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On the Anti-Adjacency Spectra of Regular Graphs

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ABSTRACT. For a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$, the anti-adjacency matrix, denoted by $A^*(G)$ is a square matrix of order n with rows and columns indexed by $V(G)$, whose $(i,j)-$ entry $(i \neq j)$ is 1, if the vertices v_i and v_j are not adjacent and 0, otherwise. The diagonal entries of $A^*(G)$ is 1. The eigenvalues obtained from $A^*(G)$ are called the anti-adjacency eigenvalues of the graph G and the corresponding spectra is called the anti-adjacency spectra, denoted by a-spec(G). In this paper, we discuss the anti-adjacency spectra of connected and disconnected regular graphs and their complement graphs.

1. INTRODUCTION

For definitions and concepts in graph theory, we refer to [14, 6]; for the concepts and results in linear algebra, see [8, 13]. For further topics in spectral graph theory, see [3, 12]. Unless mentioned otherwise, all graphs mentioned in this paper are simple, finite and undirected.

Studying the structural properties of graphs using different matrices and their associated spectra has been an area of research since a few decades. Structural properties of graphs and their characterisation is possible by studying the eigenvalues and eigenspaces of different matrices associated with graphs.

Some commonly studied matrices include adjacency matrix, Laplacian matrix, incidence matrix and distance matrix. Various results and characterisation of graph properties have been obtained in term of the eigenvalues and spectra of the adjacency matrix.

The anti-adjacency matrix, in contrast to the adjacency matrix had been defined for directed graphs (see [3]). The anti-adjacency eigenvalues for various directed graph classes had been studied in the literature (see [1, 2, 4, 7, 10, 11, 15]). While the anti-adjacency eigenvalues have been explored primarily in the context of directed graph classes, their spectral properties for undirected graphs remain largely unexamined. Existing studies on undirected graphs have focused on determinant and structure of anti-adjacency matrix with less focus on the eigenvalues obtained from this matrix. This lack of investigation presents a clear research gap, highlighting the potential for new insights through the study of the anti-adjacency spectra of undirected graph classes. Spectral study of the anti-adjacency matrix can be used to study the graph properties from a dual perspective; that is, in terms of the adjacency as well as non-adjacency of the vertices.

The anti-adjacency spectra for some undirected graph classes and the relation between the anti-adjacency eigenvalues and some graph parameters have been studied in [9].

Motivated by the studies mentioned above, in this paper, we focus on the anti-adjacency spectra of connected and disconnected regular graphs and their complements. We also characterise regular graphs, based on the anti-adjacency spectra.

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2. ANTI-ADJACENCY MATRIX OF GRAPHS

For a matrix A, the transpose of the matrix is denoted by A^T. The trace of the matrix A is denoted by $tr(A)$. We denote the column vector of order n and all 1's by 1_n. We denote the matrix of all 1's of corresponding order by J.

Definition 2.1. [5] Let G be a graph on n vertices, with $V(G) = \{v_1, \ldots, v_n\}$ and adjacency matrix $A(G)$. Then, the *anti-adjacency matrix*, $A^*(G)$ is a square matrix of order n, with rows and columns indexed by the $V(G)$, and each $(i,j)-$ entry $(i \neq j)$ defined as follows:

$$a^*_{i,j} = \begin{cases} 1 & \text{if there is no edge between } v_i \text{ and } v_j, \\ 0 & \text{if there is an edge between } v_i \text{ and } v_j. \end{cases}$$

The diagonal entries $(i = j)$ of $A^*(G)$ is equal to 1.

More specifically, the anti-adjacency matrix of G is given by, $A^*(G) = J - A(G)$. If A^* and I are the anti-adjacency matrix of G and the identity matrix of order n, respectively, then the polynomial $\phi(G,\eta) = |A^* - \eta I|$ is called the *anti-adjacency characteristic polynomial* of G. The eigenvalues $\eta_1, \eta_2, \ldots, \eta_n$ of $A^*(G)$ are called the *anti-adjacency eigenvalues* of the graph G. The eigenvectors corresponding to these eigenvalues are called *anti-adjacency eigenvectors*. The *anti-adjacency spectrum* of a graph G, denoted by a-spec(G), is the list of distinct eigenvalues with their multiplicities. If $\eta_1, \eta_2, \ldots, \eta_t$ are the distinct anti-adjacency eigenvalues of the graph G with multiplicities $m_1, m_2, \ldots, m_t$, respectively, then, we write the anti-adjacency spectrum as a-spec$(G) = \{\eta_1^{(m_1)}, \eta_2^{(m_2)}, \ldots, \eta_t^{(m_t)}\}$ (see [9]).

The anti-adjacency eigenvalues are always real, and hence are assumed to be arranged in non-increasing order as $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$. Hence, $\eta_1 (= \eta_{\max}(G))$ is the maximum anti-adjacency eigenvalue and $\eta_n (= \eta_{\min}(G))$ is the minimum anti-adjacency eigenvalue.

In the context of adjacency matrix, the *main eigenvalue* of a graph G on n vertices, is that eigenvalue of G, whose corresponding eigenvector is not orthogonal to the vector 1_n (see [12]). Similarly, we define the *main anti-adjacency eigenvalue* for a graph G on n vertices as follows:

Definition 2.2. The anti-adjacency eigenvalue of a graph G on n vertices, is said to be *main anti-adjacency eigenvalue* if the associated anti-adjacency eigenvector is not orthogonal to the vector 1_n. Otherwise, the anti-adjacency eigenvalue is non-main.

In the study of adjacency spectra of graphs, the *adjacency algebra* of a graph G, is the algebra generated by the adjacency matrix $A(G)$ (see [3]). In a similar context, we define the *anti-adjacency algebra* of a graph G as follows:

Definition 2.3. For a graph G, with anti-adjacency matrix A^*, the *anti-adjacency algebra* of G is defined as the set of all linear combinations of the powers of A^*; that is, the set of all linear combinations of the matrices $I, A^*, (A^*)^2, \ldots$.

The following result on the anti-adjacency eigenvalues is significant and relevant in the context of our study:

Theorem 2.1. *For a graph G on n vertices, having at least one pair non-adjacent vertices, $-\Delta(G) \leq \eta_{\min}(G) \leq 0$, where $\Delta(G)$ is the maximum degree of G.*

Proof. Let G be the graph on n vertices and $A^*(G)$ be the corresponding anti-adjacency matrix. Let $S \subseteq V(G)$ be the set of pairwise non-adjacent vertices in G. Without loss of generality, let the first $|S|$ rows (and columns) of $A^*(G)$ correspond to the vertices of S. Then, the matrix J of order $|S|$ forms the upper left principal submatrix of $A^*(G)$. Thus, by interlacing properties relating the eigenvalues of a symmetric matrix and of its principal submatrix, we have, $\eta_{\min}(G)$ is bounded by the minimum eigenvalue of $J_{|S|}$. Hence, $\eta_{\min}(G) \leq 0$.

To prove to left side inequality, let $X = (x_1, x_2, \ldots, x_n)^T$ be the eigenvector for the eigenvalue $\eta_n = \eta_{\min}(G)$. Then, $A^*(G)X = \eta_n X$. Therefore, from the $i-$th equation of this vector equation, we get

$$\eta_n x_i = \sum_{j \not\sim i} x_j; i = 1, 2, \ldots, n.$$

Taking the absolute value on both sides and using triangle inequality, we have

$$|\eta_n||x_i| \leq \sum_{j \not\sim i} |x_j|; i = 1, 2, \ldots, n.$$

We consider the i-th equation such that the right hand side of the above equation is minimum. This is possible only when the corresponding vertex has less anti-adjacency, or in other words, more adjacency; that is, we consider the i-th equation corresponding to the vertex of maximum degree, $\Delta(G)$. Then, we have

$$|\eta_n||x_i| \leq \Delta(G)|x_i|; deg(v_i) = \Delta(G).$$

Therefore, $|\eta_n| = |\eta_{\min}(G)| \leq \Delta(G)$, proving the left side inequality. □

Remark 2.1. In this paper, the anti-adjacency eigenvectors of graphs need not be directly obtained from the vector equation $A^*X = \eta X$, but can also be a linear combination of the vectors present in the same eigenspace of the anti-adjacency eigenvalue η.

3. Anti-Adjacency Spectra of Regular Graphs

The present section discusses the anti-adjacency spectra of k-regular graphs. Let G be a simple undirected graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$, and let A^* be the anti-adjacency matrix of G. The graph G is said to be k-regular if all the vertices has the same degree, equal to k.

The following theorem discusses the value of the maximum anti-adjacency eigenvalue $\eta_{\max}(G)$ and its associated anti-adjacency eigenvector for a k-regular graph G, possibly disconnected.

Theorem 3.2. *For any k- regular graph G, possibly disconnected, on n vertices, $\eta_1 = \eta_{\max}(G) = n - k$ with algebraic multiplicity 1. The anti-adjacency eigenvector associated with the anti-adjacency eigenvalue $n - k$ in G is 1_n.*

Proof. Let A^* be the anti-adjacency matrix of G. Since G is k- regular, by the definition of the anti-adjacency matrix, every row of A^* has exactly $n - k$ units. Therefore, $n - k$ is an anti-adjacency eigenvalue of G.

By Corollary 2.6 in [9], we have for any graph G on n vertices $n - \Delta(G) \leq \eta_{\max}(G) \leq n - \delta(G)$. In the present context, we have $\delta(G) = \Delta(G) = k$. Therefore, $n - k \leq \eta_{\max}(G) \leq n - k$.

Combining the above arguments, we get, for any k- regular graph G on n vertices, $\eta_{\max}(G) = n - k$.

We shall now prove that the algebraic multiplicity of the anti-adjacency eigenvalue $n - k$ is 1. Since A^* is a real and symmetric matrix, it is enough to prove that the geometric multiplicity of $n - k$ is 1; that is, the dimension of the null space of $A^* - \eta_{\max} I$ is 1. Let E denote the eigenspace of the eigenvalue $\eta_{\max} = n - k$. Then, $E = \{X \in \mathbb{R}^n | (A^* - \eta_{\max} I)X = 0\}$. The geometric multiplicity of $\eta_{\max}$ is the dimension of E.

It can be noted that E is the null space of $A^* - \eta_{\max} I$. By the rank-nullity theorem, the dimension of E is given by

$$dim(E) = n - dim(\mathcal{R}(A^* - \eta_{\max} I))$$

where $\mathcal{R}(A^* - \eta_{\max} I)$ is the range of $A^* - \eta_{\max} I$.

In other words, the geometric multiplicity, $dim(E)$ can be evaluated by calculating the number of linearly independent columns of $A^* - \eta_{\max} I$. This can be done by calculating the reduced row echelon form of $A^* - \eta_{\max} I$.

It can be easily verified that the row reduced echelon form of $A^* - \eta_{\max} I$ has exactly $n-1$ non-zero rows. Hence, we get, $dim(\mathcal{R}(A^* - \eta_{\max} I)) = n - 1$. Consequently, $dim(E) = 1$.

Let $X = (x_1, x_2, \ldots, x_n)^T$ be the anti-adjacency eigenvector associated with the anti-adjacency eigenvalue $n - k$. Then, by the equation $A^* X = (n - k)X$, the only possible value for x_i is 1, for $i = 1, 2, \ldots, n$, completing the proof. □

Let η be an anti-adjacency eigenvalue of the graph G, mentioned in Theorem 3.2, such that 1_n is the corresponding anti-adjacency eigenvector. Then, $A^* 1_n = \eta 1_n$. It holds only when $deg(v_i) = n - \eta; 1 \leq i \leq n$ and $\eta = n - k$; that is, G is k- regular graph. Combining this observation with Theorem 3.2, we get the following characterisation of k- regular graph G, which can possibly be disconnected also.

Theorem 3.3. *A graph G of order n, possibly disconnected, is k- regular if and only if 1_n is the anti-adjacency eigenvector of G corresponding to the anti-adjacency eigenvalue $n - k$.*

We know that, $tr(A^*) = \sum_{i=1}^{n} \eta_i = n$ and $tr(A^*)^2 = \sum_{i=1}^{n} \eta_i^2 = n^2 - 2m$ (see [9]). Hence, the following result is immediate.

Theorem 3.4. *A graph G of order n, possibly disconnected, is k- regular if and only if $n - k = \eta_1$ and $n\eta_1 = \sum_{i=1}^{n} \eta_i^2$.*

By Theorem 3.4, we infer that the regularity of a graph can be recognised from its anti-adjacency spectrum.

The following theorem discusses the value of the minimum anti-adjacency eigenvalue $\eta_{\min}(G)$ and its associated anti-adjacency eigenvector for a disconnected k-regular graph $G = \bigcup_{i=1}^{r} G_i (r > 1)$.

Note that, for a connected k-regular graph, there is no generalisation for the minimum anti-adjacency eigenvalue.

Theorem 3.5. *For any disconnected k- regular graph $G = \bigcup_{i=1}^{r} G_i (r > 1)$ on n vertices, such that $|V(G_i)| = n_i; i = 1, \ldots, r$, $\eta_n = \eta_{\min}(G) = -k$ with algebraic multiplicity $r - 1$. The anti-adjacency eigenvectors associated with the anti-adjacency eigenvalue $-k$ in G are given by column vectors of n rows indexed by $V(G) = \bigcup_{i=1}^{r} V(G_i)$, in order, with first n_1 rows having the value $\frac{-n_j}{n_1}$, the rows corresponding to the vertices of the component G_j, $j = 2, \ldots, r$, having the value 1, and 0 otherwise.*

Proof. Let $G = \bigcup_{i=1}^{r} G_i (r > 1)$ be the disconnected k- regular graph on n vertices, such that $|(V(G_i)| = n_i; i = 1, \ldots, r$. Let $A^*(G_i)$ denote the anti-adjacency matrix of G_i, $i = 1, \ldots, r$. The anti-adjacency matrix of G is given by A^*.

Let X_j, $j = 2, \ldots, r$, denote the column vector of n rows indexed by $V(G) = \bigcup_{i=1}^{r} V(G_i)$, in order, with first n_1 rows having the value $\frac{-n_j}{n_1}$, the rows corresponding to the vertices of the component G_j, $j = 2, \ldots, r$, having the value 1, and 0 otherwise.

The anti-adjacency matrix A^*, for $G = \bigcup_{i=1}^{r} G_i (r > 1)$, is given by,

$$A^* = \begin{pmatrix} A^*(G_1) & J_{n_1 \times n_2} & J_{n_1 \times n_3} & \ldots & J_{n_1 \times n_{r-1}} & J_{n_1 \times n_r} \\ J_{n_2 \times n_1} & A^*(G_2) & J_{n_2 \times n_3} & \ldots & J_{n_2 \times n_{r-1}} & J_{n_2 \times n_r} \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ J_{n_r \times n_1} & J_{n_r \times n_2} & J_{n_r \times n_3} & \ldots & J_{n_r \times n_{r-1}} & A^*(G_r) \end{pmatrix};$$

that is, A^* is a block matrix, such that the matrices $A^*(G_i)$, $i = 1, \ldots, r$ are the diagonal blocks, and the matrix J of respective orders are the off-diagonal blocks.

Consider the product $A^* X_j$; $2 \leq j \leq r$. We obtain a column vector of n rows indexed by $V(G)$ with entries as follows:

The elements in the first n_1 rows of the resulting vector are obtained by the matrix multiplication of first n_1 rows of A^* with X_j. The first n_1 rows of A^* are given by the block matrix:

$$\begin{pmatrix} A^*(G_1) & J_{n_1 \times n_2} & J_{n_1 \times n_3} & \ldots & J_{n_1 \times n_{r-1}} & J_{n_1 \times n_r} \end{pmatrix}.$$

Hence, the elements in the first n_1 rows of the resulting vector is given by $\frac{-n_j}{n_1}(n_1 - k) + n_j = \frac{n_j}{n_1} k$.

The elements in the rows corresponding to the vertices of the component G_j in the resulting vector are obtained by the matrix multiplication of the rows corresponding to the vertices of the component G_j in A^* with X_j. The rows corresponding to the vertices of the component G_j in A^* are given by the block matrix:

$$\begin{pmatrix} J_{n_j \times n_1} & J_{n_j \times n_2} & \ldots & A^*(G_j) & \ldots & J_{n_j \times n_r} \end{pmatrix}.$$

Hence, the elements in the rows corresponding to the vertices of the component G_j in the resulting vector is given by $n_1 \left(\frac{-n_j}{n_1} \right) + (n_j - k) = -k$.

The elements in the rows, other than discussed above, in the resulting vector are obtained by the matrix multiplication of the rows corresponding to the vertices of G, other than G_1 and G_j, with X_j. Hence, the elements in these rows are given by $n_1 \left(\frac{-n_j}{n_1} \right) + n_j = 0$.

It can be easily verified that $-k$ is a common factor between all the elements of the resulting vector. Factoring out $(-k)$ results in the matrix X_j. Hence, we obtain $A^* X_j = (-k) X_j$, for $j = 2, \ldots, r$. Therefore, $(-k)$ is an anti-adjacency eigenvalue of G, with X_j, for $j = 2, \ldots, r$ as the corresponding anti-adjacency eigenvectors.

Since, G is k- regular, we have $\Delta(G) = k$. Combining the above observation and Theorem 2.1, we have, $\eta_{\min}(G) = -k$.

We shall now prove that the algebraic multiplicity of the anti-adjacency eigenvalue $-k$ is $r - 1$. For this, it is enough to prove that the geometric multiplicity of $-k$ is $r - 1$; that is, the dimension of the null space of $A^* - \eta_{\min} I$ is $r - 1$. Let E denote the eigenspace of the eigenvalue $\eta_{\min} = -k$. Then, $E = \{X \in \mathbb{R}^n | (A^* - \eta_{\min} I) X = 0\}$. The geometric multiplicity of $\eta_{\min}$ is the dimension of E.

It can be noted that E is the null space of $A^* - \eta_{\min} I$. By the rank-nullity theorem, the dimension of E is given by

$$dim(E) = n - dim(\mathcal{R}(A^* - \eta_{\min} I))$$

where $\mathcal{R}(A^* - \eta_{\min} I)$ is the range of $A^* - \eta_{\min} I$.

In other words, the geometric multiplicity, $dim(E)$ can be evaluated by calculating the number of linearly independent columns of $A^* - \eta_{\min} I$. This can be done by calculating the reduced row echelon form of $A^* - \eta_{\min} I$.

It can be easily verified that the row reduced echelon form of $A^* - \eta_{\min} I$ has exactly $n - r + 1$ non-zero rows. Hence, we get, $dim(\mathcal{R}(A^* - \eta_{\min} I)) = n - r + 1$. Consequently, $dim(E) = r - 1$. □

The following corollary is trivial by Theorem 3.5.

Corollary 3.1. *The number of components of a k- regular graph is equal to one more than the algebraic multiplicity of $-k$, as the anti-adjacency eigenvalue.*

The following theorem discusses the nature of the anti-adjacency eigenvectors for a k-regular graph, possibly disconnected.

Theorem 3.6. *For any k- regular graph G on n vertices, the sum of entries in the anti-adjacency eigenvector associated with the anti-adjacency eigenvalue η_p, for $p = 2, \ldots, n$ ($\eta_p \neq \eta_{\max}(G)$) vanishes.*

Proof. Let G be a k- regular graph on n vertices and let A^* be its anti-adjacency matrix. Let η_p, for $p = 2, \ldots, n$ be an anti-adjacency eigenvalue of G, such that $\eta_p \neq \eta_{\max}(G)$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the associated anti-adjacency eigenvector. Then, $A^*X = \eta_p X$.

The i-th equation of this vector equation is given by

$$\eta_p x_i = \sum_{j \not\sim i} x_j; i = 1, 2, \ldots n.$$

or,

$$\eta_p x_i - \sum_{j \not\sim i} x_j = 0; i = 1, 2, \ldots n. \tag{3.1}$$

A^* is a real symmetric matrix with a constant row (or column) sum equal to $n - k$. Hence, adding all the n equations of 3.1, we get

$$\eta_p \left(\sum_{i=1}^{n} x_i \right) - (n-k) \left(\sum_{i=1}^{n} x_i \right) = 0$$

This is true only if $\eta_p = n - k$ or $\sum_{i=1}^{n} x_i = 0$. Since, $\eta_p \neq \eta_{\max}(G) = (n - k)$, we have, $\sum_{i=1}^{n} x_i = 0$, completing the proof. □

By Theorem 3.6, for any graph G of order n, the anti-adjacency eigenvectors associated with any anti-adjacency eigenvalues η_p, for $p \neq 1$, (that is, $\eta_p \neq \eta_{\max}(G)$), are orthogonal to the vector 1_n. Thus, these anti-adjacency eigenvalues are non-main.

In the case of a k- regular graph, every anti-adjacency eigenvector which is not spanned by 1_n, is orthogonal to 1_n. Consequently, we have the following corollary.

Corollary 3.2. *Graphs, possibly disconnected, with exactly one main anti-adjacency eigenvalue are regular graphs with the same being the largest one.*

For a disconnected graph $G = \bigcup_{i=1}^{r} G_i (r > 1)$ on n vertices, Theorem 3.2 and Theorem 3.5 combine to give r anti-adjacency eigenvalues of G; that is, $\eta_1 = n - k, \eta_{n-r+2} = \eta_{n-r+3} = \ldots = \eta_{n-1} = \eta_n = -k$. The following theorem discusses the remaining $n - r$ eigenvalues of G; that is, the value of η_p, for $p = 2, \ldots, n - r + 1$.

Theorem 3.7. *The anti-adjacency eigenvalues* $\eta_p, p = 2, \ldots, n-r+1$, *for any disconnected* k-*regular graph* $G = \bigcup_{i=1}^{r} G_i (r > 1)$ *on* n *vertices, with* $|(V(G_i)| = n_i; i = 1, \ldots, r$, *are precisely the last* $n_i - 1$ *anti-adjacency eigenvalues, in the non-increasing order, of its components* G_i *and their multiplicities are sums of the corresponding multiplicities in each component* $G_i; i = 1, \ldots, r$.

Proof. Let $G = \bigcup_{i=1}^{r} G_i (r > 1)$ be the disconnected k- regular graph on n vertices, such that $|(V(G_i)| = n_i; i = 1, \ldots, r$. Let $A^*(G_i)$ denote the anti-adjacency matrix of G_i, $i = 1, \ldots, r$. The anti-adjacency matrix of G is given by $A^*(G)$.

Let $S(G_i)$ be the set of the last $n_i - 1$ anti-adjacency eigenvalues, in the non-increasing order, of the component G_i, and $\mathcal{X}(G_i)$ be the set of anti-adjacency eigenvectors associated with the anti-adjacency eigenvalues in $S(G_i)$, $i = 1, \ldots, r$.

We shall prove that the elements of $\bigcup_{i=1}^{n} S(G_i)$ are the anti-adjacency eigenvalues of G. Let $\eta \in \bigcup_{i=1}^{n} S(G_i)$. Then, η is an anti-adjacency eigenvalue of one of the components of G, say G_i. Let $X \in \mathcal{X}(G_i)$ be the anti-adjacency eigenvector associated with η.

Construct a column vector Y of n rows indexed by $V(G)$, such that the block corresponding to vertices of G_i is the column vector X and the remaining rows has the entry 0.

Consider the product $A^*(G)Y$. We obtain a column vector of n rows indexed by $V(G)$ with entries as follows:

The rows corresponding to the vertices of G_i in the resulting vector is precisely the block matrix ηX. The entries in the remaining rows is the sum of entries of X. By Theorem 3.6, we have, the entries in the remaining rows is 0.

Hence, $A^*(G)Y = \eta Y$. Therefore, the elements of $\bigcup_{i=1}^{n} S(G_i)$ are the anti-adjacency eigenvalues of G, and the corresponding anti-adjacency eigenvectors are given by the column vector Y, as shown. $\square$

4. Anti-Adjacency Algebra of Regular Graphs

In the following theorem, we characterise regular graphs, possibly disconnected, in terms of their anti-adjacency algebra.

Theorem 4.8. *A graph* G *of order* n, *possibly disconnected, is regular if and only if the matrix* J *belongs to the anti-adjacency algebra of* G.

Proof. Let G be the graph on n vertices, possibly disconnected. Let A^* be its anti-adjacency matrix. Let $deg(v_i)$ denote the degrees of the vertex v_i in G, for $i = 1, 2, \ldots, n$.

First assume that G is k- regular graph. Then, by Theorem 3.2, $n - k$ is an anti-adjacency eigenvalue of G. Let $\psi(G, \eta)$ be the minimal polynomial of A^*. We have, $\psi(G, \eta) = (\eta - (n-k))f(\eta)$, where $f(\eta)$ is some polynomial in η. Since $\psi(G, A^*) = 0$. Therefore, $A^* f(A^*) = (n-k)f(A^*)$. This is true when each column of $f(A^*)$ is an anti-adjacency eigenvector of A^* associated with the anti-adjacency eigenvalue $n-k$. By Theorem 3.2, each column of $f(A^*)$ must be 1_n or a multiple of 1_n. Also, $f(A^*)$ is a linear combination of powers of a symmetric matrix A^*. Thus, $f(A^*)$ will also be symmetric. Combining with the fact that each column is a multiple of 1_n, we get $f(A^*) = \beta J$, for some constant β. Therefore, J belongs to the anti-adjacency algebra of G.

Conversely, assume that J belongs to the anti-adjacency algebra of G. Then, J can be expressed as a linear combination of powers of A^* as: $J = \beta_0 I + \beta_1 A^* + \ldots + \beta_m (A^*)^m$. From this equation, we note that $JA^* = A^* J$.

We have the $(i,j)-th$ entry of JA^* is $n-deg(v_j)$ and the $(i,j)-th$ entry of A^*J is $n-deg(v_i)$. By the equality $JA^* = A^*J$, we have $deg(v_i) = deg(v_j)$, for all i, j. Therefore, G is regular, completing the proof. $\square$

We can calculate the value of the constant β in Theorem 4.8 as follows: Let $n-k = \eta_1, \eta_2, \ldots, \eta_t$ be the distinct anti-adjacency eigenvalues of G in the decreasing order. Then, the minimal polynomial of A^*, $\psi(G,\eta)$ is given by

$$\psi(G,\eta) = (\eta-(n-k))(\eta-\eta_2)\ldots(\eta-\eta_t) = (\eta-(n-k))f(\eta)$$

As mentioned in the proof of Theorem 4.8, we have $f(A^*) = \beta J$.

We have, the eigenvalues of $f(A^*)$ are given by $f(n-k), f(\eta_2), f(\eta_3), \ldots, f(\eta_t)$. Also, $f(\eta_2) = f(\eta_3) = \ldots = f(\eta_t) = 0$.

Comparing the maximum eigenvalue of $f(A^*)$ and βJ, we have $f(n-k) = \beta n$, giving $\beta = \frac{f(n-k)}{n}$.

By the proof of Theorem 4.8, we also note that the polynomial $P(x)$ which results in $P(A^*) = J$ is given by $P(x) = \frac{1}{\beta}f(x) = \frac{n}{f(n-k)}f(x)$. On simplification, we get,

$$P(x) = n\prod_{i=2}^{t}\frac{x-\eta_i}{(n-k)-\eta_i}.$$

Combining the above observation with Theorem 4.8, we have the following result:

Theorem 4.9. *Let G be a graph on n vertices, possibly disconnected, with the anti-adjacency matrix A^*. There exists a polynomial $P(x)$ such that $P(A^*) = J$, if and only if G is regular. If G is k- regular, then*

$$P(x) = n\prod_{i=2}^{t}\frac{x-\eta_i}{(n-k)-\eta_i}.$$

where $\eta_2, \ldots, \eta_t$ are distinct anti-adjacency eigenvalues of G in the decreasing order.

We demonstrate the formation of the polynomial $P(x)$ by an example. Consider the $3-$regular disconnected graph G as shown in the figure 1.

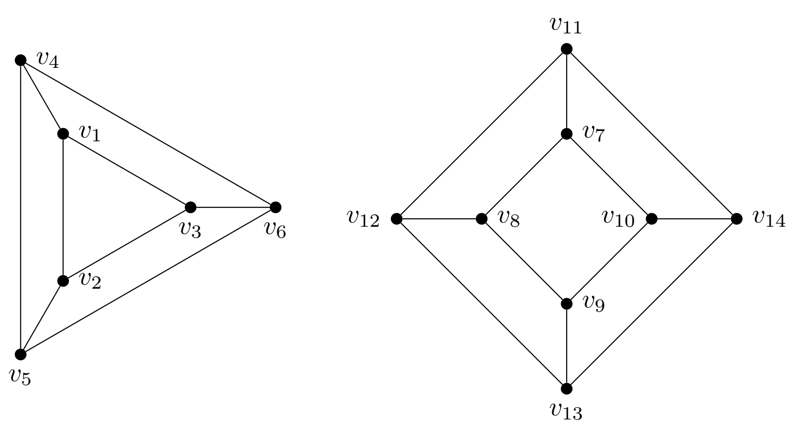


FIGURE 1. A $3-$regular disconnected graph G

Let A^* be the anti-adjacency matrix of G. Then, we have

$$a-spec(G) = \{11^{(1)}, 3^{(1)}, 2^{(2)}, 1^{(3)}, 0^{(2)}, -1^{(4)}, -3^{(1)}\}.$$

The minimal polynomial is given by $\psi(G,\eta) = (\eta - 11)f(\eta)$, such that

$$f(\eta) = (\eta-3)(\eta-2)(\eta-1)(\eta)(\eta+1)(\eta+3).$$

The constant β from the proof of theorem 4.8 is given by $\beta = \frac{f(11)}{14} = \frac{8\times 9\times 10\times 11\times 12\times 14}{14} = 95040$. It can be simultaneously noted that $f(A^*) = 95040J$.

Therefore, we have, $P(x) = \frac{(x-3)(x-2)(x-1)(x)(x+1)(x+3)}{95040} = \frac{x^6-2x^5-10x^4+20x^3+9x^2-18x}{95040}$.

Hence, the polynomial P which results in $P(A^*) = J$, for the graph G, is given by

$$P(x) = \frac{x^6 - 2x^5 - 10x^4 + 20x^3 + 9x^2 - 18x}{95040}.$$

5. Anti-Adjacency Spectra of Complement of Regular Graphs

Let G be a k- regular graph of order n, possibly disconnected, whose anti-adjacency matrix is denoted by A^*. Let $n-k = \eta_1, \eta_2, \dots, \eta_n$ be the anti-adjacency eigenvalues of G, in non-increasing order.

Let $\overline{G}$ be the complement of G whose anti-adjacency matrix is denoted by $\overline{A^*}$. Let $\overline{\eta}_1, \overline{\eta}_2, \dots, \overline{\eta}_n$ be the anti-adjacency eigenvalues of $\overline{G}$, in non-increasing order.

It can be easily verified that $A^* + \overline{A^*} = J + I$, where I is the identity matrix of order n.

The following theorem discusses about the anti-adjacency eigenvalues of the graph $\overline{G}$, in terms of $\eta_i; i = 1, \dots, n$.

Theorem 5.10. *If G is a k- regular graph on n vertices,possibly disconnected, then*

(i) the vector 1_n is an anti-adjacency eigenvector of G and $\overline{G}$, with $n-k$ and $k+1$ being the respective anti-adjacency eigenvalues.

(ii) if $X \neq 1_n$ is an anti-adjacency eigenvector of G with anti-adjacency eigenvalue η, then its corresponding anti-adjacency eigenvalue in $\overline{G}$ is $1-\eta$.

Proof. Let A^* and $\overline{A^*}$ be the anti-adjacency matrices of G and $\overline{G}$, respectively. If G is k-regular, then $\overline{G}$ is $(n-1-k)-$ regular. Therefore, by Theorem 3.3, the vector 1_n is an anti-adjacency eigenvector for both G and $\overline{G}$, corresponding to the anti-adjacency eigenvalue $n-k$ for G, and $n-(n-k-1) = k+1$ for $\overline{G}$. It can be easily verified that $k+1 = \eta_{\max}(\overline{G})$.

Let $X \neq 1_n$, be an anti-adjacency eigenvector of G with anti-adjacency eigenvalue η. By Theorem 3.6, we have $JX = 0$. Consider $\overline{A^*}X$.

$$\begin{aligned}\overline{A^*}X &= (J + I - A^*)X \\ &= JX + X - A^*X \\ &= 0 + X - \eta X \\ &= (1-\eta)X\end{aligned}$$

Therefore, X is an anti-adjacency eigenvector of $\overline{G}$ with $1-\eta$ being the corresponding anti-adjacency eigenvalue. □

By Theorem 5.10, it can be easily verified that G and $\overline{G}$ have the same anti-adjacency eigenvectors.

By Theorem 3.5, we infer that if G is a disconnected k- regular graph with r components, then $\eta_{\min}(G) = -k$, with algebraic multiplicity being $r-1$. Also we have, $1 - \eta_{\min}(G) = 1-(-k) = k+1 = \eta_{\max}(\overline{G})$. Moreover, the anti-adjacency matrix of $\overline{G}$ in this case is a block-diagonal matrix, with r blocks and each block matrix having $k+1$ as the constant row (or column) sum. Hence, the algebraic multiplicity of $k+1$ as the anti-adjacency eigenvector of $\overline{G}$ is r. Therefore, we make the following observation:

Observation 5.11. If G is a disconnected k- regular graph on n vertices with r components $G_i; i = 1, \ldots, r$, then $\eta_{\max}(\overline{G}) = k+1$, with algebraic multiplicity r. The corresponding anti-adjacency eigenvectors are column vectors X_i of n rows, indexed by the vertices of G, with rows corresponding to the vertices of graph G_i having entry 1 and rest of the rows having entry as 0.

It can be noted that 1_n is also an anti-adjacency eigenvector of $\overline{G}$ corresponding to the anti-adjacency eigenvalue $k+1$ in the previous observation, since 1_n is the linear combination of $X_i; i = 1, \ldots, r$.

We know that $\overline{\overline{G}} = G$. Therefore, if G is a k- regular connected graph of order n, such that $\overline{G}$ is disconnected, then the algebraic multiplicity of $n-k$ as the anti-adjacency eigenvalue of G is more than 1, in contrast to Theorem 3.2. Consequently, the associated anti-adjacency eigenvectors would be of the form discussed in Observation 5.11.

Note that if G is a k- regular connected graph of order n, such that $\overline{G}$ is still connected, then Theorem 3.2 holds true.

By Theorem 5.10, we have, if $n-k = \eta_1, \eta_2, \ldots, \eta_n$ are the anti-adjacency eigenvalues of a k- regular graph G of order n, then $k+1 = \overline{\eta}_1, 1-\eta_2 = \overline{\eta}_2, \ldots, 1-\eta_n = \overline{\eta}_n$ are the anti-adjacency eigenvalues of $\overline{G}$.

The following theorem discusses the anti-adjacency characteristic polynomial of $\overline{G}$, in terms of the anti-adjacency characteristic polynomial of G, where G is a k- regular graph on n vertices.

Theorem 5.12. *For a k- regular graph G of order n, possibly disconnected,*

$$\phi(\overline{G}, \eta) = (-1)^n \frac{\eta - k - 1}{\eta + n - k - 1} \phi(G, 1-\eta).$$

Proof. We have, $n-k = \eta_1, \eta_2, \ldots, \eta_n$ are the anti-adjacency eigenvalues of G. Hence,

$$\phi(G, \eta) = (\eta - (n-k))(\eta - \eta_2) \ldots (\eta - \eta_n).$$

Since, $k+1 = \overline{\eta}_1, 1-\eta_2 = \overline{\eta}_2, \ldots, 1-\eta_n = \overline{\eta}_n$ are the anti-adjacency eigenvalues of $\overline{G}$, we have,

$$\phi(\overline{G}, \eta) = (\eta - (k+1))(\eta - (1-\eta_2)) \ldots (\eta - (1-\eta_n))$$

or

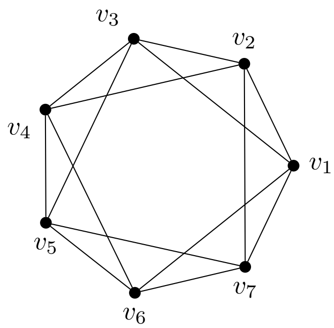
$$\phi(\overline{G}, \eta) = (\eta - (k+1))(\eta - 1 + \eta_2) \ldots (\eta - 1 + \eta_n).$$

Therefore,

$$\begin{aligned} \frac{\phi(\overline{G}, \eta)}{\phi(G, 1-\eta)} &= \frac{(\eta - (k+1))(\eta - 1 + \eta_2) \ldots (\eta - 1 + \eta_n)}{(1 - \eta - (n-k))(1 - \eta - \eta_2) \ldots (1 - \eta - \eta_n)} \\ &= (-1)^n \frac{\eta - k - 1}{\eta + n - k - 1}, \end{aligned}$$

completing the proof. □

We illustrate the result in Theorem 5.12 by an example as follows. Consider a $4-$ regular graph G as shown in the figure.



$$A^*(G) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We have, $\phi(G,\eta) = -\eta^7 + 7\eta^6 - 14\eta^5 + 21\eta^3 - 7\eta^2 - 7\eta + 3$. Therefore, $\phi(G, 1-\eta) = \eta^7 - 7\eta^5 + 14\eta^3 - 7\eta + 2$. Also,

$$(-1)^n \frac{\eta - k - 1}{\eta + n - k - 1} = (-1)^7 \frac{\eta - 5}{\eta + 2}.$$

Therefore,

$$(-1)^n \frac{\eta - k - 1}{\eta + n - k - 1} \phi(G, 1-\eta) = -\eta^7 + 7\eta^6 - 7\eta^5 - 21\eta^4 + 28\eta^3 + 14\eta^2 - 21\eta + 5.$$

It can be verified that the right hand side of the above equation is same as $\phi(\overline{G}, \eta)$.

The following result gives the lower bound for $\eta_n = \eta_{\min}(G)$ in terms of the regularity of the graph G, similar to Theorem 2.1.

Proposition 5.1. *If G is a k- regular graph with anti-adjacency eigenvalue $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$, then $\eta_n \geq -k$.*

Proof. Let G be a k- regular graph with anti-adjacency eigenvalue $n - k = \eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$. Then, $k + 1 \geq 1 - \eta_2 \geq \ldots \geq 1 - \eta_n$ are the anti-adjacency eigenvalues of $\overline{G}$. In particular, $k + 1 \geq 1 - \eta_n$. That is, $\eta_n \geq -k$. □

6. Conclusions

In this paper, the anti-adjacency spectra of connected and disconnected regular graphs have been studied. The notion of anti-adjacency algebra, analogous to the adjacency algebra of graphs had been introduced. Characterisation of regular graphs in terms of anti-adjacency algebra had been done. The anti-adjacency spectra of complement of connected and disconnected regular graphs has also been discussed. The results can be extended to other classes of regular graphs and graphs derived from regular graphs.

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