mathcal{P}$, where $A \subseteq \mathcal{P}$. Let $A = \{I_\alpha : \alpha \in \Delta\}$ where Δ is an index set and $I = \bigcap_{I_\alpha \in A} I_\alpha$. Then I is a ideal of T. Let $J \in \overline{A}$ then $\bigcap_{I_\alpha \in A} I_\alpha \subseteq J \Rightarrow I \subseteq J$. Therefore $J \in X(I)$ and so $\overline{A} \subseteq X(I)$. If $J \in X(I)$ then $I \subseteq J \Rightarrow \bigcap_{I_\alpha \in A} I_\alpha \subseteq J$. Therefore $J \in \overline{A}$ and hence $X(I) \subseteq \overline{A}$. Thus $\overline{A} = X(I)$. □

Corollary 3.1. *Any open set in $\mathcal{P}$ is of the form $Y(I)$ where I is a ideal of T.*

Let I be a ideal of T, we define for any $a \in T$, $X(a) = \{I \in \mathcal{P} : a \in I\}$ and $Y(a) = \mathcal{P} \setminus X(a) = \{I \in \mathcal{P} : a \notin I\}$.

Theorem 3.4. *The set $\{Y(a) : a \in T\}$ forms a base for open sets for the hull-kernel topology τ on $\mathcal{P}$.*

Proof. Let G be any open set in τ i.e. $G \in \tau$. Then by Corollary 3.1, we have $G = Y(I)$ where I is an ideal of T. For any $J \in G = Y(I)$ we have $I \nsubseteq J$. This implies that there exists $a \in I$ such that $a \notin J$. Hence $J \in Y(a)$. Therefore $G \subseteq Y(a)$. Now to show that $Y(a) \subseteq G$. Let $K \in Y(a)$. Then $a \notin K$. This gives that, $I \nsubseteq K$. Therefore $K \in Y(I) = G$. Hence $Y(a) \subseteq G$. Thus we get $J \in Y(a) \subseteq G$. Then $G = \bigcup_{a \in T} Y(a)$. Therefore $\{Y(a) : a \in T\}$ forms an open base for the hull-kernel topology τ on $\mathcal{P}$. □

Theorem 3.5. *The structure space $(\mathcal{P}, \tau)$ is a T_0-space.*

Proof. Let I and J be two distinct elements of $\mathcal{P}$. Then there is an element a either in $I \setminus J$ or in $J \setminus I$. Assume that $a \in I \setminus J$. But then $J \in Y(a)$ and $I \notin Y(a)$ i. e. $Y(a)$ is a neighborhood of J not containing I. Hence $(\mathcal{P}, \tau)$ is a T_0-space. □

Theorem 3.6. *The structure space $(\mathcal{P}, \tau)$ is a T_1-space if and only if no element of $\mathcal{P}$ is contained in any other element of $\mathcal{P}$.*

Proof. Suppose that $(\mathcal{P}, \tau)$ is a T_1-space. Let I and J be any two distinct elements of $\mathcal{P}$. Then each I and J has a neighborhood not containing the other. Since I and J are arbitrary elements of $\mathcal{P}$, this shows that no element of $\mathcal{P}$ is contained in any other element of $\mathcal{P}$.

Conversely, suppose that no element of $\mathcal{P}$ is contained in any other element of $\mathcal{P}$. Let I and J be any two distinct elements of $\mathcal{P}$. Then by assumption either $I \not\subset J$ and $J \not\subset I$. Therefore there exist $a, b \in T$ such that $a \in I$, $a \notin J$ and $b \in J$, $b \notin I$. Then we have $I \in Y(b)$ but $I \notin Y(a)$ and $J \in Y(a)$ but $J \notin Y(b)$, it means that, each of I and J has a neighborhood not containing the other. Hence $(\mathcal{P}, \tau)$ is a T_1-space. □

Corollary 3.2. *If $(\mathcal{P}, \tau)$ is a Hausdorff space, then no prime ideal contains any other prime ideal. Alternatively, If the space $(\mathcal{P}, \tau)$ is a Hausdorff space then the set of all minimal prime ideals and maximal ideals coincide.*

Corollary 3.3. *Let M be the set of all proper maximal ideals of a partially ordered ternary semigroup T with identity. Then $(\mathsf{M}, \tau_{\mathsf{M}})$ is a T_1-space, where τ_{M} is the induced topology on M from $(\mathcal{P}, \tau)$.*

Theorem 3.7. *The structure space $(\mathcal{P}, \tau)$ is a Hausdorff space if and only if for any two distinct pair of elements I and J of $\mathcal{P}$ there exist $a, b \in T$ such that $a \notin I$, $b \notin J$ and there does not exist any element K of $\mathcal{P}$ such that $a \notin K$ and $b \notin K$.*

Proof. Suppose that $(\mathcal{P}, \tau)$ is a Hausdorff space. Then for any two distinct pair of elements I and J of $\mathcal{P}$ there exists two basic open sets $Y(a)$ and $Y(b)$ such that $I \in Y(a), J \in Y(b)$ and $Y(a) \cap Y(b) = \emptyset$. Now $I \in Y(a)$ and $J \in Y(b)$ imply that $a \notin I$ and $b \notin J$. Let if possible there exist K in $\mathcal{P}$ such that $a \notin K$ and $b \notin K$. Then $K \in Y(a) \cap Y(b)$, a contradiction, since $Y(a) \cap Y(b) = \emptyset$. Thus there does not exist any element K of $\mathcal{P}$ such that $a \notin K$ and $b \notin K$.

Conversely, Suppose that the given condition holds. To show the space $(\mathcal{P}, \tau)$ is a Hausdorff space. Let I and J be two distinct elements of $\mathcal{P}$. Then by assumption there exists $a, b \in T$ such that $a \notin I, b \notin J$ and there does not exist any K of $\mathcal{P}$ such that $a \notin K$ and $b \notin K$. Then $I \in Y(a), J \in Y(b)$ and $Y(a) \cap Y(b) = \emptyset$. Hence $(\mathcal{P}, \tau)$ is a Hausdorff space. □

Theorem 3.8. *If $(\mathcal{P}, \tau)$ is a Hausdorff space containing more than one element then there exist $a, b \in T$ such that $\mathcal{P} = Y(a) \cup Y(b) \cup X(I)$ where I is the ideal generated by a, b in T.*

Proof. Suppose that $(\mathcal{P}, \tau)$ is a Hausdorff space containing more than one element. Let $J, K \in \mathcal{P}$ such that $J \neq K$. Since $(\mathcal{P}, \tau)$ is a Hausdorff space, there exists two basic open sets $Y(a)$ and $Y(b)$ such that $J \in Y(a), K \in Y(b)$ and $Y(a) \cap Y(b) = \emptyset$. Let I be the ideal generated by $a, b \in T$. Then I is the smallest ideal containing a and b. Let $L \in \mathcal{P}$. Then either $a \in L, b \notin L$ or $a \notin L, b \in L$ or $a, b \in L$. The case, $a \notin L, b \notin L$ is not possible, since $a \notin L, b \notin L$ implies that $L \in Y(a)$ and $L \in Y(b)$ that is $L \in Y(a) \cap Y(b)$ which is not possible because $Y(a) \cap Y(b) = \emptyset$. Now in the first case, $L \in Y(b)$ and hence $\mathcal{P} \subseteq Y(a) \cup Y(b) \cup X(I)$. In the second case, $L \in Y(a)$ and hence $\mathcal{P} \subseteq Y(a) \cup Y(b) \cup X(I)$. In the third case, $L \in X(I)$ and hence $\mathcal{P} \subseteq Y(a) \cup Y(b) \cup X(I)$. Therefore $\mathcal{P} \subseteq Y(a) \cup Y(b) \cup X(I)$. But $Y(a) \cup Y(b) \cup X(I) \subseteq \mathcal{P}$. Hence $\mathcal{P} = Y(a) \cup Y(b) \cup X(I)$. □

Theorem 3.9. *The structure space $(\mathcal{P}, \tau)$ is a regular space if and only if for any $I \in \mathcal{P}$ and $a \notin I$ for $a \in T$ there exist an ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$.*

Proof. Suppose that $(\mathcal{P}, \tau)$ is a regular space. Let $I \in \mathcal{P}$ and $a \notin I$ for $a \in T$. As $a \notin I$, we have $I \in Y(a)$ and $Y(a)$ is an open set of $\mathcal{P}$ implies $X(a) = \mathcal{P} \setminus Y(a)$ is a closed set of $\mathcal{P}$ not containing I. As $(\mathcal{P}, \tau)$ is a regular space, there exist two disjoints open sets say G and H such that $I \in G, \mathcal{P} \setminus Y(a) \subseteq H$ and $G \cap H = \emptyset$. $\mathcal{P} \setminus Y(a) \subseteq H$ implies that $\mathcal{P} \setminus H \subseteq Y(a)$. Since H is an open set of $\mathcal{P}$ implies $\mathcal{P} \setminus H$ is a closed set and hence there exist a ideal J of T such that, $\mathcal{P} \setminus H = X(J)$, by Theorem 3.3. So we find that $X(J) \subseteq Y(a)$. Again $G \cap H = \emptyset$, we have $H \subseteq \mathcal{P} \setminus G$. Since G is open set, $\mathcal{P} \setminus G$ is closed and hence there exists a ideal K of T such that $\mathcal{P} \setminus G = X(K)$ i. e. $H \subseteq X(K)$. Since $I \in G, I \notin \mathcal{P} \setminus G = X(K)$. This implies that $K \nsubseteq I$. Thus there exists $b \in K (\subset T)$ such that $b \notin I$. So $I \in Y(b)$. Now we show that $H \subseteq X(b)$. Let $M \in H \subseteq X(K)$. Then $K \subseteq M$. Since $b \in K$, it gives that $b \in M$ and hence $M \in X(b)$. Therefore $H \subseteq X(b)$. This implies that $\mathcal{P} \setminus X(b) \subseteq \mathcal{P} \setminus H = X(J) \Rightarrow Y(b) \subseteq X(J)$. Thus we get for any $I \in \mathcal{P}$ there exist an ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$.

Conversely, suppose that for any $I \in \mathcal{P}$ and $a \notin I, a \in T$ there exist an ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$. To show the space $(\mathcal{P}, \tau)$ is a regular space. Let $I \in \mathcal{P}$ and $X(K)$ be any closed set not containing I. Since $I \notin X(K)$, we have $K \nsubseteq I$. This implies that there exists $a \in K$ such that $a \notin I$. Now by the given condition, there exists a ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$. Since $a \in K, Y(a) \cap X(K) = \emptyset$. This implies that $X(K) \subseteq \mathcal{P} \setminus Y(a) \subseteq \mathcal{P} \setminus X(J)$. Since $X(J)$ is a closed set, $\mathcal{P} \setminus X(J)$ is an open set containing the closed set $X(K)$. Therefore $Y(b) \cap (\mathcal{P} \setminus X(J)) = \emptyset$. So we find that $Y(b)$ and $\mathcal{P} \setminus X(J)$ are two disjoints open sets containing I and $X(K)$ respectively. Therefore $(\mathcal{P}, \tau)$ is a regular space. □

Corollary 3.4. *The structure space $(\mathcal{P}, \tau)$ is a T_3-space if and only if for any $I \in \mathcal{P}$ and $a \notin I$ for $a \in T$ there exist an ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$.*

Theorem 3.10. *The structure space $(\mathcal{P}, \tau)$ is a compact space if and only if for any collection $\{a_i\}_{i\in\Delta}$ (where Δ is indexing set) of T there exists a finite subcollection $\{a_1, a_2, \ldots, a_n\}$ in T such that $I \in \mathcal{P}$ there exist a_i such that $a_i \notin I$.*

Proof. Suppose that $(\mathcal{P}, \tau)$ is a compact space. Then the open cover $\{Y(a_i) : a_i \in T\}$ of $(\mathcal{P}, \tau)$ has a finite subcover $\{Y(a_i) : i = 1, 2, \ldots, n\}$. Let I be any element of $\mathcal{P}$. Then $I \in \{Y(a_i) : i = 1, 2, \ldots, n\}$. Therefore $I \in Y(a_i)$ for some $a_i \in T$. Hence $a_i \notin I$. Thus $\{a_1, a_2, \ldots, a_n\}$ is the required finite subcollection of elements of T such that for any $I \in \mathcal{P}$ there exist a_i such that $a_i \notin I$.

Conversely, suppose that the given condition holds. To show the space $(\mathcal{P}, \tau)$ is a compact space. Let $\{Y(a_i) : a_i \in T\}$ be an open cover of $(\mathcal{P}, \tau)$. Assume that no finite subcollection of $\{Y(a_i) : a_i \in T\}$ be forms a cover of $\mathcal{P}$. This means that for any finite set $\{a_1, a_2, \ldots, a_n\}$ of elements of T, $Y(a_1) \cup Y(a_2) \cup \ldots \cup Y(a_n) \neq \mathcal{P} \Rightarrow \mathcal{P} \setminus [Y(a_1) \cup Y(a_2) \cup \ldots \cup Y(a_n)] \neq \emptyset \Rightarrow X(a_1) \cap X(a_2) \cap \ldots \cap X(a_n) \neq \emptyset$. This implies there exist $I \in \mathcal{P}$ such that $I \in X(a_1) \cap X(a_2) \cap \ldots \cap X(a_n)$ gives that $a_1, a_2, \ldots, a_n \in I$. Which is a contradiction to our hypothesis. Hence our assumption $\{Y(a_i) : a_i \in T\}$ has no finite subcover which covers $\mathcal{P}$ is wrong. Therefore $\{Y(a_i) : a_i \in T\}$ has finite subcover which covers $\mathcal{P}$. Hence $(\mathcal{P}, \tau)$ is a compact space. □

Corollary 3.5. *If T is finitely generated, then the space $(\mathcal{P}, \tau)$ is compact.*

Proof. Let $\{a_1, a_2, \ldots, a_n\}$ be a finite set of generators of T. Then for any $I \in \mathcal{P}$ there exist a_i such that $a_i \notin I$. Hence by Theorem 3.10, $(\mathcal{P}, \tau)$ is a compact space. □

The arbitrary intersection of all prime ideals of T is a semiprime ideal of T, provided it is non-empty. We give a necessary condition for the intersection of prime ideals of T to be a prime ideal in the following theorem,

Theorem 3.11. *Let $\{I_i\}_{i\in\Delta}$ (where Δ is any indexing set) be a family of all prime ideals of T such that $\{I_i\}_{i\in\Delta}$ forms a chain of ideals then $\bigcap_{i\in\Delta} I_i$ is a prime ideal of T.*

Proof. Let $\{I_i\}_{i\in\Delta}$ (where Δ is any indexing set) be a family of all prime ideals of T. It is clear that $\bigcap_{i\in\Delta} I_i$ is an ideal of T. Let I_1, I_2 and I_3 be any three ideals of T such that $I_1I_2I_3 \subseteq \bigcap_{i\in\Delta} I_i$. If either $I_1 \subseteq I_i\ \forall i \in \Delta$ or $I_2 \subseteq I_i\ \forall i \in \Delta$ or $I_3 \subseteq I_i\ \forall i \in \Delta$ then either $I_1 \subseteq \bigcap_{i\in\Delta} I_i$ or $I_2 \subseteq \bigcap_{i\in\Delta} I_i$ or $I_3 \subseteq \bigcap_{i\in\Delta} I_i$. If possible, let $I_1, I_2, I_3 \nsubseteq \bigcap_{i\in\Delta} I_i$ then there exist i, j and k such that $I_1 \nsubseteq \bigcap_{i\in\Delta} I_i, I_2 \nsubseteq \bigcap_{j\in\Delta} I_j$ and $I_3 \nsubseteq \bigcap_{k\in\Delta} I_k$. Since $\{I_i\}_{i\in\Delta}$ form a chain of ideals, let $I_i \subseteq I_j \subseteq I_k$. This implies that $I_2, I_3 \nsubseteq I_i$. Since $I_1I_2I_3 \subseteq I_i$ and I_i is prime ideal of T, we must have either $I_1 \subseteq I_i$ or $I_2 \subseteq I_i$ or $I_3 \subseteq I_i$. Which is a contradiction. Therefore, either $I_1 \subseteq \bigcap_{i\in\Delta} I_i$ or $I_2 \subseteq \bigcap_{i\in\Delta} I_i$ or $I_3 \subseteq \bigcap_{i\in\Delta} I_i$. Hence $\bigcap_{i\in\Delta} I_i$ is a prime ideal of T. □

Definition 3.14. The structure space $(\mathcal{P}, \tau)$ of T is called irreducible if for any decomposition $\mathcal{P} = \mathcal{U} \cup \mathcal{V} \cup \mathcal{W}$, where $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ are closed subsets of $\mathcal{P}$ then either $\mathcal{P} = \mathcal{U}$ or $\mathcal{P} = \mathcal{V}$ or $\mathcal{P} = \mathcal{W}$.

Theorem 3.12. *Let $\mathcal{U}$ be a closed subset of $\mathcal{P}$. Then $\mathcal{U}$ is irreducible if and only if $\bigcap_{I_i\in\mathcal{U}} I_i$ is a prime ideal of T.*

Proof. Assume that $\mathcal{U}$ is irreducible. To prove that $\bigcap_{I_i \in \mathcal{U}} I_i$ is a prime ideal of T. Let A, B and C be any three ideals of T such that $ABC \subseteq \bigcap_{I_i \in \mathcal{U}} I_i$. Then $ABC \subseteq I_i,\ \forall\, i$. As I_i is a prime ideal of T, we have $A \subseteq I_i$ or $B \subseteq I_i$ or $C \subseteq I_i,\ \forall\, i$. Then $I_i \in \mathcal{U} \cap \overline{A}$ or $I_i \in \mathcal{U} \cap \overline{B}$ or $I_i \in \mathcal{U} \cap \overline{C}$ give $I_i \in (\mathcal{U} \cap \overline{A}) \cup (\mathcal{U} \cap \overline{B}) \cup (\mathcal{U} \cap \overline{C})$. Therefore $\mathcal{U} = (\mathcal{U} \cap \overline{A}) \cup (\mathcal{U} \cap \overline{B}) \cup (\mathcal{U} \cap \overline{C}) = \big[(\mathcal{U} \cap \overline{A}) \cup (\mathcal{U} \cap \overline{B})\big] \cup (\mathcal{U} \cap \overline{C})$. But $(\mathcal{U} \cap \overline{A}), (\mathcal{U} \cap \overline{B})$ and $(\mathcal{U} \cap \overline{C})$ are closed subsets of $\mathcal{U}$ and $\mathcal{U}$ is irreducible imply, $\mathcal{U} = (\mathcal{U} \cap \overline{A}) \cup (\mathcal{U} \cap \overline{B})$ or $\mathcal{U} = (\mathcal{U} \cap \overline{C}) \Rightarrow \mathcal{U} = (\mathcal{U} \cap \overline{A})$ or $\mathcal{U} = (\mathcal{U} \cap \overline{B})$ or $\mathcal{U} = (\mathcal{U} \cap \overline{C})$. Hence $\mathcal{U} \subseteq \overline{A}$ or $\mathcal{U} \subseteq \overline{B}$ or $\mathcal{U} \subseteq \overline{C}$. This shows that, $A \subseteq \bigcap_{I_i \in \mathcal{U}} I_i$ or $B \subseteq \bigcap_{I_i \in \mathcal{U}} I_i$ or $C \subseteq \bigcap_{I_i \in \mathcal{U}} I_i$. Therefore $\bigcap_{I_i \in \mathcal{U}} I_i$ is a prime ideal of T.

Conversely, suppose that $\bigcap_{I_i \in \mathcal{U}} I_i$ is a prime ideal of T. To show that $\mathcal{U}$ is irreducible. Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are closed subsets of $\mathcal{U}$ such that $\mathcal{U} = \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$. Then $\bigcap_{I_i \in \mathcal{U}} I_i \subseteq \bigcap_{I_i \in \mathcal{X}} I_i$, $\bigcap_{I_i \in \mathcal{U}} I_i \subseteq \bigcap_{I_i \in \mathcal{Y}} I_i$ and $\bigcap_{I_i \in \mathcal{U}} I_i \subseteq \bigcap_{I_i \in \mathcal{Z}} I_i$. We have,

$$\bigcap_{I_i \in \mathcal{U}} I_i = \bigcap_{I_i \in \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}} I_i = \Big(\bigcap_{I_i \in \mathcal{X}} I_i\Big) \cap \Big(\bigcap_{I_i \in \mathcal{Y}} I_i\Big) \cap \Big(\bigcap_{I_i \in \mathcal{Z}} I_i\Big)$$

Now,

$$\Big(\bigcap_{I_i \in \mathcal{X}} I_i\Big)\Big(\bigcap_{I_i \in \mathcal{Y}} I_i\Big)\Big(\bigcap_{I_i \in \mathcal{Z}} I_i\Big) \subseteq \Big(\bigcap_{I_i \in \mathcal{X}} I_i\Big) \cap \Big(\bigcap_{I_i \in \mathcal{Y}} I_i\Big) \cap \Big(\bigcap_{I_i \in \mathcal{Z}} I_i\Big) = \bigcap_{I_i \in \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}} I_i = \bigcap_{I_i \in \mathcal{U}} I_i.$$

Since, $\bigcap_{I_i \in \mathcal{U}} I_i$ is prime ideal of T, then we have $\bigcap_{I_i \in \mathcal{X}} I_i \subseteq \bigcap_{I_i \in \mathcal{U}} I_i$ or $\bigcap_{I_i \in \mathcal{Y}} I_i \subseteq \bigcap_{I_i \in \mathcal{U}} I_i$ or $\bigcap_{I_i \in \mathcal{Z}} I_i \subseteq \bigcap_{I_i \in \mathcal{U}} I_i$. Therefore $\bigcap_{I_i \in \mathcal{U}} I_i = \bigcap_{I_i \in \mathcal{X}} I_i$ or $\bigcap_{I_i \in \mathcal{U}} I_i = \bigcap_{I_i \in \mathcal{Y}} I_i$ or $\bigcap_{I_i \in \mathcal{U}} I_i = \bigcap_{I_i \in \mathcal{Z}} I_i$. Now for any $I_k \in \mathcal{U}$. Then we have $\bigcap_{I_i \in \mathcal{U}} I_i = \bigcap_{I_i \in \mathcal{X}} I_i \subseteq I_k$ or $\bigcap_{I_i \in \mathcal{U}} I_i = \bigcap_{I_i \in \mathcal{Y}} I_i \subseteq I_k$ or $\bigcap_{I_i \in \mathcal{U}} I_i = \bigcap_{I_i \in \mathcal{Z}} I_i \subseteq I_k$. Since $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are closed subsets of $\mathcal{U}$, so either $I_i \subseteq I_k$ for all $I_i \in \mathcal{X}$ or $I_i \subseteq I_k$ for all $I_i \in \mathcal{Y}$ or $I_i \subseteq I_k$ for all $I_i \in \mathcal{Z}$. Thus $I_k \in \overline{\mathcal{X}} = \mathcal{X}$ or $I_k \in \overline{\mathcal{Y}} = \mathcal{Y}$ or $I_k \in \overline{\mathcal{Z}} = \mathcal{Z}$, since $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are closed subsets of $\mathcal{U}$. Therefore $\mathcal{U} \subseteq \mathcal{X}$ or $\mathcal{U} \subseteq \mathcal{Y}$ or $\mathcal{U} \subseteq \mathcal{Z}$. Hence $\mathcal{U} = \mathcal{X}$ or $\mathcal{U} = \mathcal{Y}$ or $\mathcal{U} = \mathcal{Z}$. Consequently, $\mathcal{U}$ is irreducible. □

For any subset $\mathcal{U}$ of $\mathcal{P}$, we define $R(\mathcal{U}) = \bigcap_{I_j \in \mathcal{U}} I_j$. Clearly $R(\mathcal{P}) = \bigcap_{I_j \in \mathcal{P}} I_j$ is $\mathcal{P}$-radical of T. Always $R(\mathcal{P}) \subseteq R(\mathcal{U})$. We know that $\mathcal{U} \subseteq \mathcal{P}$ is dense in $\mathcal{P}$ if $\overline{\mathcal{U}} = \mathcal{P}$.

Theorem 3.13. *The subset $\mathcal{U}$ of $\mathcal{P}$ is dense in $\mathcal{P}$ if and only if $R(\mathcal{U}) = R(\mathcal{P})$.*

Proof. Suppose that the subset $\mathcal{U}$ of $\mathcal{P}$ is dense in $\mathcal{P}$. Since $\mathcal{U} \subseteq \mathcal{P}$, we have $R(\mathcal{P}) \subseteq R(\mathcal{U})$. To show that $R(\mathcal{U}) \subseteq R(\mathcal{P})$. As $\overline{\mathcal{U}} = \mathcal{P}$ gives $\overline{\mathcal{U}} = \{I \in \mathcal{P} : \bigcap_{I_\alpha \in \mathcal{U}} I_\alpha \subseteq I\} = \mathcal{P}$. $A \in \mathcal{P}$

implies $A \in \overline{\mathcal{U}}$. Then $R(\mathcal{U}) \subseteq A$. As this is true for each $A \in \mathcal{P}$, we get $R(\mathcal{U}) = \bigcap_{I_\alpha \in \mathcal{U}} I_\alpha \subseteq \bigcap_{I_\alpha \in \mathcal{P}} I_\alpha = R(\mathcal{P})$. Hence $R(\mathcal{U}) = R(\mathcal{P})$.

Conversely, suppose that $R(\mathcal{U}) = R(\mathcal{P})$. To show that $\overline{\mathcal{U}} = \mathcal{P}$. Assume that $\mathcal{P} \setminus \overline{\mathcal{U}} \neq \emptyset$. Then there is a prime ideal say A of T such that $A \in \mathcal{P} \setminus \overline{\mathcal{U}}$ that is $A \in \mathcal{P}$ and $A \notin \overline{\mathcal{U}}$. $A \notin \overline{\mathcal{U}}$ implies there exists any open set say $Y(I)$ containing A such that $Y(I) \cap \big(\overline{\mathcal{U}} \setminus \{A\}\big) = \emptyset$. That is open set of $\mathcal{P}$ containing A does not contains any other element of $\mathcal{U}$ other than A. Therefore $R(\mathcal{P}) = \bigcap_{I_\alpha \in \mathcal{P}} I_\alpha \subset R(\mathcal{U}) = \bigcap_{I_\alpha \in \mathcal{U}} I_\alpha$. Then $R(\mathcal{U}) \neq R(\mathcal{P})$, which contradicts our hypothesis. Thus $\mathcal{P} \setminus \overline{\mathcal{U}} = \emptyset$. Hence $\overline{\mathcal{U}} = \mathcal{P}$ i.e. $\mathcal{U}$ is dense in $\mathcal{P}$. □

Definition 3.15. A partially ordered ternary semigroup T is called a Noetherian partially ordered ternary semigroup if it satisfies the ascending chain condition for ideals of T, for any sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ of ideals of T, then there exists a positive integer m such that $I_m = I_{m+1} = \ldots$

Theorem 3.14. [6] *A topological space is compact if and only if each family of closed sets which has the finite intersection property has a non-void intersection.*

Theorem 3.15. *If T is a Noetherian partially ordered ternary semigroup then the structure space $(\mathcal{P}, \tau)$ is countably compact.*

Proof. Let $\{X(I_n)\}_{n=1}^{\infty}$ be a countable collection of closed sets in $\mathcal{P}$ with finite intersection property. Let us consider the following ascending chain of prime ideals of T,

$$\langle I_1 \rangle \subseteq \langle I_1 \cup I_2 \rangle \subseteq \langle I_1 \cup I_2 \cup I_3 \rangle \subseteq \ldots$$

Since T is a Noetherian partially ordered ternary semigroup there exist a positive integer m such that,

$$\langle I_1 \cup I_2 \cup \ldots \cup I_m \rangle = \langle I_1 \cup I_2 \cup \ldots \cup I_{m+1} \rangle = \ldots$$

Thus it follows that $\langle I_1 \cup I_2 \cup \ldots \cup I_m \rangle \in \bigcap_{n=1}^{\infty} X(I_n)$. Hence $\bigcap_{n=1}^{\infty} X(I_n) \neq \emptyset$ and hence $(\mathcal{P}, \tau)$ is countably compact. □

Corollary 3.6. *If T is a Noetherian partially ordered ternary semigroup and $(\mathcal{P}, \tau)$ is second countable then $(\mathcal{P}, \tau)$ is compact.*

Proof. Proof follows from Theorem 3.15 and the fact that a second countable space is compact if it is countably compact. □

The set of all idempotent elements of T is denoted by $E(T)$, i.e. $E(T) = \{a \in T : aaa = a\}$.

Definition 3.16. An ideal I of T is said to be full ideal if $E(T) \subseteq I$.

Definition 3.17. An ideal I of T is said to be a prime full ideal if it is both prime and full ideal.

Let $\mathcal{F}$ be the family of all prime full ideals of T. Then we see that $\mathcal{F}$ is a subset of $\mathcal{P}$ and $(\mathcal{F}, \tau_{\mathcal{F}})$ is a topological space where $\tau_{\mathcal{F}}$ is the subspace topology.

Theorem 3.16. *The space $(\mathcal{F}, \tau_{\mathcal{F}})$ is a compact space if $E(T) \neq \{0\}$.*

Proof. Let $\{X(I_i)\}_{i\in\Delta}$ (where Δ is any indexing set) be any collection of closed sets in $\mathcal{F}$ with finite intersection property. Let I be the prime full ideal generated by $E(T)$. Since any prime full ideal J of T contains $E(T)$, then J contains I. Hence $I \in \bigcap_{i\in\Delta} X(I_i) \neq \emptyset$. Consequently, the space $(\mathcal{F}, \tau_{\mathcal{F}})$ is a compact space. □

Theorem 3.17. *The space $(\mathcal{F}, \tau_{\mathcal{F}})$ is a connected space if $E(T) \neq \{0\}$.*

Proof. Let I be the prime ideal generated by $E(T)$. Since any prime full ideal J contains $E(T)$, J contains I. Hence I belongs to any closed set $X(K)$ of $\mathcal{F}$. Consequently, any two closed sets of $\mathcal{F}$ are not disjoint. Hence $(\mathcal{F}, \tau_{\mathcal{F}})$ is a connected space. □

4. Conclusion

This paper is a continuation of the study of ideals in a partially ordered ternary semigroup. We have mainly focused here on the space of prime ideals of partially ordered ternary semigroups. In this article, we consider the set $\mathcal{P}$ of all prime ideals in a partially ordered ternary semigroup T endowed with the topology τ. We investigated various topological properties of space $(\mathcal{P}, \tau)$. This topological space $(\mathcal{P}, \tau)$ is referred to as the structure space of the partially ordered ternary semigroup T. Furthermore, we have introduced the concept of prime full ideals in partially ordered ternary semigroups and proved that the space of prime full ideals of a partially ordered ternary semigroup is compact. The study of bi-ideals in a ternary semigroup will be considered in future work.

Acknowledgments. The authors express gratitude to the anonymous reviewers for their invaluable feedback and suggestions, which have significantly enhanced the paper's overall quality and clarity of presentation.

References

[1] Daddi, V. R.; Pawar, Y. S. On ordered ternary semigroups. *Kyungpook Math. J.* **52** (2012), no. 4, 375–381.
[2] Iampan, A. Characterizing the minimality and maximality of ordered lateral ideals in ordered ternary semigroups. *J. Korean Math. Soc.* **46** (2009), 775–784.
[3] Jagtap, R. D.; Pawar, Y. S. Structure space of prime ideals of Γ-semirings. *Palest. J. Math.* **5** (2016), no. 1, 166–170.
[4] Kar, S. On structure space of ternary semirings. *Southeast Asian Bull. Math.*, **31** (2007), no. 3, 537–545.
[5] Kar S.; Chattopadhyay, S. On structure space of Γ-semigroups. *Acta Univ. Palack. Olomuc., Fac. Rer. Nat., Mathematica* **47** (2008), no. 1, 37–46.
[6] Kelly, J. L. *General Topology*. New York, Van Nostrand, (1969).
[7] Kostaq, H.; Vardhami, I.; Gjino, K. On the topological structures on Γ-semirings. *European Scientific Journal* **9** (2013), no. 21, 17–30.
[8] Lehmer, D. H. A ternary analogue of abelian groups. *Amer. J. Math.* **54** (1932), no. 2, 329–338.
[9] Reddy, V. S. R.; Rao, V. S.; Anjaneyulu, A.; Rao, A. G. Ideals in partially ordered ternary semigroups. Intern. *J. Math. Sci. Technology and Humanities* **110** (2014), 1177–1194.
[10] Reddy, V. S. R.; Rao, V. S.; Anjaneyulu, A.; Rao, A. G. Complete prime po ideals and prime po ideals in po ternary semigroups. *Intern. Research J. Pure Algebra* **4** (2014), 457–470.
[11] Shabir, M.; Bashir, S. Prime ideals in ternary semigroups. *Asian-Eur. J. Math.* **2** (2009), no. 1, 141–154.
[12] Shinde, D. N.; Gophane, M. T. On irreducible pseudo symmetric ideals of a partially ordered ternary semigroup. *Quasigroups Relat. Syst.* **30** (2022), no. 1, 169–180.

Department of Mathematics
Shivaji University
Kolhapur-416004, Maharashtra, India
Email address: shindednmaths@gmail.com
Email address: mtg_maths@unishivaji.ac.in