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Chebyshev polynomial coefficient bounds for a subclass of bi-univalent functions

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ABSTRACT. In this work, considering a general subclass of bi-univalent functions and using the Chebyshev polynomials, we obtain coefficient expansions for functions in this class.

1. INTRODUCTION AND DEFINITIONS

Let $\mathbb{B}$ be the open unit disk defined as $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$, and let A denote the class of functions f that are analytic in $\mathbb{B}$ and satisfy the normalization conditions:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Each function $f \in A$ has a Taylor series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Furthermore, let $\mathcal{S}$ represent the class of all functions in A that are univalent (i.e., one-to-one) in $\mathbb{B}$.

Consider two functions, f_1 and f_2, which belong to A. We say that f_1 is *subordinated* to f_2 if there exists a Schwarz function $\mathsf{c}(z)$ that is analytic in $\mathbb{B}$, and makes $f_1(z) = f_2(\mathsf{c}(z))$ hold true. This relationship is denoted as $f_1(z) \prec f_2(z)$.

Consider f as a function in $\mathcal{S}$. The function $f(z)$ is classified as *bi-univalent* if its inverse, denoted by $f^{-1}(w)$, extends analytically to $|w| < 1$ within the w-plane. That is, according to the *Koebe-One Quarter Theorem* [6], it provides that the image of $\mathbb{B}$ under every univalent function $f \in A$ contains a disk of radius $1/4$. Thus, clearly, every such univalent function $f \in A$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f\left(f^{-1}(w)\right) = w$ $\left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

We use Σ to denote the collection of all such bi-univalent functions in $\mathbb{B}$. Here, we present several examples of functions belonging to the class Σ which have greatly renewed interest in the study of bi-univalent functions:

$$f(z) = \frac{z}{1-z}, \quad g(z) = -\log(1-z), \quad h(z) = \frac{1}{2}\log\frac{1+z}{1-z}$$

with their respective inverses

$$f^{-1}(w) = \frac{w}{1+w}, \quad g^{-1}(w) = \frac{e^w - 1}{e^w}, \quad h^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}.$$

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The theory of bi-univalent functions was first introduced by Lewin [11] in 1967, who estimated the second coefficient of bi-univalent functions, stating that $|a_2| < 1.51$. Since then, substantial research has focused on the initial coefficients of these functions, leading to a rich body of literature. However, little is known about the bounds on the general coefficients $|a_n|$ for $n \geq 4$, with only a few studies addressing this issue. Consequently, the problem of estimating $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2, 3\}$ remains open.

TABLE 1. Summary of findings on coefficients $|a_n|$

Researchers	**Estimates**
Lewin 1967	$\|a_2\| \leq 1.51$ [11]
Brannan and Clunie 1980	$\|a_2\| \leq \sqrt{2}$ [3]
Netanyahu 1969	$maks\|a_2\| = \frac{2}{3}$ [14]
Tan 1984	$\|a_2\| \leq 1.485$ [17]
Various researchers	See [1], [4], [10], [12], [15], [16]
Open Problem	Unresolved [2], [8], [9]

The Chebyshev polynomials form a prominent family of orthogonal polynomials that are indispensable in both numerical analysis and approximation theory. Named after the distinguished Russian mathematician Pafnuty Chebyshev, these polynomials are renowned for their exceptional properties, particularly in minimizing the maximum deviation, making them highly effective in polynomial interpolation and curve fitting. Among the four types of Chebyshev polynomials, the first and second kinds, denoted by $T_n(t)$ and $U_n(t)$ respectively, are the most extensively studied and applied. The Chebyshev polynomials exhibit a deep intrinsic connection with trigonometric functions, as they can be elegantly expressed using cosine and sine functions. This connection not only simplifies their theoretical analysis but also enhances their practical utility in a wide range of applications, including the solution of differential equations, optimization of approximation techniques, and spectral methods in computational mathematics [5], [13]. That is, Chebyshev polynomials are widely employed in both pure and applied mathematics, offering efficient solutions across numerous fields.

The Chebyshev polynomials of the first and second kinds are among the most well-known and widely utilized orthogonal polynomials in mathematics. When defined for a real variable x in the interval $(-1, 1)$, they take the following forms: the Chebyshev polynomials of the first kind, $T_n(t)$, are expressed as

$$T_n(t) = \cos n\theta,$$

while the polynomials of the second kind, $U_n(t)$, are given by

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where the subscript n denotes the polynomial degree and where $t = \cos\theta$.

In the context of bi-univalent functions, we introduce the following class of functions characterized by a specific subordination condition.

Definition 1.1. *A function $f \in \Sigma$ is said to be in the class $H_\Sigma(\lambda, t)$, $\lambda \geq 0$ and $t \in \left(\frac{\sqrt{2}}{2}, 1\right]$, if the following subordinations hold*

$$(1 - \lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec H(z,t) = \frac{1}{1 - 2tz + z^2} \qquad (z \in \mathbb{B}) \tag{1.2}$$

and

$$(1.3)\qquad (1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)\prec H(w,t)=\frac{1}{1-2tw+w^2}\qquad (w\in\mathbb{B}),$$

where $g(w)=f^{-1}(w)$.

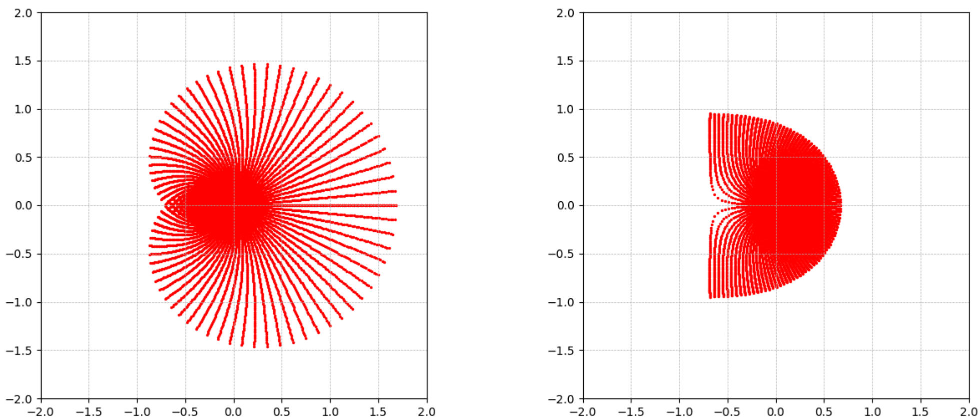
Remark 1.1. *The class* $H_\Sigma(\lambda,t)$ *is obviously not an empty set. Let* $\lambda=0$ *and* $t=1$. *Next, let consider the function*

$$f(z)=z+\frac{\sqrt{2}}{2}z^2,\quad z\in\mathbb{B},$$

and its inverse

$$g(w)=\frac{\sqrt{1+2\sqrt{2}w}-1}{\sqrt{2}},\quad w\in\mathbb{B}.$$

The following Figures show that the transformation of $\mathbb{B}$ *under* $f(z)$ *and* $g(w)$*, respectively.*



(A) The transformation of $\mathbb{B}$ under $f(z)$

(B) The transformation of $\mathbb{B}$ under $g(w)$

Remark 1.2. *If* $t=\cos\alpha,\ \alpha\in\left(-\frac{\pi}{3},\frac{\pi}{3}\right)$, *then*

$$\begin{aligned} H(z,t) &= \frac{1}{1-2tz+z^2} \\ &= 1+\sum_{n=1}^{\infty}\frac{\sin(n+1)\alpha}{\sin\alpha}z^n \quad (z\in\mathbb{B}). \end{aligned}$$

Thus

$$H(z,t)=1+2\cos\alpha z+(3\cos^2\alpha-\sin^2\alpha)z^2+\cdots\quad (z\in\mathbb{B}).$$

Following this, we write

$$H(z,t)=1+U_1(t)z+U_2(t)z^2+\cdots\quad (z\in\mathbb{B},\ \ t\in(-1,1)),$$

where $U_{n-1}=\dfrac{\sin(n\arccos t)}{\sqrt{1-t^2}}$ $(n\in\mathbb{N})$ *are the Chebyshev polynomials of the second kind. Also it is known that*

$$U_n(t)=2tU_{n-1}(t)-U_{n-2}(t),$$

and

$$\begin{aligned} &U_1(t) = 2t, \\ &U_2(t) = 4t^2 - 1, \\ &U_3(t) = 8t^3 - 4t, \\ &\vdots \end{aligned} \tag{1.4}$$

The Chebyshev polynomials $T_n(t),\ t \in [-1,1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1 - tz}{1 - 2tz + z^2} \quad (z \in \mathbb{B}).$$

However, the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are well connected by the following relationships

$$\begin{aligned} &\frac{dT_n(t)}{dt} = nU_{n-1}(t), \\ &T_n(t) = U_n(t) - tU_{n-1}(t), \\ &2T_n(t) = U_n(t) - U_{n-2}(t). \end{aligned}$$

In this paper, inspired by the earlier works, we employ Chebyshev polynomial expansions to derive estimates for the initial coefficients of bi-univalent functions in the class $H_\Sigma(\lambda, t)$. Additionally, we address the Fekete-Szegö problem for functions within this class.

2. Coefficient bounds for the function class $H_\Sigma(\lambda, t)$

Section 2 focuses on establishing coefficient bounds for the function class $H_\Sigma(\lambda, t)$, which generalizes certain well-known subclasses through the parameters λ and t.

Theorem 2.1. *A function $f(z)$ given by (1.1) is in the class $H_\Sigma(\lambda, t)$, if the next conditions hold true:*

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{\left|(1+\lambda)^2 - 4(\lambda + \lambda^2)t^2\right|}}, \quad \lambda \in \left[0, \frac{1}{3}\right) \cup [1, \infty)$$

and

$$|a_3| \le \frac{4t^2}{(1+\lambda)^2} + \frac{t}{1+2\lambda}.$$

Proof. Let $f \in H_\Sigma(\lambda, t)$. From (1.2) and (1.3), we have

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + U_1(t)\psi(z) + U_2(t)\psi^2(z) + \cdots, \tag{2.5}$$

and

$$(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots, \tag{2.6}$$

for some analytic functions ψ, v such that $\psi(0) = v(0) = 0$ and $|\psi(z)| < 1, |v(w)| < 1$ for all $z, w \in \mathbb{B}$. From the equalities (2.5) and (2.6), we obtain that

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + U_1(t)c_1 z + \left[U_1(t)c_2 + U_2(t)c_1^2\right] z^2 + \cdots, \tag{2.7}$$

and

(2.8) $$(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)=1+U_1(t)d_1w+\left[U_1(t)d_2+U_2(t)d_1^2\right]w^2+\cdots.$$

It is fairly well-known that if

$$|\psi(z)|=\left|c_1z+c_2z^2+c_3z^3+\cdots\right|<1$$

and

$$|v(w)|=\left|d_1w+d_2w^2+d_3w^3+\cdots\right|<1,$$

then

$$|c_j|\leq 1,\quad \forall j\in\mathbb{N}.$$

It follows from (2.7) and (2.8) that

(2.9) $$(1+\lambda)\,a_2=U_1(t)c_1,$$

(2.10) $$2\,(1+2\lambda)\,a_3-(1+3\lambda)\,a_2^2=U_1(t)c_2+U_2(t)c_1^2,$$

and

(2.11) $$-\,(1+\lambda)\,a_2=U_1(t)d_1,$$

(2.12) $$2\,(1+2\lambda)\left(2a_2^2-a_3\right)-(1+3\lambda)\,a_2^2=U_1(t)d_2+U_2(t)d_1^2.$$

From (2.9) and (2.11) we obtain

(2.13) $$c_1=-d_1$$

and

(2.14) $$2\,(1+\lambda)^2\,a_2^2=U_1^2(t)\left(c_1^2+d_1^2\right).$$

By adding (2.10) to (2.12), we get

(2.15) $$\left[4\,(1+2\lambda)-2\,(1+3\lambda)\right]a_2^2=U_1(t)\,(c_2+d_2)+U_2(t)\left(c_1^2+d_1^2\right).$$

By using (2.14) in equality (2.15), we have

(2.16) $$\left[2\,(1+\lambda)-\frac{2U_2(t)}{U_1^2(t)}\,(1+\lambda)^2\right]a_2^2=U_1(t)\,(c_2+d_2)\,.$$

From (1.4) and (2.16) we get

$$|a_2|\leq\frac{2t\sqrt{2t}}{\sqrt{\left|(1+\lambda)^2-4\left(\lambda+\lambda^2\right)t^2\right|}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we obtain

(2.17) $$4\,(1+2\lambda)\,a_3-4\,(1+2\lambda)\,a_2^2=U_1(t)\,(c_2-d_2)+U_2(t)\left(c_1^2-d_1^2\right).$$

Then, in view of (2.13) and (2.14) , we have from (2.17)

$$a_3=\frac{U_1^2(t)}{2\,(1+\lambda)^2}\left(c_1^2+d_1^2\right)+\frac{U_1(t)}{4\,(1+2\lambda)}\,(c_2-d_2)\,.$$

Notice that (1.4), we get

$$|a_3|\leq\frac{4t^2}{(1+\lambda)^2}+\frac{t}{1+2\lambda}.$$

□

3. Fekete-Szegö inequalities for the function class $H_{\Sigma}(\lambda, t)$

The task of identifying precise bounds for $|a_3 - \mu a_2^2|$ within any compact family of functions is termed the *Fekete-Szegö* problem. Notably, when $\mu = 1$, this formulation represents the Schwarzian derivative, which holds considerable importance in the realm of geometric function theory [7]. In this section, we derive Fekete-Szegö inequalities for the function class $H_{\Sigma}(\lambda, t)$.

Theorem 3.2. *Let $\lambda \in \left[0, \frac{1}{3}\right) \cup [1, \infty)$ and $f \in H_{\Sigma}(\lambda, t)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{t}{1+2\lambda}; & \mu \leq \frac{4(\lambda^2+5\lambda+2)-(1-\lambda)^2}{8(1+2\lambda)} \\ \dfrac{8(1-\mu)t^3}{(1+\lambda)^2 - 4t^2(\lambda+\lambda^2)}; & \mu \geq \frac{4(\lambda^2+5\lambda+2)-(1-\lambda)^2}{8(1+2\lambda)} \end{cases}.$$

Proof. From (2.16) and (2.17), we arrive

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\frac{U_1^3(t)(c_2+d_2)}{2(1+\lambda)U_1^2(t) - 2U_2(t)(1+\lambda)^2} + \frac{U_1(t)(c_2-d_2)}{4(1+2\lambda)} \\ &= U_1(t)\left[\left(h(\mu) + \frac{1}{4(1+2\lambda)}\right)c_2 + \left(h(\mu) - \frac{1}{4(1+2\lambda)}\right)d_2\right], \end{aligned}$$

where

$$h(\mu) = \frac{U_1^2(t)(1-\mu)}{2\left[(1+\lambda)U_1^2(t) - U_2(t)(1+\lambda)^2\right]}.$$

Then, in view of (1.4), we conclude that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{t}{1+2\lambda}; & h(\mu) \leq \dfrac{1}{4(1+2\lambda)} \\ 4th(\mu); & h(\mu) \geq \dfrac{1}{4(1+2\lambda)} \end{cases}.$$

Taking $\mu = 1$ we get

Corollary 3.1. *If $f \in H_{\Sigma}(\lambda, t)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{t}{1+2\lambda}.$$

□

4. Conclusions

In this paper, we explored the coefficient estimates for bi-univalent functions in the class $H_{\Sigma}(\lambda, t)$, a subclass of bi-univalent functions characterized by a subordination condition involving Chebyshev polynomials. By applying techniques inspired by the properties of these polynomials, we derived bounds for the initial coefficients of the functions in this class. Specifically, we obtained new estimates for $|a_2|$ and $|a_3|$, providing significant insight into the behavior of these coefficients under the influence of the parameters λ and t.

The use of Chebyshev polynomials allowed us to extend existing results and generalize certain well-known subclasses of bi-univalent functions. Additionally, the Fekete-Szegö problem for functions within this class was also addressed, further enriching the field of study.

The bounds obtained in this work contribute to the broader understanding of the coefficient estimation problem in bi-univalent function theory. However, there remain open questions regarding the higher-order coefficients, particularly for $n \geq 4$. Future research could focus on deriving tighter bounds for these general coefficients, exploring new subclasses, and investigating further applications of orthogonal polynomials in function theory.

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