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Recent developments in the fixed point theory of enriched contractive mappings. A survey

VASILE BERINDE[1,2] AND MĂDĂLINA PĂCURAR[3]

ABSTRACT. The aim of this note is threefold: first, to present a few relevant facts about the way in which the technique of enriching contractive mappings was introduced; secondly, to expose the main contributions in the area of enriched mappings established by the authors and their collaborators by using this technique; and third, to survey some related developments in the very recent literature which were authored by other researchers.

1. INTRODUCTION

The concept of *enriched* nonexpansive mapping has been introduced in Berinde [37] in the case of a real Hilbert space and then was extended to the more general case of a Banach space in Berinde [38], see also Berinde [36] where the technique of *enriching contractive type mappings* has been applied to strictly pseudocontractive operators.

Soon after that, the authors and their collaborators applied successfully the same technique for some other classes of contractive type mappings in Hilbert spaces, Banach spaces, convex metric spaces, CAT(0) spaces etc., see Berinde [39], [40], [42], Berinde and Păcurar [45], [47], [48], [46], [49], [50], [51], Berinde et al. [44], Abbas et al. [1], [2], Salisu et al. [114],...

Many other authors were attracted to use this technique and therefore some interesting developments on enriched mappings were obtained, most of them included in the list of current References. The intensive use of the technique of enriching contractive type mappings suggested us to undertake the task of offering a comprehensive exposure to date on the subject. So, our aim in this paper is threefold:

(1) to present a few relevant facts about the way in which the technique of enriching contractive mappings was introduced;
(2) to expose the main contributions in the area of enriched mappings established by the authors and their collaborators;
(3) to survey some related developments which were authored by other researchers.

The paper is organized as follows: in Section 2 we give a brief account on how the technique of enriching contractive type mappings has been (re)-discovered and present some facts about the origins of this technique.

Section 3 is devoted to the exposition of some classes of enriched mappings, Section 4 exposes the classes of enriched nonexpansive mappings in Hilbert and Banach spaces, Section 5 gives an account on the concepts of unsaturated and saturated classes of contractive mappings, Section 6 surveys the brand new results on enriched contractions in

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Corresponding author: Vasile Berinde; vberinde@cunbm.utcluj.ro

quasi-Banach spaces, while Section 7 deals with other developments in the area of enriched contractive type mappings by other authors.

2. THE TECHNIQUE OF ENRICHING CONTRACTIVE TYPE MAPPINGS

We start by presenting how the technique of enriching contractive type mappings was discovered (in fact, re-discovered, see explanations following later in this section).

For a rather long period of time, in connection with the study of various fixed point iterative schemes, partially surveyed in the monograph Berinde [35], we were thinking about finding a way to compare the very many classes of nonexpansive type mappings existing in literature, because such a complete comparison did not exist.

Basically, at the beginning, we were trying to compare, by appropriate examples, the following four important classes of nonexpansive type mappings:

- nonexpansive mappings
- quasi nonexpansive mappings
- strictly pseudocontractive mappings
- demicontractive mappings,

see Berinde [43], as at that time we were interested to deepen the knowledge on the great generality and value of demicontractive mappings.

On the way of performing such a comparison, we discovered by chance a method of deriving new constructive fixed point theorems, which we have called the *technique of enriching contractive type mappings*.

The starting point consisted of some known facts in the metrical fixed point theory. To present them, let $(X, \|\cdot\|)$ be a real normed space, $C \subset X$ a closed and convex set and $T : C \to C$ a self mapping. Denote by

$$Fix\,(T) = \{x \in C : Tx = x\}$$

the set of fixed points of T and, for $\lambda \in (0, 1)$, let us also denote

$$T_\lambda := (1 - \lambda)I + \lambda T.$$

T_λ is usually named as the *averaged mapping* of T (a term coined in Baillon et al. [26]).

It is easy to see that

$$Fix\,(T) = Fix(T_\lambda) \tag{2.1}$$

for all $\lambda \in (0, 1)$.

A mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C. \tag{2.2}$$

We also recall that a mapping $T : C \to C$ is called *asymptotically regular* (on C) if, for any $x \in C$,

$$\|T^{n+1}x - T^n x\| \to 0 \text{ as } n \to \infty.$$

It is known that if T is nonexpansive, then in general T is not asymptotically regular but its averaged perturbation, T_λ, is asymptotically regular, a result that apparently was first established by Krasnoselskii [85], in uniformly convex Banach spaces, and then used and developed by Browder and Petryshyn [55], [56] in Hilbert spaces and by Ishikawa [75] in Banach spaces.

This property is extremely important as it enables us to compute the fixed points of a nonexpansive mapping T by means of its averaged perturbation T_λ, which, from the point of view of the convergence of fixed point iterative schemes, has **richer** properties than T.

After contemplating for a long time the above enriching process of nonexpansive mappings with respect to asymptotical regularity, by using instead of T its averaged perturbation T_λ, we were naturally conducted to formulate the following

Open problem: Does one obtain a richer class of mappings if one uses $T_\lambda = (1-\lambda)I + \lambda T$ instead of T in a certain contraction condition from metrical fixed point theory?

Fortunately, the answer was in the affirmative: indeed, by using $T_\lambda = (1-\lambda)I + \lambda T$ instead of T in some contraction conditions from metrical fixed point theory, several richer classes of mappings were derived, see the results exposed in Sections 3-6.

We first searched an answer to the above question in the case of nonexpansive mappings, by considering inequality (2.2) with T_λ instead of T, that is, by introducing the inequality

$$\|T_\lambda x - T_\lambda y\| \leq \|x-y\|, \text{ for all } x, y \in C, \tag{2.3}$$

to define the class of *enriched nonexpansive mappings* in Hilbert spaces in Berinde [37].

Of course, at the very first steps, we were convinced that we were the first ones to discover this nice technique but, a few years later, after a careful documentation and analysis, we realized that the same technique has been applied independently and tacitly by other mathematicians long time ago, e.g., Browder and Petryshyn [56] and Hicks and Kubicek [72], see Berinde and Păcurar [50].

Without explicitly indicating their method of derivation, Browder and Petryshyn [56] introduced and studied the class of *strictly pseudocontractive mappings*, while Hicks and Kubicek [72] introduced and studied the class of *demicontractive mappings*. These classes represent two important concepts in the iterative approximation of fixed points of nonexpansive type mappings, see for example Berinde [35].

Recall that a mapping T is said to be

1) *quasi-nonexpansive* if $Fix\,(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x-y\|, \text{ for all } x \in C \text{ and } y \in Fix\,(T). \tag{2.4}$$

2) *k-strictly pseudocontractive* of the Browder-Petryshyn type ([55]) if there exists a positive number $k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|x - y - Tx + Ty\|^2, \forall x, y \in C. \tag{2.5}$$

3) *k-demicontractive* ([72]) or *quasi k-strictly pseudocontractive* (see Berinde et al. [53]) if $Fix\,(T) \neq \emptyset$ and there exists a positive number $k < 1$ such that

$$\|Tx - y\|^2 \leq \|x-y\|^2 + k\|x - Tx\|^2, \tag{2.6}$$

for all $x \in C$ and $y \in Fix\,(T)$.

If we denote by $\mathcal{NE}$, $\mathcal{QNE}$, $\mathcal{SPC}$ and $\mathcal{DC}$ the classes of nonexpansive, quasi-nonexpansive, strictly pseudocontractive and demicontractive mappings, respectively, then the relationships between these classes are completely represented in the diagram in Figure 1.

A search in MathSciNet for the keywords "nonexpansive", "quasi nonexpansive", "strictly pseudocontractive" and "demicontractive" in the title of publications is showing the following impressive figures:

- 494 indexed publications with "quasi nonexpansive" in their title;
- 200 indexed publications with "strictly pseudocontractive" or "strict pseudocontraction" in their title;
- 138 indexed publications with "demicontractive" in their title;
- 4226 indexed publications with "nonexpansive" in their title.

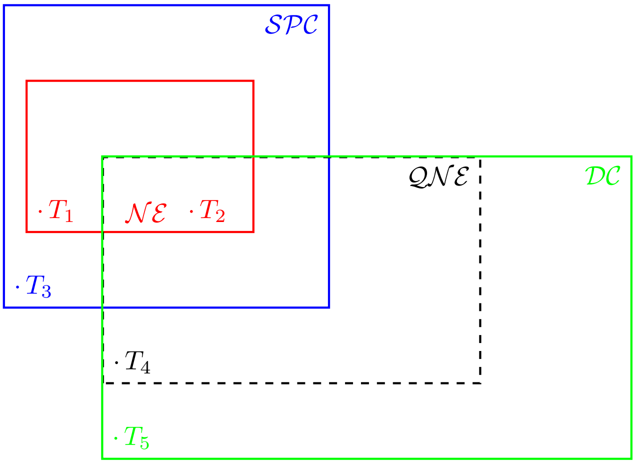

Figure 1. Diagram of the relationships between the classes $\mathcal{NE}$, $\mathcal{QNE}$, $\mathcal{SPC}$ and $\mathcal{DC}$

The diagram in Figure 1 is taken from Berinde [43], where the mappings T_1-T_5 that differentiate the four classes of mappings are presented in Examples 2.1-2.5 [43].

3. Some classes of enriched mappings

Although the first class of enriched mappings introduced in literature was, chronologically, the one corresponding to nonexpansive mappings, we start our presentation with enriched Banach contractions, which are the mappings appearing in the famous Banach contraction mapping principle - the foundation stone of metrical fixed point theory.

3.1. **Enriched contractions in Banach spaces.**

The concept of *enriched contraction* was introduced and studied in Berinde and Păcurar [45].

Definition 3.1 ([45]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $b \in [0, +\infty)$ and $\theta \in [0, b + 1)$ such that*

$$\|b(x - y) + Tx - Ty\| \leq \theta\|x - y\|, \forall x, y \in X. \tag{3.7}$$

To indicate the constants involved in (3.7) *we shall also call T a (b, θ)-enriched contraction.*

Example 3.1 ([45]).

(1) A Banach contraction T satisfies (3.7) *with $b = 0$ and $\theta = c \in [0, 1)$.*

(2) Let $X = [0, 1]$ be endowed with the usual norm and let $T : X \to X$ be defined by $Tx = 1-x$, for all $x \in [0, 1]$. Then T is not a Banach contraction but T is a $(b, 1-b)$-enriched contraction for any $b \in (0, 1)$.

An important fixed point theorem and convergence result for enriched contractions is stated in the next theorem.

Theorem 3.1 ([45]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (b, θ)-enriched contraction. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.8}$$

converges to p, for any $x_0 \in X$;

(iii) *The following estimate holds*

$$\|x_{n+i-1}-p\| \leq \frac{c^i}{1-c}\cdot\|x_n-x_{n-1}\|\,, \quad n=0,1,2,\dots;\, i=1,2,\dots, \tag{3.9}$$

where $c=\dfrac{\theta}{b+1}$.

Remark 3.1. *In the particular case $b=0$, by Theorem 3.1 we get the classical Banach contraction mapping principle in the original setting of a Banach space.*

It is possible to establish a Maia type fixed point theorem for enriched contractions defined on a linear vector space, by endowing it with a metric d which is subordinated to a norm $\|\cdot\|$. The next result has been established in Berinde [39].

Theorem 3.2 ([39]). *Let X be a linear vector space endowed with a metric d and a norm $\|\cdot\|$ satisfying the condition*

$$d(x,y)\leq\|x-y\|,\ \text{ for all } x,y\in X. \tag{3.10}$$

Suppose

(i) (X,d) is a complete metric space;

(ii) $T:X\to X$ is continuous with respect to d;

(iii) T is an enriched contraction with respect to $\|\cdot\|$, that is, there exist $b\in[0,+\infty)$ and $\theta\in[0,b+1)$ such that

$$\|b(x-y)+Tx-Ty\|\leq\theta\|x-y\|, \forall x,y\in X. \tag{3.11}$$

Then

(i) $Fix\,(T)=\{p\}$*, for some* $p\in X$;

(ii) *There exists $\lambda\in(0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by*

$$x_{n+1}=(1-\lambda)x_n+\lambda Tx_n,\ n\geq 0, \tag{3.12}$$

converges in (X,d) to p, for any $x_0\in X$;

(iii) *The estimates*

$$d(x_n,p)\leq\frac{c^n}{1-c}\cdot\|x_1-x_0\|,\ n\geq 1 \tag{3.13}$$

and

$$d(x_n,p)\leq\frac{c}{1-c}\cdot\|x_n-x_{n-1}\|,\ n\geq 1 \tag{3.14}$$

hold with $c=\dfrac{\theta}{b+1}$.

For $b=0$, by Theorem 3.2 we obtain a generalization of Maia's fixed point theorem, see Maia [87], see also Rus [107], [108], Rzepecki [110], [111].

3.2. **Enriched Kannan mappings in Banach spaces.**

The concept of *enriched Kannan mapping* was introduced and studied in Berinde and Păcurar [46], where some applications for solving split feasibility and variational inequality problems are also presented.

Definition 3.2 ([46]). *Let $(X,\|\cdot\|)$ be a linear normed space. A mapping $T:X\to X$ is said to be an enriched Kannan mapping if there exist $a\in[0,1/2)$ and $k\in[0,\infty)$ such that*

$$\|k(x-y)+Tx-Ty\|\leq a\left[\|x-Tx\|+\|y-Ty\|\right],\ \text{for all } x,y\in X. \tag{3.15}$$

To indicate the constants involved in (3.15) *we shall also call T a (k,a)-enriched Kannan mapping.*

Example 3.2.

(1) Any Kannan mapping is a $(0, a)$*-enriched Kannan mapping, i.e., it satisfies* (3.15) *with* $k = 0$.

(2) Let $X = [0, 1]$ *be endowed with the usual norm and* $T : X \to X$ *be defined by* $Tx = 1 - x$, *for all* $x \in [0, 1]$. *Then* T *is not a Kannan mapping but* T *is an enriched Kannan mapping (*T *is also an enriched contraction and nonexpansive mapping).*

The next result provides a convergence theorem for the Krasnoselskij iterative method used to approximate the fixed points of *enriched* Kannan mappings.

Theorem 3.3 **([46]).** *Let* $(X, \|\cdot\|)$ *be a Banach space and* $T : X \to X$ *a* (k, a)*-enriched Kannan mapping. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) *There exists* $\lambda \in (0, 1]$ *such that the iterative method* $\{x_n\}_{n=0}^{\infty}$, *given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.16}$$

converges to p, for any $x_0 \in X$;

(iii) *The following estimate holds*

$$\|x_{n+i-1} - p\| \leq \frac{\delta^i}{1 - \delta} \cdot \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots \tag{3.17}$$

where $\delta = \frac{a}{1-a}$.

Remark 3.2. *The notion of enriched Kannan mapping has been also extended to* enriched Bianchini mapping, *whose definition is given below, and for which corresponding existence and approximation results that generalize Theorem 3.3 were established in Berinde and Păcurar* [46].

Definition 3.3 **([46]).** *Let* $(X, \|\cdot\|)$ *be a linear normed space. A mapping* $T : X \to X$ *is said to be an enriched Bianchini mapping if there exist* $h \in [0, 1)$ *and* $k \in [0, \infty)$ *such that*

$$\|k(x - y) + Tx - Ty\| \leq h \max\{\|x - Tx\|, \|y - Ty\|\}, \text{ for all } x, y \in X. \tag{3.18}$$

To indicate the constants involved in (3.18) *we shall also call it a* (k, h)*-enriched Bianchini mapping.*

Particular cases of Theorem 3.3 recover various results due to Kannan [78], [79].

3.3. Enriched Ćirić-Reich-Rus contractions in Banach spaces.

It is possible to unify and extend Theorems 3.1 and 3.3 from the previous sections and thus obtain a fixed point theorem for the so called *enriched Ćirić-Reich-Rus contractions*. This concept has been first introduced in Berinde and Păcurar [48], in a particular case, and then was improved to the currentt version in Berinde and Păcurar [51].

Definition 3.4 **([51]).** *Let* $(X, \|\cdot\|)$ *be a linear normed space and* $T : X \to X$ *a self mapping.* T *is said to be a* (k, a, b)*-enriched Ćirić-Reich-Rus contraction if, for some* $k \in [0, \infty)$ *and* $a, b \geq 0$, *satisfying* $\dfrac{a}{k+1} + 2b < 1$, *the following condition holds:*

$$\|k(x - y) + Tx - Ty\| \leq a\|x - y\| + b\left(\|x - Tx\| + \|y - Ty\|\right), \text{ for all } x, y \in X. \tag{3.19}$$

Remark 3.3.

1) A Ćirić-Reich-Rus contraction satisfies (3.19) *with* $k = 0$.

2) If $b = 0$, *then from* (3.19) *we obtain the contraction condition* (3.7) *that defines enriched contractions, with* $k \in [0, +\infty)$ *and* $a \in [0, k + 1)$.

3) If $a = 0$, *then from* (3.19) *we obtain the contraction condition* (3.15) *satisfied by an enriched Kannan mapping.*

Theorem 3.4 ([51]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (k, a, b)-enriched Ćirić-Reich-Rus contraction in the sense of Definition 3.4. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{y_n\}_{n=0}^{\infty}$, given by

$$y_{n+1} = (1-\lambda)y_n + \lambda T y_n,\ n \geq 0, \tag{3.20}$$

converges to p, for any $y_0 \in X$;

(iii) The following estimates hold

$$\|y_n - p\| \leq \begin{cases} \alpha^n \cdot \|y_0 - p\|, n \geq 0 \\ \dfrac{\alpha}{1-\alpha} \cdot \|y_n - y_{n-1}\|, n \geq 1 \end{cases} \tag{3.21}$$

where $\alpha = \dfrac{a + (k+1)b}{(k+1)(1-b)}$.

Remark 3.4.

As mentioned before, a preliminary version of the concept of enriched Ćirić-Reich-Rus contraction in Definition 3.4 has been introduced and studied in Berinde and Păcurar [48].

Particular cases of Theorem 3.4 recover various results due to Ćirić [60], *Reich* [105] *and Rus* [106], *who independently discovered in 1971 what we call now a* Ćirić-Reich-Rus contraction.

3.4. **Enriched Chatterjea mappings in Banach spaces.**

The notion of *enriched Chatterjea mapping* was introduced and studied in Berinde and Păcurar [47].

Definition 3.5 ([47]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched Chatterjea mapping if there exist $b \in [0, 1/2)$ and $k \in [0, +\infty)$ such that*

$$\|k(x-y) + Tx - Ty\| \leq b\left[\|(k+1)(x-y) + y - Ty\| + \right.$$

$$\left. + \|(k+1)(y-x) + x - Tx\|\right], \forall x, y \in X. \tag{3.22}$$

To indicate the constants involved in (3.22) *we shall call T a (k, b)-enriched Chatterjea mapping.*

Example 3.3.

1) A Chatterjea mapping satisfies (3.22) *with $k = 0$.*

2) All Banach contractions with constant $c < \dfrac{1}{3}$, all Kannan mappings with Kannan constant $a < \dfrac{1}{4}$ and all Chatterjea mappings are enriched Chatterjea mappings, i.e., they satisfy (3.22) *with $k = 0$.*

3) T in Example 3.2 (2) is also an enriched Chatterjea mapping.

Theorem 3.5 ([47]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (k, b)-enriched Chatterjea mapping. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.23}$$

converges to p, for any $x_0 \in X$;

(iii) The following estimate holds

$$\|x_{n+i-1} - p\| \leq \frac{\delta^i}{1-\delta} \cdot \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots \tag{3.24}$$

where $\delta = \dfrac{b}{1-b}$.

Particular cases of Theorem 3.5 recover various results due to Chatterjea [58], [59].

3.5. **Enriched almost contractions in Banach spaces.**

The notion of *enriched almost contraction* has been introduced and studied in Berinde and Păcurar [52]. It is very general and unifies and extends all the previous concepts of enriched contractive type mappings.

Definition 3.6 ([52]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched almost contraction if there exist $b \in [0, \infty)$, $\theta \in (0, b+1)$ and $L \geq 0$ such that*

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\| + L\|b(x-y) + Tx - y\|, \tag{3.25}$$

for all $x, y \in X$. To indicate the constants involved in (3.25) *we shall also call T as an enriched (b, θ, L)-almost contraction.*

Example 3.4.

1) An enriched $(0, \delta, L)$-almost contraction i.e., a mapping which satisfies (3.25) *with $b = 0$ and $\theta = \delta$, is a (δ, L)-almost contraction, see Berinde* [33], [34]*;*

2) Any (b, θ)-enriched contraction is an enriched $(b, \theta, 0)$-almost contraction;

3) Any (k, a)-enriched Kannan mapping is an enriched $\left(k, \dfrac{a}{1-a}, \dfrac{2a}{1-a}\right)$-almost contraction;

4) Any (k, b)-enriched Chatterjea mapping is an enriched $\left(k, \dfrac{b}{1-b}, \dfrac{2b}{1-b}\right)$-almost contraction.

The next result unifies all main results in the previous subsections, i.e., Theorem 3.1, Theorem 3.3, Theorem 3.4 and Theorem 3.5, and many others.

Theorem 3.6 ([52]). *Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be a (b, θ, L)-almost contraction.*

Then

1) *$Fix\,(T) \neq \emptyset$;*

2) *For any $x_0 \in X$, there exists $\lambda \in (0, 1)$ such that the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$, defined by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.26}$$

converges to some $x^* \in Fix\,(T)$, for any $x_0 \in X$;

3) *The estimate* (3.24) *holds with $\delta = \dfrac{\theta}{b+1}$.*

The next example is remarkable and illustrates the great generality of enriched almost contractions and therefore of Theorem 3.6 itself.

Example 3.5 ([52]). *Let $X = \left[0, \frac{4}{3}\right]$ with the usual norm and $T : X \to X$ be given by*

$$Tx = \begin{cases} 1-x, & \text{if } x \in \left[0, \frac{2}{3}\right) \\ 2-x, & \text{if } x \in \left[\frac{2}{3}, \frac{4}{3}\right]. \end{cases} \tag{3.27}$$

Then $Fix\,(T) = \left\{\dfrac{1}{2}, 1\right\}$ and:

1) T is a $(1, \theta, 3)$-enriched almost contraction, for any $\theta \in (0, 2)$;

2) T is not an almost contraction;

3) T does not belong to the classes of enriched contractions, enriched Kannan mappings or enriched Chatterjea mappings;

4) T is neither nonexpansive nor quasi-nonexpansive;

5) T is not an enriched nonexpansive mapping (and hence not strictly pseudocontractive).

Remark 3.5. *It is important to note that, in view of Theorems 3.1-3.5, enriched contractions, enriched Kannan mappings, enriched Ćirić-Reich-Rus contractions and enriched Chatterjea mappings have all a unique fixed point, while enriched almost contractions - which include all these classes of mappings - could have two or more fixed points.*

3.6. **Enriched φ-contractions in Banach spaces.**

The notion of *enriched φ-contraction* has been introduced and studied in Berinde et al. [44].

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *comparison function* (see for example [35]), if the following two conditions hold:

(i_φ) φ is nondecreasing, i.e., $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;

(ii_φ) $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$.

It is obvious that any comparison function also possesses the following property:

(iii_φ) $\varphi(t) < t$, for $t > 0$.

Some examples of comparison functions are the following:

$$\varphi(t) = \frac{t}{t+1}, t \in [0, \infty);\ \varphi(t) = \frac{t}{2}, t \in [0, 1] \text{ and } \varphi(t) = t - \frac{1}{3}, t \in (1, \infty);$$

(one can note that a comparison function is not necessarily continuous).

Definition 3.7 ([44]). *Consider a linear normed space $(X, \|\cdot\|)$ and let $T : X \to X$ be a self mapping. T is said to be an enriched φ-contraction if one can find a constant $b \in [0, +\infty)$ and a comparison function φ such that*

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\varphi(\|x-y\|), \forall x, y \in X. \tag{3.28}$$

We shall also call T a (b, φ)-enriched contraction.

Example 3.6.

1) Any (b, θ)-enriched contraction is an enriched φ-contraction with $\varphi(t) = \dfrac{\theta}{b+1} \cdot t$.

2) Any φ-contraction is a $(0, \varphi)$-enriched contraction.

3) Consider X to be the unit interval $[0, 1]$ of $\mathbb{R}$ endowed with the usual norm and the function $T : X \to X$ given by $Tx = 1 - x$, for all $x \in [0, 1]$. Then T is neither a contraction nor a φ-contraction but T is an enriched φ-contraction (as it is an enriched contraction).

Theorem 3.7 ([44]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ an enriched (b, φ)-contraction. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^\infty$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.29}$$

and $x_0 \in X$ arbitrary, converges strongly to p.

Remark 3.6.

An enriched (b, φ)-contraction with $b = 0$ is a usual φ-contraction, a concept that was studied previously in Berinde [28], [29], [30], [31], [32], [34] *and by many other authors.*

Now, let us consider the auxiliary functions $\psi : \mathbb{R}_+ \to [0, 1)$ satisfying the following property:

(g) If $\{t_n\} \subset \mathbb{R}_+$ and $\psi(t_n) \to 1$ as $n \to \infty$, then $t_n \to 0$ as $n \to \infty$.

Let $\mathcal{P}$ denote the set of all auxiliary functions ψ satisfying condition (g) above. It is easy to check that $\mathcal{P} \neq \emptyset$, as the function $\psi(t) = \exp(-t)$, for $t \geq 0$, belongs to $\mathcal{P}$.

The next result is a very general fixed point theorem, that includes many other fixed point results as particular cases, see Berinde and Păcurar [49].

Theorem 3.8 ([49])**.** *Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be an enriched ψ-contraction, i.e., a mapping for which there exists a function $\psi \in \mathcal{P}$ such that*

$$\|b(x-y)+Tx-Ty\| \le (b+1)\psi(\|x-y\|)\|x-y\|, \forall x, y \in X. \tag{3.30}$$

Then

(i) $Fix\,(T)=\{p\}$, for some $p \in X$.

(ii) The sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iterative process

$$x_{n+1}=(1-\lambda)x_n+\lambda T x_n,\ n \ge 0, \tag{3.31}$$

and $x_0 \in X$ arbitrary, converges strongly to p.

3.7. **Cyclic enriched φ-contractions in Banach spaces.**

The class of enriched φ-contractions has been extended further to *cyclic enriched φ-contractions* in Berinde et al. [44]. To present it, we need the following prerequisites, see Rus [109].

Let X be a nonempty set, m a positive integer and $T : X \to X$ an operator. By definition, $\bigcup_{i=1}^m X_i$ is a *cyclic representation of X with respect to T* if

(i) $X_i \neq \emptyset$, $i=1,2,...,m$;

(ii) $T(X_1) \subset X_2, \dots, T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$.

Let (X,d) be a metric space, m a positive integer, $A_1, \dots, A_m$ nonempty and closed subsets of X and $Y=\bigcup_{i=1}^m A_i$. An operator $T : X \to X$ is called a *cyclic φ-contraction* if

(a) $\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;

(b) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(Tx,Ty) \le \varphi(d(x,y)), \tag{3.32}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i=1,2,\dots,m$, where $A_{m+1}=A_1$.

Definition 3.8. *Consider a linear normed space $(X, \|\cdot\|)$, $T : X \to X$ be a self mapping and let $\bigcup_{i=1}^m A_i$ be a cyclic representation of X with respect to T. If one can find a constant $b \in [0,+\infty)$ and a comparison function φ such that*

$$\|b(x-y)+Tx-Ty\| \le (b+1)\varphi(\|x-y\|), \forall x \in A_i \text{ and } \forall y \in A_{i+1}, \tag{3.33}$$

for $i=1,2,\dots,m$, where $A_{m+1}=A_1$, then T is said to be a cyclic enriched φ-contraction.

Example 3.7.

1) Any cyclic φ-contraction is a cyclic enriched φ-contraction (with $b=0$);

2) Any enriched contraction is a cyclic enriched φ-contraction (with $m=1$).

A comparison function φ is said to be a (c)-comparison function (see [31]) if there exist $k_0 \in \mathbb{N}$, $\delta \in (0,1)$ and a convergent series of nonnegative terms $\sum\limits_{k=1}^{\infty} v_k$ such that

$$\varphi^{k+1}(t) \le \delta\varphi^k(t)+v_k,\ k \ge k_0,\ t \in \mathbb{R}_+. \tag{3.34}$$

It is known (see for example Lemma 1.1 [93]) that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c)-comparison function, then $s : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$s(t)=\sum_{k=1}^{\infty}\varphi^k(t),\ t \in \mathbb{R}_+, \tag{3.35}$$

is increasing and continuous at 0.

Theorem 3.9 ([44])**.** *Let $(X, \|\cdot\|)$ be a Banach space, m a positive integer, $A_1, \dots, A_m$ nonempty and closed subsets of X, $Y=\bigcup_{i=1}^m A_i$ and $T : X \to X$ a cyclic enriched φ-contraction with φ a (c)-comparison function. Then*

(i) T has a unique fixed point $p \in \bigcap_{i=1}^m A_i$;

(ii) there exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^\infty$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.36}$$

and $x_0 \in X$ arbitrary, converges strongly to p;

(iii) the following estimates hold

$$\|x_n - p\| \leq s\left(\varphi^n(\|x_0 - x_1\|)\right), n \geq 1;$$
$$\|x_n - p\| \leq s\left(\varphi(\|x_n - x_{n+1}\|)\right), n \geq 1;$$

(iv) for any $x \in Y$:

$$\|x_n - p\| \leq s(\lambda\|x - Tx\|),$$

where s is defined by (3.35).

Remark 3.7. *If $m = 1$, then by Theorem 3.9 one obtains Theorem 3.8 from the previous section.*

3.8. **Enriched contractions in convex metric spaces.**

The notion of *enriched contraction* in convex metric spaces has been introduced and studied in Berinde and Păcurar [49]. To introduce it we need the following beautiful concept of convexity introduced by Takahashi [125].

Definition 3.9. *Let (X,d) be a metric space. A continuous function $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X if, for all $x, y \in X$ and any $\lambda \in [0,1]$,*

$$d(u, W(x,y;\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y), \text{ for any } u \in X. \tag{3.37}$$

A metric space (X,d) endowed with a convex structure W is called a Takahashi convex metric space and is usually denoted by (X,d,W).

Remark 3.8. *Any linear normed space and each of its convex subsets are convex metric spaces, with the natural convex structure*

$$W(x,y;\lambda) = \lambda x + (1-\lambda)y, x, y \in X; \lambda \in [0,1] \tag{3.38}$$

but the reverse is not valid.

Definition 3.10 ([49]). *Let (X,d,W) be a convex metric space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $c \in [0,1)$ and $\lambda \in [0,1)$ such that*

$$d(W(x,Tx;\lambda), W(y,Ty;\lambda)) \leq c d(x,y), \text{ for all } x, y \in X. \tag{3.39}$$

To specify the parameters c and λ involved in (3.39)*, we also call T a (λ, c)-enriched contraction.*

Example 3.8.

Any $(0,c)$-enriched contraction is a usual Banach contraction and therefore an enriched contraction.

The next result is a significant extension of Theorem 3.1 from the case of a Banach space setting to that of an arbitrary complete convex metric space.

Theorem 3.10 ([49]). *Let (X,d,W) be a complete convex metric space and let $T : X \to X$ be a (λ, c)-enriched contraction. Then*

(i) $Fix\,(T) = \{p\}$, for some $p \in X$.

(ii) The sequence $\{x_n\}_{n=0}^\infty$ obtained from the iterative process

$$x_{n+1} = W(x_n, Tx_n; \lambda),\ n \geq 0, \tag{3.40}$$

converges to p, for any $x_0 \in X$.

(iii) The following estimate holds

$$d(x_{n+i-1}, p) \leq \frac{c^i}{1-c} \cdot d(x_n, x_{n-1}) \quad n = 1, 2, \ldots;\ i = 1, 2, \ldots \tag{3.41}$$

3.9. **Enriched Prešić contractions.**

The notion of *enriched Prešić contraction* has been introduced and studied in Păcurar [94].

Definition 3.11 ([94]). *Let $(X, +, \cdot)$ be a linear vector space, k a positive integer and $T : X^k \to X$ an operator. For $\lambda_0, \lambda_1, \dots, \lambda_k \geq 0$, with $\sum\limits_{i=0}^{k} \lambda_i = 1$ and $\lambda_k \neq 0$, the operator $T_\lambda : X^k \to X$, defined by*

$$T_\lambda(x_0, x_1, \dots, x_{k-1}) = \lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_{k-1} x_{k-1} + \lambda_k T(x_0, x_1, \dots, x_{k-1}) \tag{3.42}$$

will be called the averaged mapping corresponding to T.

Remark 3.9. *One can easily see that, for $k = 1$, the above definition reduces to $T_\lambda(x_0) = \lambda_0 x_0 + \lambda_1 T(x_0)$, for $x_0 \in X$, where $\lambda_0 + \lambda_1 = 1$, that is, the averaged mapping $T_\lambda : X \to X$ extensively used in the previous sections.*

Remark 3.10. *As in the case of the averaged mapping corresponding to an operator defined on X, it is not difficult to show that $x^* \in X$ is a fixed point of $T^k : X \to X$ if and only if it is a fixed point of the corresponding $T_\lambda : X^k \to X$, for some $\lambda_i \geq 0$, $i = 0, 1, \dots, k$, with $\sum\limits_{i=0}^{k} \lambda_i = 1$ and $\lambda_k \neq 0$.*

Indeed, supposing $x^* \in X$ such that $T_\lambda(x^*, x^*, \dots, x^*) = x^*$, it follows that

$$\lambda_0 x^* + \lambda_1 x^* + \cdots + \lambda_{k-1} x^* + \lambda_k T(x^*, x^*, \dots, x^*) = x^*,$$

so

$$(1 - \lambda_k) x^* + \lambda_k T(x^*, x^*, \dots, x^*) = x^*.$$

Since $\lambda_k \neq 0$, it follows immediately that $T(x^*, x^*, \dots, x^*) = x^*$. The inverse is obvious.

Definition 3.12 ([94]). *Let $(X, \|\cdot\|)$ be a linear normed space and k a positive integer. A mapping $T : X^k \to X$ is said to be an enriched Prešić operator if there exist $b_i \geq 0, i = 0, 1, \dots, k-1$ and $\theta_i \geq 0, i = 0, 1, \dots, k-1$ with $\sum\limits_{i=0}^{k-1} (\theta_i - b_i) < 1$ such that:*

$$\left\| \sum_{i=0}^{k-1} b_i (x_i - x_{i+1}) + T(x_0, x_1, \dots, x_{k-1}) - T(x_1, x_2, \dots, x_k) \right\| \leq \sum_{i=0}^{k-1} \theta_i \|x_i - x_{i+1}\|,$$

for all $x_0, x_1, \dots, x_k \in X$.

Remark 3.11.

1) For $k = 1$ this reduces to the definition of an enriched Banach contraction;

2) If $b_0 = b_1 = \cdots = b_{k-1} = 0$ in the above definition, then we obtain the definition of a Prešić operator, see Păcurar [94].

The next result states that an enriched Prešić operator posseses a unique fixed point, which can be obtained by means of some appropriate iterative methods.

Theorem 3.11 ([94]). *Let $(X, \|\cdot\|)$ be a Banach space, k a positive integer and $T : X^k \to X$ an enriched Prešić operator with constants $b_i, \theta_i, i = 0, 1, \dots, k-1$. Then:*

1) *T has a unique fixed point $x^* \in X$ such that $T(x^*, x^*, \dots, x^*)$;*
2) *There exists $a \in (0, 1]$ such that the iterative method $\{y_n\}_{n \geq 0}$ given by*

$$y_n = (1 - a) y_{n-1} + a T(y_{n-1}, y_{n-1}, \dots, y_{n-1}), n \geq 1,$$

converges to the unique fixed point x^*, starting from any initial point $y_0 \in X$.

3) *There exist* $\lambda_0, \lambda_1, \ldots, \lambda_k \geq 0$ *with* $\sum_{i=0}^{k} \lambda_i = 1$ *and* $\lambda_k \neq 0$ *such that the iterative method* $\{x_n\}_{n\geq 0}$ *given by*

$$x_n = \lambda_0 x_{n-k} + \lambda_1 x_{n-k+1} + \cdots + \lambda_{k-1} x_{n-1} + \lambda_k T(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1})$$

or simply

$$x_n = T_\lambda(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), n \geq 1,$$

converges to x^**, for any initial points* $x_0, x_1, \ldots, x_{k-1} \in X$.

4. Enriched nonexpansive mappings

4.1. Enriched nonexpansive mappings in Hilbert spaces.

The notion of *enriched nonexpansive mapping* has been introduced and studied in Berinde [37].

Definition 4.13 ([37]). *Let* $(X, \|\cdot\|)$ *be a linear normed space. A mapping* $T : X \to X$ *is said to be an enriched nonexpansive mapping if there exists* $b \in [0, \infty)$ *such that*

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\|x-y\|, \forall x, y \in X. \tag{4.43}$$

To indicate the constant involved in (4.43) *we shall also call* T *as a* b*-enriched nonexpansive mapping.*

It is important to note that inequality (4.43) in Definition 4.13 was derived from (2.3) by denoting $b = \dfrac{1}{\lambda} - 1$.

Any nonexpansive mapping T is an enriched nonexpansive mapping, i.e., it satisfies (4.43) with $b = 0$, but the reverse is not true, as shown by the next example.

Example 4.9 ([37]).

Let $X = \left[\dfrac{1}{2}, 2\right]$ *be endowed with the usual norm and* $T : X \to X$ *be defined by* $Tx = \dfrac{1}{x}$*, for all* $x \in \left[\dfrac{1}{2}, 2\right]$. *Then*

(i) T *is Lipschitz continuous with Lipschitz constant* $L = 4$ *(and so* T *is not nonexpansive);*

(ii) T *is a* $3/2$*-enriched nonexpansive mapping.*

For the sake of completeness, we recall that a mapping $T : C \to H$, where C is a bounded closed convex subset of a Hilbert space H, is called *demicompact* if it has the property that whenever $\{u_n\}$ is a bounded sequence in H and $\{Tu_n - u_n\}$ is strongly convergent, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which is strongly convergent.

The next result states that any enriched nonexpansive mapping which is also demicompact has a nonempty convex fixed point set and that one can approximate its fixed points by means of a Krasnoselskij type iterative scheme.

Theorem 4.12 ([37]). *Let* C *be a bounded closed convex subset of a Hilbert space* H *and* $T : C \to C$ *be a* b*-enriched nonexpansive and demicompact mapping. Then the set* $Fix\,(T)$ *of fixed points of* T *is a nonempty convex set and there exists* $\lambda \in (0, 1)$ *such that, for any given* $x_0 \in C$*, the Krasnoselskij iteration* $\{x_n\}_{n=0}^{\infty}$ *given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\; n \geq 0, \tag{4.44}$$

converges strongly to a fixed point of T.

If we denote by $\mathcal{ENE}$ the class of enriched nonexpansive mappings, then by the previous example it follows that we have the following strict inclusion relationship

$$\mathcal{NE} \subsetneq \mathcal{ENE}$$

and, by the diagram in Figure 1, we also deduce that

$$\mathcal{NE} \subsetneq \mathcal{SPC}.$$

It was then natural to raise the following

Problem. Find the relationship between the classes $\mathcal{ENE}$ and $\mathcal{SPC}$.

For the case of enriched nonexpansive mappings defined on a real Hilbert space, the answer is given by the next theorem which is a version of Theorem 8 in Berinde and Păcurar [50].

Theorem 4.13. *In a real Hilbert space, $\mathcal{ENE} = \mathcal{SPC}$.*

Proof. Let $T \in \mathcal{SPC}$. Then T satisfies (2.5) with some $k \in (0, 1)$. We have

$$\|x - y - (Tx - Ty)\|^2 = \langle x - y - (Tx - Ty), x - y - (Tx - Ty)\rangle \tag{4.45}$$

$$= \|x - y\|^2 - 2\langle x - y, Tx - Ty\rangle + \|Tx - Ty\|^2$$

and so (2.5) is equivalent to

$$\|Tx - Ty\|^2 \leq \frac{1+k}{1-k} \cdot \|x - y\|^2 - \frac{2k}{1-k} \cdot \langle x - y, Tx - Ty\rangle.$$

By adding to both sides of the previous inequality the quantity

$$\left(\frac{k}{1-k}\right)^2 \cdot \|x - y\|^2 + \frac{2k}{1-k}\langle x - y, Tx - Ty\rangle,$$

we deduce that (2.5) is equivalent to

$$\left\|\frac{k}{1-k}(x - y) + Tx - Ty\right\|^2 \leq \left(\left(\frac{k}{1-k}\right)^2 + \frac{1+k}{1-k}\right)\|x - y\|^2. \tag{4.46}$$

Now, by denoting $b = \dfrac{k}{1-k} > 0$, it follows that inequality (4.46) is equivalent to

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)\|x - y\|, \, \forall x, y \in C,$$

and this shows that $T \in \mathcal{NE}$.

The converse follows by runing backwardly the previous implications. □

4.2. Enriched nonexpansive mappings in Banach spaces.

Remark 4.12. *The equality in Theorem 4.13 is no more valid if we work in a Banach space. The main reason is that in a Banach space we cannot derive the fundamental identity* (4.45) *in the proof of Theorem 4.13, as it is expressed by means of the inner product in the Hilbert space H.*

Hence in a Banach space the class of enriched nonexpansive mappings and that of of strictly pseudocontractive mappings are independent and therefore the next result is an important generalization of several results in literature established for nonexpansive mappings, e.g., the ones in Browder and Petryshyn [56].

Theorem 4.14 ([38]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a b-enriched nonexpansive mapping. Suppose T satisfies Condition I, i.e., there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with the properties $f(0) = 0$ and $f(r) > r$, for $r > 0$, such that*

$$\|x - Tx\| \geq f(d(x, Fix\,(T)), \forall x \in C, \tag{4.47}$$

where

$$d(x, Fix\,(T)) = \inf\{\|x - z\| : z \in Fix\,(T)\}$$

is the distance between the point x and the set $Fix\,(T)$.

Then $Fix\,(T) \neq \emptyset$ and, for any $\lambda \in \left(0, \frac{1}{b+1}\right)$ and for any given $x_0 \in C$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{4.48}$$

converges strongly to a fixed point of T.

5. Unsaturated and saturated classes of contractive mappings

After a close examination of the fixed point results in the Sections 3 and 4, it was noted that the technique of enriching a contractive type mapping T, by means of the averaged operator T_λ, cannot effectively enlarge all classes of contractive mappings.

This observation suggested us a new interesting concept, that of *saturated* class of contractive mappings with respect to the averaged operator T_λ, a notion that has been introduced and studied in Berinde and Păcurar [50].

Definition 5.14 ([50]). *Let $(X, \|\cdot\|)$ be a linear normed space and let $\mathcal{C}$ be a subset of the family of all self mappings of X. A mapping $T : X \to X$ is said to be $\mathcal{C}$-*enriched *or* enriched with respect to $\mathcal{C}$ *if there exists $\lambda \in (0, 1]$ such $T_\lambda \in \mathcal{C}$.*

We denote by $\mathcal{C}^e$ the set of all enriched mappings with respect to $\mathcal{C}$.

Remark 5.13. *From Definition 5.14 it immediately follows that $\mathcal{C} \subseteq \mathcal{C}^e$.*

Definition 5.15 ([50]). *Let X be a linear vector space and let $\mathcal{C}$ be a subset of the family of all self mappings of X. If $\mathcal{C} = \mathcal{C}^e$, we say that $\mathcal{C}$ is a* saturated class of mappings, *otherwise $\mathcal{C}$ is said to be* unsaturated.

If we summarize and the results surveyed in the previous two sections of this paper, then we can see that the following classes of mappings:

- Banach contractions
- Kannan mappings
- Ćirić-Reich-Rus contractions
- Chatterjea mappings
- almost contractions
- φ-contractions
- cyclic enriched φ-contractions
- Prešić contractions

are all *unsaturated* in the setting of a Banach space.

Also, from the results surveyed in Section 4 we infer that

- the class of nonexpansive mappings is unsaturated in Hilbert spaces;
- the class of nonexpansive mappings is unsaturated in Banach spaces

Let us denote by $\mathcal{E}$ the enriching operator defined by the averaged perturbation of a certain class of self mappings of X, i.e., if $\mathcal{C}$ is a class of mappings, then

$$\mathcal{E}(\mathcal{C}) = \mathcal{C}^e.$$

It is easy to prove that $\mathcal{E}$ is idempotent, that is, $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$, which means that any class of enriched mappings is *saturated*.

This implies that

- the class of strictly pseudocontractive mappings is saturated in Hilbert spaces;
- the class of enriched nonexpansive mappings is saturated in Hilbert spaces;
- the class of demicontractive mappings is saturated in Hilbert spaces.

The last claim follows by Theorem 9 in Berinde and Păcurar [50].

6. ENRICHED CONTRACTIONS IN QUASI-BANACH SPACES

The notion of enriched contraction in the setting of a quasi-Banach space has been introduced and studied in Berinde [43]. We recall the following prerequisites.

Definition 6.16. *A quasi-norm on a real vector space X is a map $\|\cdot\| : X \to [0, \infty)$ satisfying the following conditions:*

(QN_0) $\|x\| = 0$ *if and only if* $x = 0$;
(QN_1) $\|\lambda x\| = |\lambda| \cdot \|x\|$, *for all* $x \in X$ *and* $\lambda \in \mathbb{R}$.
(QN_2) $\|x + y\| \leq C\,[\|x\| + \|y\|]$, *for all* $x, y \in X$, *where* $C \geq 1$ *does not depend on* x, y;

The pair $(X, \|\cdot\|)$, where $\|\cdot\|$ is a quasi-norm on a real vector space X, is said to be a quasi-normed space. If $(X, \|\cdot\|)$ is complete (with respect to the quasi norm), then it is called a *quasi-Banach space*.

Definition 6.17. *Let $(X, \|\cdot\|)$ be a linear quasi-normed space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $b \in [0, +\infty)$ and $\theta \in [0, b + 1)$ such that*

$$\|b(x - y) + Tx - Ty\| \leq \theta \|x - y\|, \forall x, y \in X. \tag{6.49}$$

To indicate the constants involved in (6.49) *we shall also call T a (b, θ)-enriched contraction.*

Remark 6.14.

1) As any Banach space is a quasi-Banach space (with $C = 1$), the enriched contractions T in a Banach space introduced in Berinde and Păcurar [45] *are enriched contractions in the sense of Definition 6.17.*

2) It is worth mentioning that, like in the case of Banach spaces, any (b, θ)-enriched contraction is continuous.

The following result is an extension of Theorem 3.1 from Banach spaces to quasi-Banach spaces.

Theorem 6.15 (Berinde [43]). *Let $(X, \|\cdot\|)$ be a quasi-Banach space and $T : X \to X$ a (b, θ)-enriched contraction. Then*

(i) $Fix\,(T) = \{p\}$;
(ii) *There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{6.50}$$

converges to p, for any $x_0 \in X$;

7. Other developments in the area of enriched contractive type mappings

The technique of enriching contractive type mappings, applied by the current authors to several classes of mappings, attracted the interest of many other researchers. In the following we summarise to date some of these contributions.

(1) Abbas et al. [1] introduced the notion of *enriched quasi-contraction*, as a generalization of the clasical Ćirić quasi-contraction, and also the class of *enriched weak contraction mappings* and established and studied the existence and iterative approximation of their fixed points.

(2) Abbas et al. [2] introduced and studied the class of *enriched multivalued contraction mappings* and also considered the data dependence problem and Ulam-Hyers stability of the fixed point problems for enriched multivalued contraction mappings. They also give applications of the obtained results to the problem of the existence of a solution of differential inclusions and dynamic programming.

(3) Abbas et al. [3] introduced the concept of *generalized enriched cyclic contraction mapping* and obtained existence of fixed points and established convergence results for Krasnoselskij iteration used for approximating fixed points of such mappings. As an application of their results, they also established the existence and uniqueness of an attractor for an iterated function system composed of generalized enriched cyclic contraction mappings.

(4) Abbas et al. [5] introduced the concept of *enriched contractive mappings of Suzuki type*. Such a mapping has a fixed point and characterizes the completeness of the underlying normed space.

(5) Abbas et al. [6] introduced the class of *enriched interpolative Kannan type operators* on Banach spaces. This class contains the classes of enriched Kannan operators, interpolative Kannan type contraction operators and some other classes of nonlinear operators. They prove a convergence theorem for the Krasnoselskii iteration method to approximate fixed point of the enriched interpolative Kannan type operatorsand, as an application of the main result, solved a variational inequality problem. The same authors propose in [7] a new class of multi-valued enriched interpolative Ćirić-Reich-Rus type contraction operators, prove a fixed point result, study the data dependence and Ulam-Hyers stability for these operators and obtain a homotopy result as an application of their results.

(6) Ali and Jubair [15] introduced and studied the so called enriched Berinde nonexpansive mappings, which are related to enriched almost contractions.

(7) Anjali and Batra [17] introduced enriched Ćirić's type and enriched Hardy-Rogers contractions for which they established fixed point theorems in Banach spaces and convex metric spaces. They showed that Ćirić's type and Hardy-Rogers contractions are unsaturated classes of mappings and also considered Reich and Bianchini contractions, which were shown to be unsaturated classes of mappings, too.

(8) Babu and Mounika [24] defined the classes of *enriched Jaggi contraction maps*, *enriched Dass and Gupta contraction maps* and *almost* (k, a, b, λ)*-enriched CRR contraction maps* in Banach spaces and proved the existence and uniqueness of fixed points of these maps.

(9) Chandok [57] obtained convergence and existence results of best proximity points for cyclic enriched contraction maps in Takahashi convex metric spaces.

(10) Debnath [61] introduced the notion of Górnicki-type pair of mappings, establish a criterion for existence and uniqueness of common fixed point for such a pair without assuming continuity of the underlying mappings and also establish a common fixed point result for a pair of enriched contractions.

(11) Dechboon and Khammahawongwe [62] established the existence and uniqueness of the best proximity point for several classes of generalized cyclic enriched contractions in convex metric spaces.
(12) Deshmukh et al. [63], amongst many other related results for enriched non-expansive maps and enriched generalized non-expansive maps, also give stability results for two iterative procedures in the class of enriched contractions.
(13) Faraji and Radenović [66] established fixed point results for enriched contractions and enriched Kannan contractions in partially ordered Banach spaces.
(14) Based on the so-called degree of nondensifiability, García [68] introduced a generalization of the (b, θ)-enriched contractions [45] and established a fixed point existence result for this new class of mappings. From their main result, and under some suitable conditions, they derived a result on the existence of fixed points for the sum of two mappings, one of them being compact.
(15) Hacioğlu and Gürsoy [73] introduced multivalued Górnicki mappings and various other new types of multivalued enriched contractive mappings, like multivalued enriched Kannan mappings, multivalued enriched Chatterjea mappings, and multivalued enriched Ćirić-Reich-Rus mappings, and established existence results for the fixed points of these multivalued contractive type mappings by using the fixed point property of the average operator of the mappings.
(16) Khan et al. [83] initiate the study of enriched mappings in modular function spaces, by introducing the concepts of *enriched ρ-contractions* and *enriched ρ-Kannan mappings* and establishing some results on the existence of fixed points of such mappings in this setting.
(17) By introducing the concept of convex structure in rectangular G_b-metric spaces, Li and Cui [86] studied the existence of fixed points of enriched type contractions in such a space.
(18) As a generalization of the main result in [45], Marchiş [88] obtained some common fixed point theorems under an enriched type contraction condition for two single-valued mappings satisfying a weak commutativity condition in Banach spaces and has shown that the unique common fixed point of these mappings can be approximated using the Krasnoselskij iteration.
(19) By using some semi-implicit relations, Mondal et al. [89] introduced *enriched $\mathcal{A}$-contractions* and *enriched $\mathcal{A}'$-contractions* and studied the existence of fixed points, the well-posedness and limit shadowing property of the fixed point problem involving these contractions. The same authors obtained laterMaia type results for enriched contractions via implicit relations [27] , thus extending the results of Berinde [39].
(20) Popescu [101] introduced a new class of Picard operators, called *Górnicki mappings*, which includes the class of enriched contractions [45], enriched Kannan mappings [46], and enriched Chatterjea mappings [47], and proved some fixed point theorems for these mappings. However, while the fixed points of enriched contractions can be approximated by means of Krasnoselskij iteration, there is no any approximation result in [101] for the case of Górnicki mappings. This rise the challenging problem of finding iterative schemes to approximate the unique fixed point of a Górnicki mapping.
(21) By using the idea of the orbital contraction condition given in [99] and considering the second iterate of the mapping in the enriched contraction condition, Nithiarayaphaks and Sintunavarat [90] introduced the class of *weak enriched contraction mappings* and approximated their fixed point by Kirk's iterative scheme.

(22) Panicker and Shukla [95] obtained stability results of fixed point sets for a sequence of enriched contraction mappings in the setting of convex metric spaces, by considering two types of convergence of sequences of mappings, namely, $(\mathcal{G})$-convergence and $(\mathcal{H})$-convergence.

(23) Panja et al. [97] introduced a new non-linear semigroup of enriched Kannan type contractions and proved the existence of a common fixed point on a closed, convex, bounded subset of a real Banach space having uniform normal structure.

(24) Phairatchatniyoman et al. Phairat used a modified Ishikawa iteration scheme to solve a fixed point problem and a split variational inclusion problem in real Hilbert spaces, for b-enriched nonexpansive mapping, and applied it for solving a split feasibility problem.

(25) Among many other related results, Prithvi and Katiyar [102] studied fractals through generalized cyclic enriched Ćirić-Reich-Rus iterated function systems.

(26) Rawat et al. [104] considered enriched ordered contractions in convex noncommutative Banach spaces, while Rawat, Bartwal and Dimri [103] defined and studied interpolative enriched contractions of Kannan type, Hardy-Rogers type and Matkowski type in the setting of a convex metric space.

(27) Ullah et al. [128] introduced and studied the class of enriched Suzuki nonexpansive mappings, which properly contains the class of Suzuki nonexpansive as well as the class of enriched nonexpansive mappings.

(28) Turcanu and Postolache [126] introduced and studied enriched Suzuki mappings, which in particular include enriched nonexpansive mappings, from Hilbert spaces to Hadamard spaces.

(29) Zhou et al. [129] introduced and studied weak enriched $\mathcal{F}$-contractions, weak enriched $\mathcal{F}'$-contraction, and k-fold averaged mapping based on Kirk's iterative algorithm of order k and proved the existence of a unique fixed point of the k-fold averaged mapping associated with weak enriched contractions considered.

(30) Other interesting contributions to the study of various classes of enriched mappings can be found in Abbas et al. [4], Abbas et al. [8]-Ali and Jubair [14], [16], Anjum and Abbas [18]-Anjum and Khan [23], Babu et al. [25], Bisht and Singh [54], Dhivya et al. [64], Eshi et al. [65], Gangwar et al. [67], Gautam et al. [69]-Górnicki and Bisht [71], Inuwa et al. [74], Jorquera Álvarez [76], Ju and Zhai [77], Kesahorm and Sintunavarat [80]-Khan [82], Khuen and Hassan [84], Okonkwo et al. [91], Omidire and Olatinwo [92], Panja et al. [96], Panwar et al. [98], Saleem et al. [112], Salisu et al. [113], Salisu et al. [115]-Suantai et al. [124], Turcanu and Postolache [127],...

8. Conclusions

1. In the first part of this paper we presented a few relevant facts about the way in which the technique of enriching contractive mappings was (re-)discovered.

2. In the main part of the paper we have exposed the main contributions in the area of enriched mappings established by the authors and their collaborators by using this technique.

3. In the last part, we also surveyed some related developments which are authored by other researchers, by considering a list of references to date.

References

[1] Abbas, M.; Anjum, R.; Berinde, V. Equivalence of Certain Iteration Processes Obtained by Two New Classes of Operators. *Mathematics* **9** (2021), no. 18, Article Number 2292.

[2] Abbas, M.; Anjum, R.; Berinde, V. Enriched Multivalued Contractions with Applications to Differential Inclusions and Dynamic Programming. *Symmetry-Basel*, **13** (2021), no. 8, Article Number 1350.

[3] Abbas, M.; Anjum, R.; Iqbal, H. Generalized enriched cyclic contractions with application to generalized iterated function system. *Chaos Solitons Fractals* **154** (2022), Paper No. 111591, 9 pp.

[4] Abbas, M.; Anjum, R.; Ismail, N. Approximation of fixed points of enriched asymptotically nonexpansive mappings in $\mathrm{CAT}(0)$ spaces. *Rend. Circ. Mat. Palermo (2)* **72** (2023), no. 4, 2409–2427.

[5] Abbas, M.; Anjum, R.; Rakočević, V. A generalized Suzuki-Berinde contraction that characterizes Banach spaces. *J. Appl. An.* https://doi.org/10.1515/jaa-2022-2007

[6] Abbas M.; Anjum R.; Riasat, S. Fixed point results of enriched interpolative Kannan type operators with applications. *Appl. Gen. Topol.* **23** (2022), no. 2, 391–404.

[7] Abbas M.; Anjum R.; Riasat, S. A new type of fixed point theorem via interpolation of operators with application in homotopy theory. *Arab. J. Math.* **12** (2023), 277–288, https://doi.org/10.1007/s40065-022-00402-z.

[8] Abbas M.; Anjum R.; Riasat, S. Fixed point results of R-enriched interpolative Kannan pair in R-convex metric spaces. *Creat. Math. Inform.* **32** (2023), no. 1, 1–11.

[9] Abbas, M.; Anjum, R.; Riasat, S. Solution of integral equation involving interpolative enriched cyclic Kannan contraction mappings. *Bangmod Int. J. Math. Comput. Sci.* **9** (2023), 1–9.

[10] Abbas, M.; Anjum, R.; Tahir, M. H. Fixed point theorems of enriched multivalued mappings via sequentially equivalent Hausdorff metric. *Topol. Algebra Appl.* https://doi.org/10.1515/taa-2022-0136.

[11] Abbas, M.; Asghar, M. W.; De la Sen, M. Approximation of the Solution of Delay Fractional Differential Equation Using AA-Iterative Scheme. *Math.* **10** (2022), no. 2, Article Number 273.

[12] Abdeljawad, T.; Ullah, K.; Ahmad, J.; Arshad, M.; Ma, Z. On the convergence of an iterative process for enriched Suzuki nonexpansive mappings in Banach spaces. *AIMS Math.* **7** (2022), no. 11, 20247–20258.

[13] Agwu, I. K.; Igbokwe, D. I. Convergence theorems and demiclosedness principle for enriched strictly pseuocontractive mappings in real Banach spaces. *Int. J. Nonlinear Anal. Appl.* In Press, 1–11.

[14] Ali, J.; Jubair, M. Fixed points theorems for enriched non-expansive mappings in geodesic spaces. *Filomat* **37** (2023), no. 11, 3403–3409.

[15] Ali, J.; Jubair, M. Existence and estimation of the fixed points of enriched Berinde nonexpansive mappings. *Miskolc Math. Notes* **24** (2023), no. 2, 541–552.

[16] Ali, J.; Jubair, M. Estimation of common fixed points of SKC mappings and an application to fractional differential equations. *J. Anal.* **32** (2024), no. 2, 889–913.

[17] Anjali; Chugh, R.; Batra, C. Fixed point theorems of enriched Ciric's type and enriched Hardy-Rogers contractions, *Numer. Algebra Control Optim.* **2023** Doi: 10.3934/naco.2023022

[18] Anjum, R.; Abbas, M. Common fixed point theorem for modified Kannan enriched contraction pair in Banach spaces and its applications. *Filomat* **35** (2021), no. 8, 2485–2495.

[19] Anjum, R., Fulga, A., Akram, M.W. Applications to Solving Variational Inequality Problems via MR-Kannan Type Interpolative Contractions. *Mathematics* **11** (2023), no. 22, Article No. 4694.

[20] Anjum, R.; Ismail, N.; Bartwal, A. Implication between certain iterative processes via some enriched mappings. *J. Anal.* **31** (2023), no. 3, 2173–2186.

[21] Anjum, R.; Abbas, M.; Safdar, H.; Din, M.; Zhou, M.; Radenovic, S. Application to Activation Functions through Fixed-Circle Problems with Symmetric Contractions, *Symmetry-Basel*, **16** (2024), no. 1, Article No. 69.

[22] Anjum, R.; Din, M.; Zhou, M. Fractals of two types of enriched (q, θ)-Hutchinson-Barnsley operators. *Chaos Solitons Fractals* **181** (2024):10 DOI: 10.1016/j.chaos.2024.114589

[23] Anjum, R.; Khan, S. H. Equivalence of certain iteration processes via averaged mappings. *J. Anal.* **32** (2024), no. 2, 1181–1198.

[24] Babu G. V. R.; Mounika, P. Fixed points of enriched contraction and almost enriched CRR contraction maps with rational expressions and convergence of fixed points. *Proc. Int. Math. Sci.* **5** (2023), no. 1, 5–16.

[25] Babu, G. V. R.; Rao, C. S.; Mounika, P. A fixed point theorem on enriched (ψ, φ_λ)-weakly contractive maps in convex metric spaces. *J. Ramanujan Soc. Math. Math. Sci.* **10** (2022), no. 1, 103–112.

[26] Baillon, J. B.; Bruck, R. E.; Reich, S. On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston J. Math.* **4** (1978), no. 1, 1–9.

[27] Bera, A.; Mondal, P.; Garai, H.; Dey, L. K. On Maia type fixed point results via implicit relation. *AIMS Math.* **8** (2023), no. 9, 22067–22080.

[28] Berinde, V. Error estimates in the approximation of the fixed points for a class of ϕ-contractions. *Studia Univ. Babeş-Bolyai Math.* **35** (1990), no. 2, 86–89.

[29] Berinde, V. Abstract ϕ-contractions which are Picard mappings. *Mathematica (Cluj)* **34(57)** (1992), no. 2, 107–111.

[30] Berinde, V. ϕ-monotone and ϕ-contractive operators in Hilbert spaces. *Studia Univ. Babeş-Bolyai Math.* **38** (1993), no. 4, 51–58.

[31] Berinde, V. *Generalized contractions and applications* (in Romanian), Editura Cub Press 22, Baia Mare, 1997.
[32] Berinde V. Approximating fixed points of weak φ-contractions using the Picard iteration. *Fixed Point Theory*, **4** (2003), 131–142.
[33] Berinde, V. Approximating fixed points of weak contractions using the Picard iteration. *Nonlinear Anal. Forum* **9** (2004), no. 1, 43–53.
[34] Berinde V, Approximation of fixed points of some nonself generalized φ-contractions. *Math. Balkanica (N.S.)* **18** (2004), 85–93.
[35] Berinde, V. *Iterative approximation of fixed points*. Second edition. Lecture Notes in Mathematics, 1912. Springer, Berlin, 2007.
[36] Berinde, V. Weak and strong convergence theorems for the Krasnoselskij iterative algorithm in the class of enriched strictly pseudocontractive operators. *An. Univ. Vest Timiş. Ser. Mat.-Inform.* **56** (2018), no. 2, 13–27.
[37] Berinde, V. Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces. *Carpathian J. Math.* **35** (2019), no. 3, 293–304.
[38] Berinde, V. Approximating fixed points of enriched nonexpansive mappings in Banach spaces by using a retraction-displacement condition. *Carpathian J. Math.* **36** (2020), no. 1, 27–34.
[39] Berinde, V. Maia type fixed point theorems for some classes of enriched contractive mappings in Banach spaces. *Carpathian J. Math.* **38** (2022), no. 1, 35–46.
[40] Berinde, V. A Modified Krasnosel'skiĭ-Mann Iterative Algorithm for Approximating Fixed Points of Enriched Nonexpansive Mappings. *Symmetry-Basel*, **14** (2022), no. 1, Article No. 123.
[41] Berinde, V. Single-Valued Demicontractive Mappings: Half a Century of Developments and Future Prospects. *Symmetry-Basel*, **15** (2023), no. 10, Article No. 1866.
[42] Berinde, V. Approximating fixed points of demicontractive mappings via the quasi-nonexpansive case. *Carpathian J. Math.* **39** (2023), no. 1, 73–85.
[43] Berinde, V. On a useful lemma that relates quasi-nonexpansive and demicontractive mappings in Hilbert spaces. *Creat. Math. Inform.* **33** (2024), no. 1, 7–21.
[44] Berinde, V.; Harjani, J.; Sadarangani, K. Existence and Approximation of Fixed Points of Enriched φ-Contractions in Banach Spaces. *Math.* **10** (2022), no. 21, Article Number 4138.
[45] Berinde, V.; Păcurar, M. Approximating fixed points of enriched contractions in Banach spaces. *J. Fixed Point Theory Appl.* **22** (2020), no. 2, Paper No. 38, 10 pp.
[46] Berinde, V.; Păcurar, M. Kannan's fixed point approximation for solving split feasibility and variational inequality problems. *J. Comput. Appl. Math.* **386** (2021), Paper No. 113217, 9 pp.
[47] Berinde, V.; Păcurar, M. Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces. *J. Fixed Point Theory Appl.* **23** (2021), no. 4, Paper No. 66, 16 pp.
[48] Berinde, V.; Păcurar, M. Fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces and convex metric spaces. *Carpathian J. Math.* **37** (2021), no. 2, 173–184.
[49] Berinde, V.; Păcurar, M. Existence and Approximation of Fixed Points of Enriched Contractions and Enriched φ-Contractions. *Symmetry-Basel*, **13** (2021), no. 3, Article Number 498.
[50] Berinde, V.; Păcurar, M. Fixed points theorems for unsaturated and saturated classes of contractive mappings in Banach spaces. *Symmetry-Basel*, **13** (2021), no. 4, Article Number 713.
[51] Berinde, V.; Păcurar, M. A new class of unsaturated mappings: Ćirić-Reich-Rus contractions. *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat.* **30** (2022), no. 3, 37–50.
[52] Berinde, V.; Păcurar, M. Krasnoselskij-type algorithms for fixed point problems and variational inequality problems in Banach spaces. *Topology Appl.* **340** (2023), Paper No. 108708, 15 pp.
[53] Berinde, V.; Petruşel, A.; Rus, I. A. Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces. *Fixed Point Theory* **24** (2023), no. 2, 525–540.
[54] Bisht, R.; Singh, J. Fixed point approximations via generalized MR-Kannan mappings in Banach spaces. *Appl. General Topology* **25** (2024), no. 1, 71–77.
[55] Browder, F. E. and Petryshyn, W. V. The solution by iteration of nonlinear functional equations in Banach spaces. *Bull. Amer. Math. Soc.* **72** (1966), 571–575.
[56] Browder, F. E. and Petryshyn, W. V. Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.* **20** (1967), 197–228.
[57] Chandok, S. Existence and convergence of best proximity points for cyclic enriched contractions. *Linear Nonlinear Anal.* **8** (2022), no. 2, 185–195.
[58] Chatterjea, S. K. Fixed-point theorems, *C. R. Acad. Bulgare Sci.* **25** (1972), 15–18.
[59] Chatterjea, S. K. Sur un théorème du point fixe, *Math. Balkanica* **6** (1976), 30–32 (1978).
[60] Ćirić, L. B. Generalized contractions and fixed-point theorems. *Publ. Inst. Math. (Beograd) (N.S.)* **12(26)** (1971), 19–26.
[61] Debnath, P. New common fixed point theorems for Górnicki-type mappings and enriched contractions. *Sao Paulo J. Math. Sci.* **16** (2022), no. 2, 1401–1408.

[62] Dechboon, P.; Khammahawong, K. Best proximity point for generalized cyclic enriched contractions, *Math. Methods Appl. Sci.* **47** (2024), no. 6, 4573–4591.

[63] Deshmukh, A.; Gopal, D.; Rakocević, V. Two new iterative schemes to approximate the fixed points for mappings. *Int. J. Nonlinear Sci. Numer. Simul.* **24** (2023), no. 4, 1265–1309.

[64] Dhivya, P.; Diwakaran, D.; Selvapriya, P. Best proximity points for proximal Górnicki mappings and applications to variational inequality problems. *AIMS Math.* **9** (2024), no. 3, 5886–5904.

[65] Eshi, D., Hazarika, B., Saikia, N., Pant, R. On Reich and Chaterjea type cyclic weakly contraction mappings in metric spaces. *J. Math. Comput. Sci.* **32** (2024), no. 4, 348–357.

[66] Faraji, H.; Radenović, S. Some fixed point results of enriched contractions by Krasnoselskij iterative method in partially ordered Banach spaces. *Trans. A. Razmadze Math. Inst.* **177** (2023), no. 1, 19–26.

[67] Gangwar, A.; Tomar, A.; Sajid, M.; Dimri, R. C. Common fixed points and convergence results for α-Krasnosel'skii mappings. *AIMS Math.* **8** (2023), no. 4, 9911–9923.

[68] García, G. A generalization of the (b,θ)-enriched contractions based on the degree of nondensifiability. *Asian-Eur. J. Math.* **15** (2022), no. 9, Paper No. 2250168, 12 pp.

[69] Gautam, P.; Kumar, S.; Verma, S.; Gulati, S. On enriched Hardy-Rogers contractions in Banach space via Krasnoselskij iteration with an application. *J. Anal.* **32** (2024), no. 2, 1145–1160.

[70] Goel, P.; Singh, D. Fixed point results for a pair of mappings in Banach space for enriched contraction condition with application in integral calculus *AIP Conf. Proc.* 3025, 030013 (2024) https://doi.org/10.1063/5.0201637

[71] Górnicki, J.; Bisht, R. K. Around averaged mappings. *J. Fixed Point Theory Appl.* **23** (2021), no. 3, Paper No. 48, 12 pp.

[72] Hicks, T. L. and Kubicek, J. D. On the Mann iteration process in Hilbert spaces, *J. Math. Anal. Appl.* **59** (1977), 498–504.

[73] Hacioğlu,E.; Gürsoy, F. Existence and data dependence results for the fixed points of multivalued mappings. *arXiv* preprint. arXiv:2108.06811

[74] Inuwa, A. Y.; Kumam, P.; Chaipunya, P.; Salisu, S. Fixed point theorems for enriched Kannan mappings in $\mathrm{CAT}(0)$ spaces. *Fixed Point Theory Algorithms Sci. Eng.* **2023**, Paper No. 13, 21 pp.

[75] Ishikawa, S. Fixed points and iteration of a nonexpansive mapping in a Banach space. *Proc. Amer. Math. Soc.* **59** (1976), no. 1, 65–71.

[76] Jorquera Álvarez, E. D. On monotone pseudocontractive operators and Krasnoselskij iterations in an ordered Hilbert space. *Arab. J. Math. (Springer)* **12** (2023), no. 2, 297–307.

[77] Ju, Y., Zhai, C. Fixed point theorems for (a,b,θ)-enriched contractions. *J. Nonlinear Funct. Anal.* **2023**, no. 1, Article No. 15.

[78] Kannan, R. Some results on fixed points, *Bull. Calcutta Math. Soc.* **60** (1968), 71–76.

[79] Kannan, R. Some results on fixed points. II. *Amer. Math. Monthly* **76** (1969), 405–408.

[80] Kesahorm, T.; Sintunavarat, W. On novel common fixed point results for enriched nonexpansive semigroups. *Thai J. Math.* **18** (2020), no. 3, 1549–1563.

[81] Kesahorm, T.; Sintunavarat, W. On numerical approximation of common fixed points for enriched Kannan semigroups and its experiment. *J. Interdisc. Math.* **25** (2022), no. 1, 15–43.

[82] Khan, S. H. Fixed points of enriched ρ-nonexpansive mappings in modular function spaces *Mat. Vesnik* **76**, (2024), no. 4, 303–310.

[83] Khan, S. H.; Al-Mazrooei, A. E. and Latif, A. Banach Contraction Principle-Type Results for Some Enriched Mappings in Modular Function Spaces. *Axioms*, **12** (2023), no. 6, Article Number 549.

[84] Khuen, W. N.; Hassan, A. S. Fixed point theorems with various enriched contraction conditions in generalized Banach spaces. *J. Al-Qadisiyah Comp. Sci. Math.* **14** (2022), no. 2, 15–26.

[85] Krasnosel'skiĭ, M. A. Two remarks about the method of successive approximations. (Russian) *Uspehi Mat. Nauk (N.S.)* **10** (1955), no. 1(63), 123–127.

[86] Li, C. B.; Cui, Y. N. Rectangular G(b)-Metric Spaces and Some Fixed Point Theorems. *Axioms* **11** (2022), no. 3, Article Number 108.

[87] Maia, M. G. Un'osservazione sulle contrazioni metriche. (Italian) *Rend. Sem. Mat. Univ. Padova* **40** (1968), 139–143.

[88] Marchiş, A. Common fixed point theorems for enriched Jungck contractions in Banach spaces. *J. Fixed Point Theory Appl.* **23** (2021), no. 4, Paper No. 76, 13 pp.

[89] Mondal, P.; Garai, H.; Dey, L. K. On some enriched contractions in Banach spaces and an application. *Filomat* **35** (2021), no. 15, 5017–5029.

[90] Nithiarayaphaks, W.; Sintunavarat, W. On approximating fixed points of weak enriched contraction mappings via Kirk's iterative algorithm in Banach spaces. *Carpathian J. Math.* **39** (2023), no. 2, 423–432.

[91] Okonkwo, A. G.; Araka, N. N.; Ibeh, K. O.; Ibeh, K. O. A Survey on Approximation of Common Fixed Point of Countably Infinite Family of Nonexpansive Mappings. *Int. J. Math. Sci. Optimization: Theory Appl.* **8** (2022), no. 2, 38–57.

[92] Omidire, O. J.; Olatinwo, M. O. Fixed Point and Common Fixed Point of Enriched Hardy-Rogers Contractions in Some Banach Spaces. DOI: 10.21203/rs.3.rs-4084674/v1 (Preprint)

[93] Păcurar, M.; Rus, I. A. Fixed point theory for cyclic ϕ-contractions. *Nonlinear Anal.* **72** (2010), no. 3-4, 1181–1187.

[94] Păcurar, M. Asymptotic stability of equilibria for difference equations via fixed points of enriched Prešić operators. *Demonstr. Math.* **56** (2023), no. 1, Paper No. 20220185, 8 pp.

[95] Panicker, R.; Shukla, R. Stability results for enriched contraction mappings in convex metric spaces. *Abstr. Appl. Anal.* **2022**, Art. ID 5695286, 7 pp.

[96] Panja, S.; Roy, K.; Saha, M. Wardowski type enriched contractive mappings with their fixed points. *J. Anal.* **32** (2024), no. 1, 269–281.

[97] Panja, S.; Saha, M.; Bisht, R. K. Existence of common fixed points of non-linear semigroups of enriched Kannan contractive mappings. *Carpathian J. Math.* **38** (2022), no. 1, 169–178.

[98] Panwar, A.; Lamba, P.; Rakočević, V.; Gopal, D. New fixed point results of some enriched contractions in CAT(0) spaces. *Miskolc Math. Notes* **24** (2023), no. 3, 1477–1493.

[99] Petruşel, A.; Petruşel, G. Fixed point results for decreasing convex orbital operators in Hilbert spaces. *J. Fixed Point Theory Appl.* **23** (2021), no. 3, Paper No. 35, 10 pp.

[100] Phairatchatniyom, P.; Kumam, P.; Berinde, V. A modified Ishikawa iteration scheme for b-enriched nonexpansive mapping to solve split variational inclusion problem and fixed point problem in Hilbert spaces. *Math. Methods Appl. Sci.* **46** (2023), no. 12, 13243–13261.

[101] Popescu, O. A new class of contractive mappings. *Acta Math. Hungar.* **164** (2021), no. 2, 570–579.

[102] Prithvi, B. V.; Katiyar, S. K. Revisiting fractal through nonconventional iterated function systems. *Chaos Solitons Fractals* **170** (2023), Paper No. 113337, 12 pp.

[103] Rawat, S.; Bartwal, A.; Dimri, R. C. Approximation and existence of fixed points via interpolative enriched contractions. *Filomat* **37** (2023), no. 16, 5455–5467.

[104] Rawat, S.; Kukreti, S.; Dimri, R. C. Fixed point results for enriched ordered contractions in noncommutative Banach spaces. *J. Anal.* **30** (2022), no. 4, 1555–1566.

[105] Reich, S. Some remarks concerning contraction mappings. *Canad. Math. Bull.* **14** (1971), 121–124.

[106] Rus, I.A. Some fixed point theorems in metric spaces. *Rend. Istit. Mat. Univ. Trieste* **3** (1971), 169–172 (1972).

[107] Rus, I. A. On a fixed point theorem of Maia. *Studia Univ. Babeş-Bolyai Math.* **22** (1977), 40–42.

[108] Rus, I.A. Basic problem for Maia's theorem. *Seminar on Fixed Point Theory. Univ. "Babeş-Bolyai" Cluj-Napoca*, **81-3** (1981), 112–115.

[109] Rus, I. A. Cyclic representations and fixed points, *Ann. T. Popoviciu Seminar Funct. Eq. Approx. Convexity* **3** (2005) 171–178.

[110] Rzepecki, B. A note on fixed point theorem of Maia. *Studia Univ. Babeş-Bolyai Math.* **25** (1980), no. 2, 65–71.

[111] Rzepecki, B. On fixed point theorems of Maia type. *Publ. Inst. Math. (Beograd) (N.S.)* **28 (42)** (1980), 179–186.

[112] Saleem, N.; Agwu, I. K.; Ishtiaq, U.; Radenović, S. Strong Convergence Theorems for a Finite Family of Enriched Strictly Pseudocontractive Mappings and Phi(T)-Enriched Lipschitizian Mappings Using a New Modified Mixed-Type Ishikawa Iteration Scheme with Error. *Symmetry-Basel*, **14** (2022), no. 5, Article Number 1032.

[113] Salisu, S.; Kumam, P.; Sriwongsa, S. On fixed points of enriched contractions and enriched nonexpansive mappings. *Carpathian J. Math.* **39** (2023), no. 1, 237–254.

[114] Salisu, S.; Berinde, V.; Sriwongsa, S.; Kumam, P. Approximating fixed points of demicontractive mappings in metric spaces by geodesic averaged perturbation techniques. *AIMS Math.* **8** (2023), no. 12, 28582–28600.

[115] Salisu, S.; Kumam, P.; Sriwongsa, S.; Inuwa, A. Y. Enriched multi-valued nonexpansive mappings in geodesic spaces. *Rend. Circ. Mat. Palermo (2)* DOI10.1007/s12215-023-00993-2.

[116] Salisu, S.; Kumam, P.; Sriwongsa, S.; Gopal, D. Enriched asymptotically nonexpansive mappings with center zero. *Filomat* **38** (2024), no. 1, 343–356.

[117] Shukla, R.; Pant, R. Some new fixed point results for monotone enriched nonexpansive mappings in ordered Banach spaces. *Adv. Theory Nonlinear Anal. Appl.* **5** (2021), no. 4, 559–567.

[118] Shukla, R.; Panicker, R. Approximating fixed points of enriched nonexpansive mappings in geodesic spaces. *J. Funct. Spaces* **2022**, Art. ID 6161839, 8 pp.

[119] Shukla, R.; Panicker, R. Some fixed point theorems for generalized enriched nonexpansive mappings in Banach spaces. *Rend. Circ. Mat. Palermo (2)* **72** (2023), no. 2, 1087–1101.

[120] Shukla, R.; Pant, R. Some fixed point results for enriched nonexpansive type mappings in Banach spaces. *Appl. Gen. Topol.* **23** (2022), no. 1, 31–43.

[121] Shukla,R.; Pant,R.; Nashine, H.K.; De la Sen, M. Approximating Solutions of Matrix Equations via Fixed Point Techniques. *Mathematics* **9** (2021), Article No. 2684.

[122] Simsek, E.; Yildirim, I. Krasnoselskii iteration process for approximating fixed points of enriched generalized nonexpansive mappings in Banach spaces. *Carpathian Math. Publ.* **14** (2022), no. 1, 86–94.

[123] Simsek, E.; Yildirim, I. On New Fixed Point Results for Some Classes of Enriched Mappings in N-Banach Spaces. *Commun. Nonlinear Anal.* **1** (2022), 1–9.
[124] Suantai, S.; Chumpungam, D.; Sarnmeta, P.. Existence of fixed points of weak enriched nonexpansive mappings in Banach spaces. *Carpathian J. Math.* **37** (2021), no. 2, 287–294.
[125] Takahashi, W. A convexity in metric space and nonexpansive mappings. I. *Kodai Math. Sem. Rep.* **22** (1970), 142–149.
[126] Turcanu, T.; Postolache, M. On Enriched Suzuki Mappings in Hadamard Spaces, *Mathematics*, **12** (2024), no. 1, Article No. 157.
[127] Turcanu, T.; Postolache, M. On a new approach of enriched operators. *Heliyon* **10** (2024) e27890.
[128] Ullah, K.; Ahmad, J.; Arshad, M.; Ma, Z. H. Approximation of Fixed Points for Enriched Suzuki Nonexpansive Operators with an Application in Hilbert Spaces. *Axioms* **11** (2022), no. 1, Article No. 14.
[129] Zhou, M.; Saleem, N.; Abbas, M. Approximating fixed points of weak enriched contractions using Kirk's iteration scheme of higher order. *J. Inequal. Appl.* **2024**, Paper No. 23, 26 pp.

[1] Department of Mathematics and Computer Science
North University Centre at Baia Mare
Technical University of Cluj-Napoca
Victoriei 76, 430072 Baia Mare Romania
Email address: vasile.berinde@mi.utcluj.ro

[2] Academy of Romanian Scientists, 3 Ilfov, 050044, Bucharest, Romania
Email address: vasile.berinde@gmail.com

[3] Department of Economics and Bussiness Administration
in German Language
Faculty of Economics and Bussiness Administration
Babeş-Bolyai University of Cluj-Napoca
T. Mihali 58-60, 400591 Cluj-Napoca, Romania
Email address: madalina.pacurar@econ.ubbcluj.ro