CREAT. MATH. INFORM.
Volume **33** (2024), No. 1,
Pages 87 - 103

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2024.01.09

Leverage Centrality on Barycentric Subdivision of Some Graphs

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ABSTRACT. The centrality index on a graph is a real-valued function on the nodes, which provides a ranking of vital nodes in the graph. Each node could be important from an angle, depending on how the concept of importance is defined. There are various measures of centrality, and each one defines a node's importance from a different perspective and provides relevant analytical information about the graph. Leverage centrality of nodes in a graph was defined by Joyce et al. in 2010 as a means to analyze connections within the brain. The definition of this measure shows that it is unique among existing measures in that it counts not just a node's degree, but also its neighbor's degrees. In this paper, we study the leverage centrality of nodes in the k^{th} barycentric subdivision of some classes of graphs. This is a new concept in literature. The process can make regular graphs irregular, and the leverage center of the edge-subdivided graphs under study was investigated.

1. INTRODUCTION

A graph that can be obtained from a given graph by breaking up each edge into one or more segments by inserting intermediate vertices between its two ends is called a subdivision graph. The barycentric subdivision subdivides each edge of the graph. This is a special subdivision, as it always results in a bipartite graph. The leverage centrality measure identifies highly influential nodes within a network by identifying their leverage centrality. Node's leverage centrality is determined by how well connected they are in comparison to their neighbor's networks.

The most popular degree centrality defines central nodes to be those having the highest number of connections, or degrees. While node degree often proves to identify critical network elements, a highly essential node in the neural network may not necessarily have ubiquitous connections to other nodes in the network as assumed by degree centrality. An increasingly popular centrality metric, eigenvector centrality, is unique in that it considers the centrality of immediate neighbors when computing the centrality of a node. However, eigenvector centrality does not account for the disparity in the degree of a node concerning its neighbors, which has different implications depending on the network's assortativity, or the tendency for nodes to be connected to similar degree nodes. Furthermore, it is computationally intensive as compared to other centrality metrics. Betweenness centrality considers nodes along the shortest geodesic paths to be the most central in the network. This centrality assumes that information travels through a network along the shortest path in a serial fashion. Despite the potential utility of this measure of centrality, it is not ideal for a system that processes information via unrestricted walk such as the brain, where information typically does not follow shortest paths as they are not predetermined. Leverage centrality captures nodes in the network which are connected to more nodes than their neighbors and, therefore, control the content and quality of the

Received: 30.10.2022. In revised form: 06.10.2023. Accepted: 13.10.2023
2020 *Mathematics Subject Classification.* 05C90, 94C15 .
Key words and phrases. Leverage center, Neural network, Barycentric subdivision, Regular graph.
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information received by their neighbors. Leverage is designed to capture the local assortative or disassortative behavior of the network, as node degree is evaluated with respect to degrees of immediate neighbors [6].

This study is motivated by its application in diagnosing brain tumors. Brain tumors can damage brain function if they grow large enough to press on surrounding tissues. The non-neuronal brain cells grow and divide faster than normal, taking over space in the brain. Since these abnormal cells form a brain hub, it is easily detected by the leverage centrality using fMRI. A hub is the best connected node and therefore is likely to have high leverage centrality since its degree is high concerning other nodes in the neighborhood. Also, we can identify an edge in a graph with a cell in an organism, and its k^{th} barycentric subdivision as the k^{th} division of the parent cell. This is quite what we need for the reproduction and the growth of the organism. Here we derive some mathematical perspectives of that division. The leverage analysis in a tissue can be used to check whether the functioning or growth takes place in a normal manner or not. In [11], we defined the leverage center of a graph and have determined the same for some special classes of graphs. As a continuous study, in this paper, we investigate the leverage center of some edge subdivided graphs.

Definition 1.1. [10] Let $G = (V, E)$ be a graph. Let $e = uv$ be an edge of G and w is not a vertex of G. The edge e is subdivided when it is replaced by edges $e_1 = uw$ and $e_2 = wv$.

Definition 1.2. [10] If every edge of a graph G is subdivided, then the resulting graph is called the barycentric subdivision of the graph G. In other words, a barycentric subdivision graph is a graph obtained by inserting a vertex of degree two into every edge of the original graph. This graph is also known as a subdivision graph.

Definition 1.3. The k^{th} barycentric subdivision is the barycentric subdivision of the $(k-1)^{th}$ barycentric subdivision of the graph.

Definition 1.4. The degree of a vertex v is the number of edges incident to v and is denoted by $deg(v)$.

Now the leverage centrality of a node v is defined as follows:

Definition 1.5. [14] Leverage centrality is a measure of the relationship between the degree of a given node v and the degree of each of its neighbors v_i averaged over all neighbors N_v and is defined as:

$$l(v) = \frac{1}{deg(v)} \sum_{v_i \in N_v} \frac{deg(v) - deg(v_i)}{deg(v) + deg(v_i)}$$

Definition 1.6. [11] The leverage center of a graph is defined as the set of nodes having the highest leverage centrality in the graph.

Definition 1.7. [11] Unicentric leverage graphs are those with unique leverage centers.

Definition 1.8. [11] Bicentric leverage graphs are those with exactly two leverage centers.

1.1. **Some Basic Propositions on Leverage Centrality.**

Proposition 1.1. [8] *Let G be a graph with n vertices. For any vertex v, $|l(v)| \leq 1 - \frac{2}{n}$. Furthermore, these bounds are tight in the cases of stars and complete graphs.*

Proposition 1.2. [4] *For any graph G, $\sum_{v \in G} l(v) \leq 0$.*

Proposition 1.3. [4] *In a graph G, a vertex of lowest degree (highest degree) cannot have a positive (negative) leverage centrality. It is possible to have all the vertices in a graph except for one to have negative leverage centrality, similarly, all but one have positive leverage centrality.*

Theorem 1.1. [4] *In a graph G of order n, the maximum number of vertices with positive leverage centrality is* $n-1$.

The leverage centrality $l(v) = 0$ for every vertex $v \in G$ if and only if G is a regular graph[4].

2. MAIN RESULTS

In this section, we outline the leverage centrality analysis of nodes in some special classes of graphs and their leverage types. Firstly we illustrate the barycentric subdivision of a star graph.

2.1. Star Graph.

The star graph $K_{1,n-1}$ has $(n-1)$ vertices with negative leverage centrality. The leverage centrality of the central vertex can be calculated as:

$$\frac{1}{n-1}\left((n-1)\frac{(n-1)-1}{(n-1)+1}\right) = 1 - \frac{2}{n}$$

Also, the leverage centrality of all the pendant vertices is

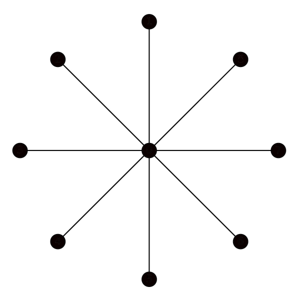

FIGURE 1. Star Graph $K_{1,8}$

$$\frac{1-(n-1)}{1+(n-1)} = -1 + \frac{2}{n}$$

Therefore in the case of star graphs, the leverage centrality meets the two extremes and only the central vertex has a positive leverage centrality [14]. Hence the central node is the leverage center for the star graph.
Now we discuss in detail the k^{th} barycentric subdivision of a star graph which we denote by $K_{1,n-1,k}$ and it divides each edge k times. Firstly we analyse the case $k = 1$. Here the central node is of degree $(n-1)$, all the inner nodes are of degree 2, and the outer nodes are of degree 1.

There are three cases for the leverage centrality analysis of $G = K_{1,n-1,1}$ depending on the types of nodes of G.

(1) Case 1: $v \in G$ is the central node
The central node has a degree of $n-1$. Also,

$$N_v = \{v_i, deg(v_i) = 2, 1 \leq i \leq n-1\}$$

where N_v represents the neighborhood of v. Then

$$\begin{aligned} l(v) &= \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{(n-1)-deg(v_i)}{(n-1)+deg(v_i)} \\ &= \frac{1}{n-1}(n-1)\frac{(n-1)-2}{(n-1)+2} \\ &= \frac{n-3}{n+1} \end{aligned}$$

(2) Case 2: $v \in G$ is an outer node
Here N_v consists of a single vertex w with $deg(w) = 2$. Hence

$$\begin{aligned} l(v) &= \frac{1-deg(w)}{1+deg(w)} \\ &= \frac{-1}{3} \end{aligned}$$

(3) Case 3: $v \in G$ adjacent to both the central and outer vertices

$$N_v = \{c, w \text{ where } deg(c) = n-1, deg(w) = 1\}$$

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{3-n}{1+n} + \frac{1}{3}\right) \\ &= \frac{-(n-5)}{3(n+1)} \end{aligned}$$

Hence in this case there are only three distinct leverage centralities.
Now we analyze the case $k = 2$.

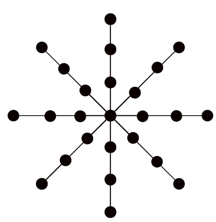

FIGURE 2. Barycentric subdivision $K_{1,8,2}$

Here instead of case 3 above, there are two special cases as explained below. Cases 1 and 2 are similar.

(1) Case 3: $v \in G$ is adjacent to the outer node
$N_v = \{u, w\}$, where $deg(u) = 1$ and $deg(w) = 2$.

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-deg(u)}{2+deg(u)} + \frac{2-deg(w)}{2+deg(w)}\right) \\ &= \frac{1}{6} \end{aligned}$$

(2) Case 4: $v \in G$ is adjacent to the central node
$N_v = \{c, w\}$, where $deg(c) = n - 1$ and $deg(w) = 2$.

$$l(v) = \frac{1}{2}\left(\frac{2 - deg(c)}{2 + deg(c)} + \frac{2 - deg(w)}{2 + deg(w)}\right)$$
$$= \frac{-(n-3)}{2(n+1)}$$

Hence there are four distinct leverage centralities in this case. In both cases when $n \geq 4$, G becomes unicentric with the central node as the leverage center.

Finally, for the generalization of the above results in the k^{th} barycentric subdivision for $k \geq 3$, we classify the newly formed vertices as the following three types:

- Type I: v is adjacent to the outer node.
- Type II: v is adjacent to the nodes of degree 2.
- Type III: v is adjacent to the central node.

Now we state our first theorem as follows.

Theorem 2.2. *Let $G = K_{1,n-1,k}$ where $n \geq 2, k \geq 3$ be the k^{th} barycentric subdivision of the star graph $K_{1,n-1}$ of order n. Then for $v \in G$,*

$$l(v) = \begin{cases} \frac{n-3}{n+1}, & \text{if } v \text{ is the central node} \\ \frac{-1}{3}, & \text{if } v \text{ is an outer node.} \\ \frac{1}{6}, & \text{if } v \text{ is a node of Type I} \\ 0, & \text{if } v \text{ is a node of Type II.} \\ \frac{-(n-3)}{2(n+1)}, & \text{if } v \text{ is a node of Type III.} \end{cases}$$

Proof. Let $G = K_{1,n-1,k}$ where $n \geq 2, k \geq 3$ be the k^{th} barycentric subdivision of the star graph $K_{1,n-1}$ of order n. There are five cases for the leverage centrality analysis of $G = K_{1,n-1,k}$ depending on the types of nodes of G. Here we need to consider only the case of $v \in G$ as a node of Type II. The remaining cases are discussed above.

If $v \in G$ is a node of Type II, then $N_v = \{x, w\}$, where $deg(x) = 2$ and $deg(w) = 2$.

$$l(v) = \frac{1}{2}\left(\frac{2 - deg(x)}{2 + deg(x)} + \frac{2 - deg(w)}{2 + deg(w)}\right)$$
$$= 0$$

□

Corollary 2.1. *Let $G = K_{1,n-1,k}$ where $n \geq 2, k \geq 3$ be the k^{th} barycentric subdivision of the star graph $K_{1,n-1}$ of order n. Then*

- *For $n = 2, k \geq 3$, G will be a bicentric leverage graph with Type I and Type III nodes as leverage centers.*
- *For $n = 3, k \geq 3$, G will be a unicentric leverage graph with Type I nodes as the leverage center.*
- *For $n \geq 4, k \geq 3$, G will be a unicentric leverage graph with the central node as leverage center.*

In [12] we can see the analogous result for the most classical betweenness centrality.

2.2. **Wheel Graph.** Let W_n be the wheel graph of degree $n+1$. We use $W_{n,k}$ to denote the k^{th} barycentric subdivision of the wheel W_n. Degree three vertices of the outer cycle are the outer nodes. The newly formed vertices are classified as:

- Type I: v is adjacent to the outer node.
- Type II: v is adjacent to the nodes of degree 2.
- Type III: v is adjacent to the central node.

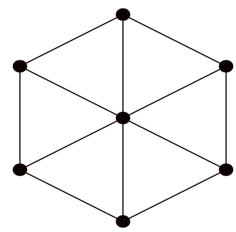

FIGURE 3. Wheel W_6

Now we have the following theorem.

Theorem 2.3. *Let $G = W_{n,k}$ where $n \geq 3, k \geq 3$ be the k^{th} barycentric subdivision of the wheel W_n of order n+1. Then for $v \in G$,*

$$l(v) = \begin{cases} \dfrac{n-2}{n+2} & \text{if } v \text{ is the central node} \\ \dfrac{1}{5} & \text{if } v \text{ is an outer node.} \\ \dfrac{-1}{10} & \text{if } v \text{ is a node of Type I} \\ 0 & \text{if } v \text{ is a node of Type II} \\ \dfrac{-(n-2)}{2(n+2)} & \text{if } v \text{ is a node of Type III} \end{cases}$$

Proof. There are five cases for the leverage centrality analysis of $G = W_{n,k}$ depending on the types of nodes of G.

(1) Case 1: $v \in G$ is the central node
The central node has a degree of n. Also,

$$N_v = \{v_i, 1 \leq i \leq n, deg(v_i) = 2\}$$

$$\begin{aligned} l(v) &= \frac{1}{n}\sum_{i=1}^{n} \frac{n - deg(v_i)}{n + deg(v_i)} \\ &= \frac{n-2}{n+2} \end{aligned}$$

(2) Case 2: $v \in G$ is an outer node
The outer node has degree 3, and its neighbors v_i are of degree 2. Hence

$$\begin{aligned} l(v) &= \frac{1}{3}\sum_{i=1}^{3} \left(\frac{3 - deg(v_i)}{3 + deg(v_i)}\right) \\ &= \frac{1}{5} \end{aligned}$$

(3) Case 3: $v \in G$ is a node of Type I
$N_v = \{u, w\}$, where $deg(u) = 3$ and $deg(w) = 2$.

$$l(v) = \frac{1}{2}\left(\frac{2-deg(u)}{2+deg(u)} + \frac{2-deg(w)}{2+deg(w)}\right) = \frac{-1}{10}$$

(4) Case 4: $v \in G$ is a node of Type II
$N_v = \{x, w\}$, where $deg(x) = 2$ and $deg(w) = 2$.

$$l(v) = \frac{1}{2}\left(\frac{2-deg(x)}{2+deg(x)} + \frac{2-deg(w)}{2+deg(w)}\right) = 0$$

(5) Case 5: $v \in G$ is a node of Type III
$N_v = \{c, w\}$, where $deg(c) = n$ and $deg(w) = 2$.

$$l(v) = \frac{1}{2}\left(\frac{2-deg(c)}{2+deg(c)} + \frac{2-deg(w)}{2+deg(w)}\right) = \frac{-(n-2)}{2(n+2)}$$

□

Remark 2.1. *When $n = 3, k \geq 3$, $W_{n,k}$ becomes bicentric leverage graph. But for $n \geq 4$, it is unicentric.*

Corollary 2.2. *Let $G = W_{n,k}$ where $n \geq 3, k \geq 3$ be the k^{th} barycentric subdivision of the wheel W_n of order n+1. Then for $k = 1$ and $v \in G$, we have the following:*

$$l(v) = \begin{cases} \frac{n-2}{n+2} & \text{if } v \text{ is the central node} \\ \frac{1}{5} & \text{if } v \text{ is an outer node.} \\ \frac{-1}{5} & \text{if } v \text{ is a node in between two outer nodes} \\ \frac{4-3n}{5(n+2)} & \text{if } v \text{ is a node in between the central node and an outer node.} \end{cases}$$

Proof. The first two cases are obvious. If v is in between two outer nodes of degree three, then

$$l(v) = \frac{2-3}{5} = \frac{-1}{5}$$

Now, if v is in between the central node and an outer node, then

$$l(v) = \frac{1}{2}\left(\frac{2-n}{2+n} + \frac{2-3}{5}\right) = \frac{4-3n}{5(n+2)}$$

□

Remark 2.2. *When $n = 3$ and $k = 1$, $W_{n,k}$ is bicentric, but for $n \geq 4$, it is unicentric.*

In [12] we can see the analogous result for the betweenness centrality.

2.3. **Sunflower network.** Sunflower network SF_n consists of a wheel with central node c and an n-cycle $\{v_0, v_1, \ldots, v_{n-1}\}$ and additional n nodes $\{u_0, u_1, \ldots, u_{n-1}\}$ where u_i is joined by links to v_i, v_{i+1} for $0 \leq i \leq n-1$, where $i+1$ is taken modulo n. v_i's are major nodes and u_i's are minor nodes. The central node c of SF_n has a node degree of n.

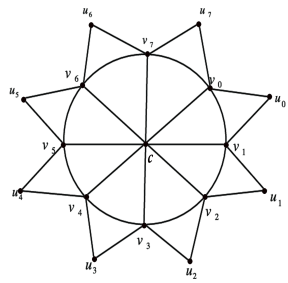

FIGURE 4. Sunflower network SF_8

In the k^{th} barycentric subdivision $SF_{n,k}$ of SF_n, the newly formed vertices can mainly be classified as follows:

- Type I: v is adjacent to the major node
- Type II: v is adjacent to the minor node
- Type III: v is adjacent to the central node

  Another type of node is those which are adjacent only to the nodes of degree two other than the minor nodes. Its leverage is zero.

Theorem 2.4. *Let $G = SF_{n,k}$, $k \geq 3$ be the k^{th} barycentric subdivision of SF_n. Then for $v \in G$,*

$$l(v) = \begin{cases} \dfrac{3}{7} & \text{if } v \text{ is a major node} \\ 0 & \text{if } v \text{ is a minor node.} \\ \dfrac{n-2}{n+2} & \text{if } v \text{ is the central node} \\ \dfrac{-3}{14} & \text{if } v \text{ is Type I} \\ 0 & \text{if } v \text{ is Type II} \\ \dfrac{-(n-2)}{2(n+2)} & \text{if } v \text{ is Type III} \end{cases}$$

Proof. For the major nodes v_i, $deg(v_i) = 5$ and all the neighbors are of degree two. Hence

$$\begin{aligned} l(v) &= \frac{1}{5}\left(\frac{5-2}{7}\right)5 \\ &= \frac{3}{7} \end{aligned}$$

For the minor nodes, $l(v) = 0$. For the central node c,

$$\begin{aligned} l(v) &= \frac{1}{n}\left(\frac{n-2}{n+2}\right)n \\ &= \frac{n-2}{n+2} \end{aligned}$$

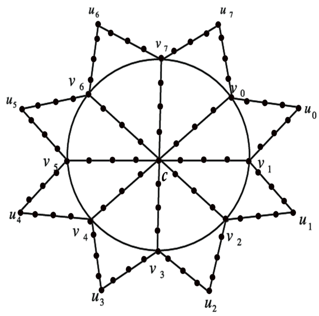

FIGURE 5. Barycentric subdivision $SF_{8,3}$

Now, for the type I vertices, since they are adjacent to the major nodes

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-5}{7}\right) \\ &= \frac{-3}{14} \end{aligned}$$

Similarly, for the Type II vertices, $l(v) = 0$. For the Type III vertices,

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-n}{2+n}\right) \\ &= \frac{-(n-2)}{2(n+2)} \end{aligned}$$

since they are adjacent to the central node. □

2.4. **Helm Network.** Helm H_n is a graph of order $2n+1$ obtained from a wheel W_n with cycle C_n having a pendant link attached to each node of the cycle. H_n consists of the node set $V(H_n) = \{v_i : 0 \leq i \leq n-1\} \cup \{u_i : 0 \leq i \leq n-1\} \cup \{c\}$ and link set $E(H_n) = \{v_i v_{i+1} : 0 \leq i \leq n-1\} \cup \{v_i u_i : 0 \leq i \leq n-1\} \cup \{v_i c : 0 \leq i \leq n-1\}$, where $i+1$ is taken modulo n. The central node c of H_n has a node degree of n. Here v_i's are major nodes and the pendant u_i's are minor nodes.

In the k^{th} barycentric subdivision $H_{n,k}$ of H_n, the newly formed vertices are classified as follows:

- Type I: v is adjacent to the major node
- Type II: v is adjacent to the minor node
- Type III: v is adjacent to the central node

Theorem 2.5. *Let $G = H_{n,k}$, $k \geq 3$ be the k^{th} barycentric subdivision of H_n. Then for $v \in G$,*

$$l(v) = \begin{cases} \frac{1}{3} & \text{if } v \text{ is a major node} \\ \frac{-1}{3} & \text{if } v \text{ is a minor node.} \\ \frac{n-2}{n+2} & \text{if } v \text{ is the central node} \\ \frac{-1}{6} & \text{if } v \text{ is Type I} \\ \frac{1}{6} & \text{if } v \text{ is Type II} \\ \frac{-(n-2)}{2(n+2)} & \text{if } v \text{ is Type III} \end{cases}$$

Proof. The major nodes v_i's are of degree 4, and all the neighbors are of degree two. Hence

$$\begin{aligned} l(v) &= \frac{1}{4}\left(\frac{4-2}{6}\right)4 \\ &= \frac{1}{3} \end{aligned}$$

For the minor nodes,

$$\begin{aligned} l(v) &= \frac{1-2}{1+2} \\ &= \frac{-1}{3} \end{aligned}$$

For the central node c,

$$\begin{aligned} l(v) &= \frac{1}{n}\left(\frac{n-2}{n+2}\right)n \\ &= \frac{n-2}{n+2} \end{aligned}$$

Now, for the type I vertices, since they are adjacent to the major nodes

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-4}{6}\right) \\ &= \frac{-1}{6} \end{aligned}$$

Similarly, for the Type II vertices,

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-1}{3}\right) \\ &= \frac{1}{6} \end{aligned}$$

For the Type III vertices,

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-n}{2+n}\right) \\ &= \frac{-(n-2)}{2(n+2)} \end{aligned}$$

since they are adjacent to the central node. □

2.5. **Fans.** If we join a node of C_n to all other nodes, then the resulting graph is called a fan and is denoted by F_n. Let $\{c, v_0, v_1, \ldots, v_{n-2}\}$ be the nodes of F_n, where v_0 and v_{n-2} are the nodes of degree two and let c be the node that is connected to all other nodes. Then c is the central node of F_n with degree $n-1$. The nodes of degree two are referred to as minor nodes and the nodes of degree three to as major nodes.

In the k^{th} barycentric subdivision $F_{n,k}$ of F_n, the newly formed vertices can mainly be classified as follows:

- Type I: v is adjacent to the major node
- Type II: v is adjacent to the minor node
- Type III: v is adjacent to the central node

Theorem 2.6. *Let $G = F_{n,k}$, $k \geq 3$ be the k^{th} barycentric subdivision of F_n. Then for $v \in G$,*

$$l(v) = \begin{cases} \frac{1}{5} & \text{if } v \text{ is a major node} \\ 0 & \text{if } v \text{ is a minor node.} \\ \frac{n-3}{n+1} & \text{if } v \text{ is the central node} \\ \frac{-1}{10} & \text{if } v \text{ is Type I} \\ 0 & \text{if } v \text{ is Type II} \\ \frac{-(n-3)}{2(n+1)} & \text{if } v \text{ is Type III} \end{cases}$$

Proof. The major nodes v_i's are of degree 3, and all the neighbors are of degree two. Hence

$$\begin{aligned} l(v) &= \frac{1}{3}\left(\frac{3-2}{5}\right)3 \\ &= \frac{1}{5} \end{aligned}$$

For the minor nodes, $l(v) = 0$. For the central node c,

$$\begin{aligned} l(v) &= \frac{1}{n-1}\left(\frac{(n-1)-2}{n-1+2}\right)(n-1) \\ &= \frac{n-3}{n+1} \end{aligned}$$

Now, for the type I vertices, since they are adjacent to the major nodes

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-3}{5}\right) \\ &= \frac{-1}{10} \end{aligned}$$

Similarly, for the Type II vertices, $l(v) = 0$

For the Type III vertices,

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-(n-1)}{2+(n-1)}\right) \\ &= \frac{-(n-3)}{2(n+1)} \end{aligned}$$

since they are adjacent to the central node. □

2.6. **Friendship Graph.** The k^{th} barycentric subdivision of the friendship graph f_n is denoted by $f_{n,k}$ where $n \geq 1$, and $k \geq 2$. In the friendship graph f_n, the central vertex has a node degree of $2n$, and all other remaining vertices of degree 2 can be further classified in a k^{th} barycentric subdivision as follows:

- Type I: v is adjacent to the corners of the triangle.
- Type II: v is adjacent to the nodes of degree 2.
- Type III: v is adjacent to the central node.

Here, by the corners of the triangle we mean the two corners other than the central node.

Theorem 2.7. *Let G= $f_{n,k}$ where $n \geq 1$, $k \geq 2$ be the k^{th} barycentric subdivision of the friendship graph f_n of order $2n+1$. Then for $v \in G$,*

$$l(v) = \begin{cases} \dfrac{n-1}{n+1} & \textit{if } v \textit{ is the central node.} \\ 0 & \textit{if } v \textit{ is a node of Type I or Type II.} \\ \dfrac{-(n-1)}{2(n+1)} & \textit{if } v \textit{ is a node of Type III.} \end{cases}$$

Proof. The central node v has a degree of $2n$, and all its neighbors are of degree 2. Hence

$$\begin{aligned} l(v) &= \frac{1}{2n}\left(\frac{2n-2}{2n+2}\right)2n \\ &= \frac{n-1}{n+1} \end{aligned}$$

Now if v is a Type I or Type II node, its neighbors are of degree 2. So $l(v) = 0$ in this case. Finally, if v is a node of Type III, it is adjacent to both the central node and a node of degree 2. Hence

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-2n}{2+2n}\right) \\ &= \frac{-(n-1)}{2(n+1)} \end{aligned}$$

□

Remark 2.3. For $n > 1$, the friendship graph f_n is unicentric.

In [12] we can see the analogous result for the betweenness centrality.

2.7. **Butterfly Graph.** The butterfly graph is a planar undirected graph with 5 vertices and 6 edges. It can be constructed by joining 2 copies of the cycle graph C_3 with a common vertex and is therefore isomorphic to the friendship graph f_2.

2.8. **Petersen Graph.** Petersen graph is 3-regular and hence the leverage centrality of each node is zero. If G denotes the Petersen graph, then the k^{th} barycentric subdivision of G is denoted by G_k. For $k \geq 3$, we classify the newly formed nodes as follows:

- Type I: v is adjacent to the corner node(inner or outer).
- Type II: v is adjacent to the nodes of degree 2.

Theorem 2.8. *Let G= G_k, $k \geq 3$ be the k^{th} barycentric subdivision of the Petersen graph. Then for $v \in G$,*

$$l(v) = \begin{cases} \frac{1}{5} & \textit{if } v \textit{ is the corner node(both inner and outer)} \\ \frac{-1}{10} & \textit{if } v \textit{ is a node of Type I.} \\ 0 & \textit{if } v \textit{ is a node of Type II.} \end{cases}$$

Proof. Both the inner and outer corner nodes are of degree 3 and its neighbors v_i are of degree 2. Hence if v is a corner node, then

$$\begin{aligned} l(v) &= \frac{1}{3}\sum_{i=1}^{3}\frac{3-deg(v_i)}{3+deg(v_i)} \\ &= \frac{1}{5} \end{aligned}$$

Now if v is a node of Type I, then one of its neighbors is the corner node which has degree 3 and the other neighbor is of degree 2. Hence

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-3}{5}\right) \\ &= \frac{-1}{10} \end{aligned}$$

The remaining case is obvious. □

Corollary 2.3. *Let G= G_k, $k = 1$ be the first barycentric subdivision of the Petersen graph. Then for $v \in G$,*

$$l(v) = \begin{cases} \frac{1}{5} & \textit{if } v \textit{ is the corner node(both inner and outer)} \\ \frac{-1}{5} & \textit{if } v \textit{ is a node of Type I.} \end{cases}$$

2.9. **Moser spindle.** The Moser spindle is a 7-node unicentric leverage graph with the degree 4 nodes as the leverage center. In the $k^{th}, k \geq 3$ barycentric subdivision of the moser spindle, we classify the nodes of degree 2 as follows:

- Type I: v is adjacent to the degree 4 node
- Type II: v is adjacent to the nodes of degree 2
- Type III: v is adjacent to the nodes of degree 3

Theorem 2.9. *Let G= G_k, $k \geq 3$ be the k^{th} barycentric subdivision of the Moser spindle. Then for $v \in G$,*

$$l(v) = \begin{cases} \frac{1}{3} & \textit{if } v \textit{ is the degree 4 node.} \\ \frac{1}{5} & \textit{if } v \textit{ is a degree 3 node.} \\ \frac{-1}{6} & \textit{if } v \textit{ is a node of Type I.} \\ 0 & \textit{if } v \textit{ is a node of Type II.} \\ \frac{-1}{10} & \textit{if } v \textit{ is a node of Type III.} \end{cases}$$

Proof. If v is the degree 4 node, then $N_v = \{v_i : deg(v_i) = 2, 1 \leq i \leq 4\}$

$$l(v) = \frac{1}{4}\sum_{i=1}^{4}\frac{4 - deg(v_i)}{4 + deg(v_i)} = \frac{1}{3}$$

If v is a degree 3 node, then $N_v = \{v_i : deg(v_i) = 2, 1 \leq i \leq 3\}$

$$l(v) = \frac{1}{3}\sum_{i=1}^{3}\frac{3 - deg(v_i)}{3 + deg(v_i)} = \frac{1}{5}$$

Now if v is a node of Type I, then $N_v = \{u, w\}$ where $deg(u) = 4$ and $deg(w) = 2$

$$l(v) = \frac{1}{2}\left(\frac{2 - deg(u)}{2 + deg(u)} + \frac{2 - deg(w)}{2 + deg(w)}\right) = \frac{-1}{6}$$

If v is a node of Type II, then clearly $l(v) = 0$. Finally if v is a node of Type III, then $N_v = \{u, w\}$ where $deg(u) = 3$ and $deg(w) = 2$

$$l(v) = \frac{1}{2}\left(\frac{2 - deg(u)}{2 + deg(u)}\right) = \frac{-1}{10}$$

□

Corollary 2.4. *Let G= G_k, $k = 1$ be the first barycentric subdivision of the Moser spindle. Then for $v \in G$,*

$$l(v) = \begin{cases} \frac{1}{3} & \text{if } v \text{ is the degree 4 node.} \\ \frac{1}{5} & \text{if } v \text{ is a degree 3 node.} \\ \frac{-4}{15} & \text{if } v \text{ is in between the degree 4 and a degree 3 node} \\ \frac{-1}{5} & \text{if } v \text{ is in between two degree 3 nodes} \end{cases}$$

Proof. The first two cases are obvious. If v is in between the degree 4 and a degree 3 node, then $N_v = \{u, w\}$ where $deg(u) = 4$ and $deg(w) = 3$

$$l(v) = \frac{1}{2}\left(\frac{2 - deg(u)}{2 + deg(u)} + \frac{2 - deg(w)}{2 + deg(w)}\right) = \frac{-4}{15}$$

If v is in between two degrees 3 nodes, then $N_v = \{u, w\}$ where $deg(u) = 3$ and $deg(w) = 3$

$$l(v) = \frac{1}{2}\left(\frac{2 - deg(u)}{2 + deg(u)} + \frac{2 - deg(w)}{2 + deg(w)}\right) = \frac{-1}{5}$$

□

Corollary 2.5. *Let G= G_k, $k = 2$ be the second barycentric subdivision of the Moser spindle. Then for $v \in G$, we get the same result as the above theorem except in the case that the Type II nodes are not there.*

2.10. **Bull Graph.** The bull graph is a planar undirected graph with 5 vertices and 5 edges, in the form of a triangle with two disjoint pendant edges. It is a bicentric leverage graph with a centrality of $\frac{7}{30}$. Two corners of the triangle have a degree of 3 and the other corner is of degree 2. In the k^{th} barycentric subdivision, the newly formed nodes can be classified as:

- Type I: v is adjacent to the degree 3 corner node
- Type II: v is adjacent to the nodes of degree 2.
- Type III: v is adjacent to the pendant node

Theorem 2.10. *Let G= G_k, $k \geq 2$ be the k^{th} barycentric subdivision of the Bull Graph. Then for $v \in G$,*

$$l(v) = \begin{cases} \dfrac{1}{5} & \text{if } v \text{ is a degree 3 corner node.} \\ \dfrac{-1}{10} & \text{if } v \text{ is a node of Type I.} \\ 0 & \text{if } v \text{ is a node of Type II.} \\ \dfrac{1}{6} & \text{if } v \text{ is a node of Type III.} \\ \dfrac{-1}{3} & \text{if } v \text{ is a pendant vertex.} \end{cases}$$

Proof. If v is a degree 3 corner node, then $N_v = \{v_i : deg(v_i) = 2, 1 \leq i \leq 3\}$

$$\begin{aligned} l(v) &= \frac{1}{3}\sum_{i=1}^{3}\frac{3 - deg(v_i)}{3 + deg(v_i)} \\ &= \frac{1}{5} \end{aligned}$$

Now if v is a node of Type I ,then $N_v = \{u, w\}$ where $deg(u) = 3$ and $deg(w) = 2$

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2 - deg(u)}{2 + deg(u)} + \frac{2 - deg(w)}{2 + deg(w)}\right) \\ &= \frac{-1}{10} \end{aligned}$$

If v is a node of Type II, then clearly $l(v) = 0$. Now if v is a node of Type III, then $N_v = \{u, w\}$ where $deg(u) = 2$ and $deg(w) = 1$

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2 - deg(w)}{2 + deg(w)}\right) \\ &= \frac{1}{6} \end{aligned}$$

If v is the pendant node, then

$$\begin{aligned} l(v) &= \frac{1 - 2}{3} \\ &= \frac{-1}{3} \end{aligned}$$

□

Theorem 2.11. *Let G= G_k, $k = 1$ be the first barycentric subdivision of the Bull Graph. Then for $v \in G$,*

$$l(v) = \begin{cases} \frac{1}{5} & \text{if } v \text{ is a degree 3 corner node.} \\ \frac{-1}{10} & \text{if } v \text{ is adjacent to degree 2 and degree 3 corners} \\ \frac{-1}{5} & \text{if } v \text{ is in between degree 3 nodes.} \\ 0 & \text{if } v \text{ is a node of Type II.} \\ \frac{1}{15} & \text{if } v \text{ is a node of Type III.} \\ \frac{-1}{3} & \text{if } v \text{ is a pendant vertex.} \end{cases}$$

Definition 2.9. *A null leverage graph is a graph in which all the vertices are of leverage zero.*

Before concluding, we narrate an application of barycentric subdivision by converting a null leverage graph to a non-null leverage graph. $C_n(C_n)$ is the graph obtained by taking the barycentric subdivision of the cycle C_n and joining each newly inserted vertices of two incident edges by an edge [13]. It resembles C_n inscribed in C_n. We label the vertices of

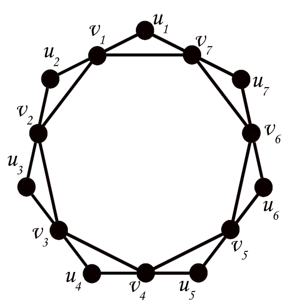

FIGURE 6. $C_7(C_7)$

the underlying cycle C_n as $\{u_1, u_2, ..., u_n\}$ and that of the inner cycle as $\{v_1, v_2, ..., v_n\}$. Since the cycle is 2-regular, the leverage of all the vertices is zero. But the newly formed graph $C_n(C_n)$ is a non-null leverage graph. Since

$$\begin{aligned} l(u_i) &= \frac{1}{2}\left(\frac{2-4}{6}\right)2 \\ &= \frac{-1}{3} \end{aligned}$$

and

$$\begin{aligned} l(v_i) &= \frac{1}{4}\left(\frac{4-2}{6}\right)2 \\ &= \frac{1}{6} \end{aligned}$$

So the leverage centers are changed from u_i to $v_i, i = 1, 2, ..., n$ in $C_n(C_n)$.

3. Conclusions

The leverage centrality analysis in the k^{th} barycentric subdivision of some classes of graphs has been investigated and we found some regularity in centrality for $k \geq 3$. When we compare the result with the most classical betweenness centrality, we find that the centrality varies with k in the case of betweenness centrality, as it is based on the number of shortest paths. Using barycentric subdivision and some graph operations, we can study and compare the leverage centers of various graphs, the number of distinct leverage centralities, the bounds of leverage centralities, etc. Also, this study can be extended to other centrality measures.

Acknowledgments. Our thanks go out to an anonymous referee who provided helpful feedback.

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