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A new Mann-type iterative algorithm for maximal monotone mappings in classical Banach spaces

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ABSTRACT. Let $E = L_q(1 < q < \infty)$ with dual space E^*. Let $A : E \to E^*$ be a bounded and maximal monotone mapping such that $A^{-1}(0) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated iteratively by the following algorithm: $x_1 \in E$,

$$x_{n+1} = (1 - \lambda_n\theta_n)x_n - \lambda_n J^{-1}(Ax_n),\ n \geq 1,$$

where J is the normalized duality mapping from E into E^* and $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0, 1)$ with suitable conditions. The main contribution of this paper is to prove that the sequence $\{x_n\}$ converges strongly to x^* which belongs to $A^{-1}(0)$. Numerical simulations are provided to illustrate the results.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. An operator $A : H \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle_H \geq 0 \text{ for all } x, y \in H, \tag{1.1}$$

and is called *strongly monotone* if there exists $k > 0$ such that

$$\langle Ax - Ay, x - y \rangle_H \geq k\|x - y\|_H^2 \text{ for all } x, y \in H. \tag{1.2}$$

Interest in monotone operators stems mainly from their usefulness in numerous applications. Many problems in nonlinear analysis and optimization theory can be formulated as follows: *find u such that* $0 \in Au$. This problem has been investigated by many researchers (see for instance, Brézis and Lions [4], Martinet [24], Minty[27], Reich [38], Rockafellar [40], Takahashi and Ueda [43] and others). Such a problem is connected with the *convex minimization problem*. In fact, if $f : H \to (-\infty, +\infty]$ is a proper, lower-semicontinuous convex function, then, it is known that the multi-valued map $T := \partial f$ (the subdifferential of f) is *maximal monotone* (see, e.g., [27], [40]), where for $w \in H$,

$$\begin{aligned} w \in \partial f(x) &\Leftrightarrow f(y) - f(x) \geq \langle y - x, w \rangle \ \ \forall\ y \in H \\ &\Leftrightarrow x \in \underset{H}{\operatorname{Argmin}}(f(\cdot) - \langle \cdot, w \rangle). \end{aligned}$$

In particular, the inclusion $0 \in \partial f(x)$ is equivalent to $f(x) = \min_{y \in H} f(y)$. Several existence theorems have been established for the equation $Au = 0$ when A is of the monotone-type (see e.g., Deimling [18], Pascali and Sburian [32]).

The extension of the monotonicity definition to operators from a Banach space into its dual space has been the starting point for the development of nonlinear functional analysis. The monotone maps constitute the most manageable class because of the very simple structure of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts since they can be found in many functional equations.

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Many of them appear also in calculus of variations as subdifferential of convex functions (see for example, Pascali and Sburian [32], p. 101).

The *first* involves mappings from E to E^*. Here and in the sequel, $\langle \cdot, \cdot \rangle$ stands for the duality pairing between (a possible normed linear space) E and its dual E^*. A mapping $A : D(A) \subset E \to E^*$ is called *monotone* if for all $x, y \in D(A)$,

$$\langle x - y, Ax - Ay \rangle \geq 0. \tag{1.3}$$

It is said to be *strongly monotone* if there exists a positive constant k such that for all $x, y \in D(A)$,

$$\langle x - y, Ax - Ay \rangle \geq k\|x - y\|^2. \tag{1.4}$$

The mapping $A : D(A) \subset E \to E^*$ is called maximal monotone if it is monotone and its graph $G(A)$ is not properly contained is any other graph of monotone map.
Note that if E is a real Hilbert space H, then $H = H^*$ and (1.3) coincides with (1.1).

The *second* extension of the notion of monotonicity to real normed spaces involves mappings E *into itself*. A mapping $A : D(A) \subset E \to E$ is called *accretive* if and only if for all $x, y \in D(A)$, the following inequality is satisfied:

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\|, \ \forall\ s > 0. \tag{1.5}$$

Due to result by Kato [21] it has been shown that $A : D(A) \subset E \to E$ is accretive if and only if for all $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that the following inequality holds:

$$\langle Ax - Ay, j(x - y) \rangle \geq 0, \tag{1.6}$$

where $J : E \to 2^{E^*}$ is the normalized duality mapping of E defined by:

$$J(x) := \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 \text{ and } \|f\|_{E^*} = \|x\| \right\}.$$

Here, if E is a real Hilbert space, J becomes the identity map and condition (1.6) reduces to (1.1). Hence, in real Hilbert spaces, *accretive* operators become *monotone*. Consequently, accretive operators can be regarded as extension of Hilbert space monotonicity condition to real normed spaces.

A mapping $A : D(A) \subset E \to E$ is called *strongly accretive* if there exists a constant $k > 0$ such that for every $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq k\|x - y\|^2.$$

If $A : E \to E$ is of accretive-type and $Au = 0$ has a solution, then in order to approximate a solution of $Au = 0$ Browder [5] introduced a pseudo-contractive operator $T : E \to E$ defined by $T := I - A$, where I is the identity map on E. It is trivial to observe that zeros of A correspond to fixed points of T. For Lipschitz strongly pseudo-contractive maps, Chidume [11] proved the following theorem.

Theorem C1. (Chidume, [11]) Let $E = L_p,\ 2 \leq p < \infty$, and $K \subset E$ be nonempty closed convex and bounded. Let $T : K \to K$ be a strongly pseudo-contractive and Lipschitz map. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n,\ n \geq 0$, where $\{\lambda_n\} \subset (0, 1)$ satisfies the following conditions: $(i)\ \sum_{n=1}^{\infty} \lambda_n = \infty,\ (ii) \sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T.

By setting $T := I - A$ in Theorem C1, the following theorem for approximating a solution of $Au = 0$, where A is a strongly accretive and bounded operator, can be proved.

Theorem C2. Let $E = L_p$ with $2 \leq p < \infty$, and $A : E \to E$ be a strongly accretive and bounded map. Assume $A^{-1}(0) \neq \emptyset$. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = x_n - \lambda_n A x_n,\ n \geq 0$, where $\{\lambda_n\} \subset (0,1)$ satisfies the following conditions: $(i)\ \sum_{n=1}^{\infty} \lambda_n = \infty,\ (ii) \sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique solution of $Au = 0$.

The main tool used in the proof of Theorem C1 is an inequality of Bynum [6]. This theorem signalled the return to extensive research efforts on inequalities in Banach spaces and their applications to iterative methods for solutions of nonlinear equations. Consequently, Theorem C1 has been generalized and extended in various directions, leading to flourishing areas of research, for the past thirty years or so, for numerous authors (see e.g.,Censor and Reich [7], Chidume [11], Chidume [9, 10], Chidume and Ali [13], Chidume and Chidume [14, 15], Chidume and Osilike [16], Deng [17], Liu [23], Weng [44], Liu [33], Reich [34, 35, 39], Reich and Sabach [36, 37], Xiao [46], Xu [50, 48, 49], Berinde *et al.* [3], Moudafi [28, 29, 30], Moudafi and Thera [31], Xu and Roach [52], Xu *et al.* [51], Zhu [54] and a host of other authors). Recent monographs emanating from these researches include those by Berinde [2], Chidume [8], Goebel and Reich [20], and William and Shahzad [45].

Unfortunately, the success achieved in using geometric properties developed from the mid 1980s to early 1990s in approximating zeros of *accretive-type mappings* has not carried over to approximating zeros of *monotone-type operators* in general Banach spaces. Part of the problem is that since A maps E to E^*, for $x_n \in E$, Ax_n is an element of E^*. Consequently, a recursion formula containing x_n and Ax_n may not be well defined. Attempts have been made to overcome this difficulty by introducing the inverse of the normalized duality mapping in the recursion formulas for approximating zeros of monotone-type mappings.

In the case of Banach spaces, for finding zeros point of a maximal monotone mappings by using the proximal point algorithm, Kohshada and Takahashi [22] introduced the following iterative sequence for a monotone mapping $A : E \to 2^{E^*}$:

$$x_1 = u \in E,\ x_{n+1} = J^{-1}\Big(\alpha_n Ju + (1-\alpha) J J_{r_n} x_n\Big),\ n \geq 1, \tag{1.7}$$

where $J_{r_n} := (J + r_n A)^{-1}$, and J is the duality mapping from E into E^*, $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$ and $\lim_{n\to\infty} r_n = \infty$. They proved that if E is smooth and uniformly convex and A maximal monotone with $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element of $A^{-1}(0)$. This result extends the theorem proposed by Kohshada and Takahashi [22] to Banach spaces. However, the algorithm requires the computation of $(J + r_n A)^{-1} x_n$ at each step of the process, which makes difficult its implementation for applications. Following the work of Kohshada and Takahashi [22], Zegeye introduced in [53] an iterative scheme for approximating zeros of maximal monotone mappings defined in uniformly smooth and 2-uniformly convex real Banach spaces. In fact, he proved the following theorem.

Theorem Z (Zegeye [53]). *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual E^*. Let $A : E \to E^*$ be a Lipschitz continuous and monotone mapping with constant $L > 0$ and $A^{-1}(0) \neq \emptyset$. For given $u, x_1 \in E$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = J^{-1}\Big(\beta_n Ju + (1-\beta_n)(Jx_n - \alpha_n A x_n)\Big) \text{ for all } n \geq 1,$$

where J is the normalized duality mapping from E into E^* and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying (i) $\lim_{n\to\infty}\beta_n = 0$, (ii) $\sum\beta_n = \infty$ and (iii) $\alpha_n = o(\beta_n)$. Suppose that $B_{min} \cap \left(AJ^{-1}\right)^{-1}(0) \neq \emptyset$. Then $\{x_n\}$ converges strongly to $x^* \in A^{-1}(0)$ and that $R(Ju) = Jx^* \in \left(AJ^{-1}\right)^{-1}(0)$, where R is a sunny generalized nonexpansive retraction of E^* onto $\left(AJ^{-1}\right)^{-1}(0)$.

Motivated by approximating zeros of monotone mappings, Chidume et. al. [12] proposed a Krasnoselskii-type scheme and proved a strong convergence theorem in $L_p,\ 2 \leq p < \infty$. In fact, they obtained the following result.

Theorem CA (Chidume et. al. [12]). *Let $X = L_p,\ 2 \leq p < \infty$ and $A : X \to X^*$ be a Lipschitz map. Assume that there exists a constant $k \in (0,1)$ such that A satisfies the following condition:*

$$\big\langle Ax - Ay, x - y\big\rangle \geq k\|x - y\|^{\frac{p}{p-1}}, \tag{1.8}$$

and that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in X$, define the sequence $\{x_n\}$ iteratively by:

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), n \geq 1, \tag{1.9}$$

where $\lambda \in (0, \delta_p)$ and δ_p is some positive constant. Then, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

In [12], they also proved a similar result for the class of Lipschitz and strongly monotone mappings in L_p spaces for $1 < p \leq 2$.

Remark 1.1. *Theorem CA is proved in L_p spaces, $2 \leq p < \infty$ with Lipschitz mapping satisfying condition (1.8). The method of proof used in (1.8) is not extendable to the class of strongly monotone mappings.*

Following the works of Chidume et. al [12] and motivated by approximating zeros of monotone-type mappings, several strong convergence results have been established by various authors using the algorithm (1.9) proposed by Chidume et. al in [12] (see, e.g., Diop et. al. [19], Mendy et. al. [25], Mendy et. al [26], Sow et. al. [41]).
Recently, Mendy et. al. [25] proposed a pertubed version of the Mann-type algorithm (1.9) proposed by Chidume et. al. [12] and proved strong convergence theorems for approximating zeros of bounded and maximal monotone mappings defined in 2-uniformly convex and q-uniformly smooth (or p-uniformly convex and 2-uniformly smooth) real Banach spaces. In fact, they proved the following theorem.

Theorem MA (Mendy et. al. [25]). *For $q > 1$, let E be a 2- uniformly convex and q-uniformly smooth real Banach space and E^* its dual space. Let $A : E \to E^*$ be a bounded and maximal monotone mapping such that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, let $\{x_n\}$ be the sequence defined iteratively by:*

$$x_{n+1} = J^{-1}(Jx_n - \lambda_n Ax_n - \lambda_n\theta(Jx_n - Jx_1)),\ n \geq 1, \tag{1.10}$$

where λ and $\{\theta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:

$$(i)\ \ \theta_n \to 0,\ \lambda_n = o(\theta_n);\quad (ii)\ \sum_{n=1}^{\infty}\lambda_n\theta_n = \infty;\quad (iii)\ \sum_{n=0}^{\infty}\alpha_n^2 < \infty,\ \limsup\frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n\theta_n} \leq 0.$$

Then, there exists $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0,\ \forall\, n \geq 1$, x_n converges strongly to $x^* \in A^{-1}(0)$.

Notice that in the algorithm (1.10), at each iteration, one must compute at the same time Jx_n and Jx_1 and the inverse of J at $Jx_n - \lambda_n Ax_n - \lambda_n\theta(Jx_n - Jx_1)$. Likewise, the algorithm (1.9) requires to compute Jx_n and the inverse $J^{-1}(Jx_n - \lambda Ax_n$ at each iteration. This can make the implementation of the algorithm uncomfortable and inaccurate due to a number of errors that arise during the process.
It is our purpose in this paper to introduce a new Mann-type algorithm to approximate the zero of a bounded maximal monotone mapping defined in q-uniformly smooth and p-uniformly convex real Banach spaces. This class of Banach spaces includes all the L_p and Sobolev spaces. The algorithm proposed in this work is simpler than (1.9) proposed by Chidume et. al. in [12] and its modified version (1.10) by Mendy et. al. in [25] in the sense that it does not require further calculus in the implementation. The last section is devoted to numerical simulations to illustrate the results.

2. Preliminaries

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if the limit

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t} \tag{2.11}$$

exists for each $x, y \in S$. Further, E is said to be Fréchet differentiable if it is smooth and the limit in (2.11) is attained uniformly for $y \in S_E$. Finally E is uniformly smooth if it is smooth and the limit in (2.11) is attained uniformly for each $x, y \in S_E$. If E is a normed linear space of dimension ≥ 2, then, the *modulus of smoothness* of E, ρ_E, is defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\} \text{for all } \tau > 0.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*.

Classical spaces with such properties are L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or\ \ell_p)\ or\ W_p^m \text{ is } \begin{cases} 2 - \text{uniformly smooth and } p - \text{uniformly convex} & \text{if } 2 \leq p < \infty; \\ 2 - \text{uniformly convex and } p - \text{uniformly smooth} & \text{if } 1 < p < 2. \end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\|_{E^*} = \|x\|^{q-1}\right\},$$

where J_2 is called the *normalized duality mapping* and is denoted simply by J.
It is well known that E is smooth if and only if J is single valued. Moreover, if E is a reflexive, smooth and strictly convex real Banach space, then J^{-1} is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E. Further, in L^p, W_p^m spaces $1 < p < \infty$, the functions J and J_p are properly known as stated in the following remark.

Remark 2.2. *(see e.g. Alber and Ryazantseva* [1, p.36]*) Let $1 < p < \infty$, one has the following*

(i) $L_p: \ Jx = \|x\|_{L_p}^{2-p}|x|^{p-2}x,$ *and for all* $x \neq 0, \ J_p(x) = \|x\|^{p-2}Jx.$

(ii) W_m^p : $Jx = \|x\|_{W_m^p}^{2-p} \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha \Big(|D^\alpha x|^{p-2} D^\alpha x\Big)$, *and for all* $x \neq 0$, *one has*

$$J_p x = \|x\|_{W_m^p}^{2-p} Jx, \text{ where for } \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n, \text{ one defines } |\alpha| = \sum_{i=1}^n \alpha_i.$$

In the sequel, we shall need the followings results.

Lemma 2.1 (Xu [47]). *Let* $\{\rho_n\}$ *be a sequence of non-negative real numbers satisfying the following inequality*

$$\rho_{n+1} \leq (1-\alpha_n)\rho_n + \alpha_n \sigma_n + \gamma_n, \tag{2.12}$$

where $\{\alpha_n\}, \{\sigma_n\}$ *and* $\{\gamma_n\}$ *are real sequences satisfying:*

(i) $\{\alpha_n\} \subset]0,1[$ *and* $\sum \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;

(iii) $\gamma_n \geq 0$ *and* $\sum \gamma_n < \infty$.

Then, the sequence (ρ_n) *converges to zero as* $n \to \infty$.

Lemma 2.2. *(See, e.g., Chidume* [8]*) Let* $q > 1$ *be a fixed real number and* E *be a smooth Banach space. Then the following statements are equivalent:*
i) E *is q-uniformly smooth*
ii) *There is a constant* $d_q > 0$ *such that for all* $x, y \in E$

$$\|x+y\|^q \leq \|x|^q + q\langle y, J_q(x)\rangle + d_q\|y\|^q.$$

Lemma 2.3. *(See, e.g., Chidume* [8]*) Let* E *be a real normed space and* $J_q : E \to 2^{E^*}, 1 < q < \infty$, *be the generalized duality map. Then, for any* $x, y \in E$, *the following inequality holds:*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle \tag{2.13}$$

for all $j_q(x+y) \in J_q(x+y)$.

Theorem 2.1 (Takahashi [42]). *Let* E *be a uniformly convex real Banach space with Fréchet differentiable norm and* E^* *be its dual space. Let* $A : E^* \to 2^E$ *be a multivalued maximal monotone mapping with* $A^{-1}(0) \neq \emptyset$. *Then, the following holds: for* $u \in E$,

$$\lim_{\lambda\to\infty} (I + \lambda AJ)^{-1}u \text{ exists and belongs to } (AJ)^{-1}(0), \tag{2.14}$$

where J *is the normalized duality mapping from* E *into* E^*. *Moreover, if* $Ru := \lim_{\lambda\to\infty}(I + \lambda AJ)^{-1}u$, *then* R *is a sunny generalized nonexpansive retraction of* E *onto* $(AJ)^{-1}(0)$.

3. Main results

We begin with the following immediate consequence of the above Theorem 2.1.

Lemma 3.4. *Let* E *be a uniformly convex and uniformly smooth real Banach space with dual space* E^* *and* $A : E \to E^*$ *be a maximal monotone mapping such that* $A^{-1}(0) \neq \emptyset$. *Given* $u \in E$ *and a sequence of positive real numbers* $\{\theta_n\}_n$ *satisfying* $\theta_n \to 0$ *as* $n \to +\infty$, *there exists a sequence* $\{y_n\}$ *in* E *such that:*

$$\theta_n(Jy_n - Ju) + Ay_n = 0, \ \forall\, n \geq 1,$$
$$y_n \to y^* \text{ with } y^* \in A^{-1}(0),$$

where J *is the normalized duality mapping from* E *into* E^*.

Proof. Since E uniformly convex and uniformly smooth, then the duality mapping J from E into E^* is single valued, onto and one to one and its inverse J^{-1} is the duality mapping of E^*. Therefore, from Theorem 2.1, it follows that

$$\lim_{\lambda\to\infty} J_\lambda u \text{ exists and belongs to } (AJ^{-1})^{-1}(0),$$

where $J_\lambda u := (I^* + \lambda AJ^{-1})^{-1}Ju$, and I^* the identity map on E^*. For each $n \geq 1$, define $z_n := J_{t_n}u$ where $t_n = \theta_n^{-1}$. Then, the sequence $\{y_n\}$ given by $y_n := J^{-1}z_n$ for all $n \geq 1$ satisfies the above relations. □

By taking $u = 0$ in Lemma 3.4, we have the following corollary.

Corollary 3.1. *Let E be a uniformly convex and uniformly smooth real Banach space and $A : E \to E^*$ be a maximal monotone mapping such that $A^{-1}(0) \neq \emptyset$. Then, there exists a sequence $\{y_n\}$ in E such that:*

$$\theta_n Jy_n + Ay_n = 0, \ \forall\, n \geq 1, \tag{3.15}$$

$$y_n \to y^* \ \textit{with} \ y^* \in A^{-1}(0) \tag{3.16}$$

where J is the normalized duality mapping from E into E^*.

Theorem 3.2. *Let $p > 1$ and $q > 1$ be real numbers and let E be a q-uniformly smooth and p-uniformly convex real Banach space. Let $A : E \to E^*$ be a bounded and maximal monotone map such that $A^{-1}(0) \neq \emptyset$. Assume that $J^{-1}A$ is accretive. Define the sequence $\{x_n\}$ as follows: For $x_1 \in E$ given randomly,*

$$x_{n+1} = (1 - \lambda_n\theta_n)x_n - \lambda_n J^{-1}(Ax_n), \ n \geq 1, \tag{3.17}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are real decreasing sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim \theta_n = 0$,

(ii) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, \quad \lambda_n^{q-1} = o(\theta_n)$,

(iii) $\limsup \dfrac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n\theta_n} \leq 0 \quad$ *and* $\quad \sum_{n=1}^{\infty} \lambda_n^q < \infty.$

There exists a real positive constant γ_0 such that if $\lambda_n^{q-1} \leq \gamma_0\theta_n$ for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to some $x^* \in A^{-1}(0)$.

Proof. Let $x^* \in A^{-1}(0)$, consider the real $r > 0$ large enough so that one has

$$\sup\{\|x^*\|^q, \|x_1 - x^*\|^q\} < \frac{r^q}{2}.$$

Define the following constants:

$$\mathrm{M} := \sup\left\{\|J^{-1}Ax + \theta x\|^q : x \in B(x^*, r), \ 0 < \theta < 1\right\} + 1 < +\infty,$$

and

$$\gamma_0 := \frac{r^q}{2\mathrm{d_q}\mathrm{M^q}}.$$

The remainder of the proof is devided in two steps.
Step 1: We prove by induction that the sequence $\{x_n\}$ is bounded.

By definition $x_1 \in B(x^*, r)$. Suppose that $x_n \in B(x^*, r)$. Using Lemma 2.2 and the fact that $J^{-1}A$ is accretive, we have the following computations

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &= \|x_n - x^* - \lambda_n(J^{-1}Ax_n + \theta_n x_n)\|^q \\
&\le \|x_n - x^*\|^q - q\lambda_n\langle J^{-1}Ax_n + \theta_n x_n, J_q(x_n - x^*)\rangle + \lambda_n^q \mathrm{d_q}\|J^{-1}Ax_n + \theta_n x_n\|^q \\
&\le \|x_n - x^*\|^q - q\lambda_n\langle J^{-1}Ax_n, J_q(x_n - x^*)\rangle - q\lambda_n\theta_n\langle x_n - x^*, J_q(x_n - x^*)\rangle \\
&+ q\lambda_n\theta_n\langle x^*, J_q(x_n - x^*)\rangle + \mathrm{d_q}\mathrm{M^q}\lambda_n^q \\
&\le \|x_n - x^*\|^q - q\lambda_n\theta_n\|x_n - x^*\|^q + q\lambda_n\theta_n\|x^*\| \cdot \|x_n - x^*\|^{q-1} + \mathrm{d_q}\mathrm{M^q}\lambda_n^q.
\end{aligned}$$

By Young's inequality, for $q' > 1$ with $\frac{1}{q'} + \frac{1}{q} = 1$, we have

$$\|x^*\| \cdot \|x_n - x^*\|^{q-1} \le \frac{1}{q}\|x^*\|^q + \frac{1}{q'}\|x_n - x^*\|^{q'(q-1)}.$$

Using this, we obtain the following:

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &\le [1 - \lambda_n\theta_n]\|x_n - x^*\|^q + \lambda_n\theta_n\|x^*\|^q + \lambda_n\theta_n\mathrm{d_q}\mathrm{M^q}\gamma_0 \\
&\le [1 - \lambda_n\theta_n]r^q + 2\lambda_n\theta_n\frac{r^q}{2} = r^q.
\end{aligned}$$

This implies that $x_{n+1} \in B(x^*, r)$. Therefore, $\{x_n\}$ is bounded.

Step 2: We prove that the sequence $\{x_n\}$ converges to some $x^* \in A^{-1}(0)$. Using Lemma 2.2 again, the fact $J^{-1}A$ is accretive and Corollary 3.1 we have the following estimates

$$\begin{aligned}
\|x_{n+1} - y_n\|^q &= \|x_n - y_n - \lambda_n(J^{-1}Ax_n + \theta_n x_n)\|^q \\
&\le \|x_n - y_n\|^q - q\lambda_n\langle J^{-1}Ax_n + \theta_n x_n, J_q(x_n - y_n)\rangle + \mathrm{d_q}\lambda_n^q\|J^{-1}Ax_n + \theta_n x_n\|^q \\
&\le \|x_n - y_n\|^q - q\lambda_n\langle J^{-1}Ax_n + \theta_n x_n, J_q(x_n - y_n)\rangle + \mathrm{d_q}\mathrm{M^q}\lambda_n^q \\
&= \|x_n - y_n\|^q - q\lambda_n\langle J^{-1}Ax_n - J^{-1}Ay_n + J^{-1}Ay_n + \theta_n x_n, J_q(x_n - y_n)\rangle + \mathrm{d_q}\mathrm{M^q}\lambda_n^q \\
&\le \|x_n - y_n\|^q - q\lambda_n\langle J^{-1}Ay_n + \theta_n x_n, J_q(x_n - y_n)\rangle + \mathrm{d_q}\mathrm{M^q}\lambda_n^q \\
&= \|x_n - y_n\|^q - q\lambda_n\theta_n\langle x_n - y_n, J_q(x_n - y_n)\rangle + \mathrm{d_q}\mathrm{M^q}\lambda_n^q \\
&= (1 - q\lambda_n\theta_n)\|x_n - y_n\|^q + \mathrm{d_q}\mathrm{M^q}\lambda_n^q.
\end{aligned}$$

Hence,

$$\|x_{n+1} - y_n\|^q \le (1 - q\lambda_n\theta_n)\|x_n - y_n\|^q + \mathrm{d_q}\mathrm{M^q}\lambda_n^q. \tag{3.18}$$

Next, by Lemma 2.3 and Swartz inequality, we have

$$\begin{aligned}
\|x_n - y_n\|^q &\le \|x_n - y_{n-1}\|^q + q\langle y_{n-1} - y_n, J_q(x_n - y_n)\rangle \\
&\le \|x_n - y_{n-1}\|^q + q\|y_{n-1} - y_n\| \cdot \|x_n - y_n\|^{q-1}. \qquad (3.19)
\end{aligned}$$

On the other hand, since $J^{-1}A$ is accretive, by using inequality (1.5) and Corollary 3.1 we have

$$\begin{aligned}\|y_{n-1}-y_n\| \leq& \|y_{n-1}-y_n+\frac{1}{\theta_n}(J^{-1}Ay_{n-1}-J^{-1}Ay_n)\|\\ =&\|y_{n-1}-y_n+\frac{1}{\theta_n}(\theta_n y_n-\theta_{n-1}y_{n-1})\|\\ =&\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)\|y_{n-1}\|. \end{aligned} \tag{3.20}$$

From (3.18), (3.19), (3.20) and the fact that $\{x_n\}$ and $\{y_n\}$ are bounded, we obtain

$$\|x_{n+1}-y_n\|^q \leq (1-q\lambda_n\theta_n)\|x_n-y_{n-1}\|^q + K\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)+\mathrm{d_q M^q}\lambda_n^q, \tag{3.21}$$

for some constant $K>0$.

Therefore, using Lemma 2.1 with $\alpha_n = q\lambda_n\theta_n$, $\sigma_n = \frac{K\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)}{\lambda_n\theta_n}$ and $\gamma_n = \mathrm{d_q M^q}\lambda_n^q$, it follows that $\|x_n-y_{n-1}\|$ converges to zero. That is x_n converges to $x^*\in A^{-1}(0)$. □

Remark 3.3. *Real sequences that satisfy conditions* (i)-(iii) *could be* $\lambda_n=(n+1)^{-a}$ *and* $\theta_n=(n+1)^{-b}$, $n\geq 1$ *with* $0<b<(q-1)a$, $\frac{1}{q}<a<1$ *and* $a+b<1$. *In fact,* (i), (ii) *and the second part of* (iii) *are easy to check. For the first part of condition* (iii), *using the fact that* $(1+x)^s\leq 1+sx$, *for* $x>-1$ *and* $0<s<1$, *we have*

$$\begin{aligned}0\leq& \frac{\left(\frac{\theta_{n-1}}{\theta_n}-1\right)}{\lambda_n\theta_n} = \left[\left(1+\frac{1}{n}\right)^b-1\right]\cdot(n+1)^{a+b}\\ \leq& b\cdot\frac{(n+1)^{a+b}}{n} = b\cdot\frac{n+1}{n}\cdot\frac{1}{(n+1)^{1-(a+b)}}\to 0 \text{ as } n\to\infty.\end{aligned}$$

In the next corollaries we deduce the convergence in L_p-spaces, $1<p<\infty$.

Corollary 3.2. *Let* $E=L_q,\ 1<q<2$ *and* E^* *be its dual space. Let* $A:E\to E^*$ *be a bounded and maximal monotone map such that* $A^{-1}(0)\neq\emptyset$. *Assume that* $J^{-1}A$ *is accretive. Let the sequence* $\{x_n\}$ *be defined as follows: For arbitrary chosen* $x_1\in E$,

$$x_{n+1}=(1-\lambda_n\theta_n)x_n-\lambda_n J^{-1}(Ax_n),\ n\geq 1, \tag{3.22}$$

where $\{\lambda_n\}$ *and* $\{\theta_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim\theta_n=0$,

(ii) $\sum_{n=1}^{\infty}\lambda_n\theta_n=\infty,\ \ \lambda_n=o(\theta_n)$,

(iii) $\limsup\frac{\frac{\theta_{n-1}}{\theta_n}-1}{\lambda_n\theta_n}\leq 0$ *and* $\sum_{n=1}^{\infty}\lambda_n^q<\infty$.

There exists a real positive constant γ_0 *such that if* $\lambda_n^{q-1}\leq\gamma_0\theta_n$ *for all* $n\geq 1$, *the sequence* $\{x_n\}$ *converges strongly to some* $x^*\in A^{-1}(0)$.

Proof. Observing that L_q-spaces, $1<q<2$ are q-uniformly smooth and 2-uniformly convex, the result is a direct application of Theorem 3.2. □

Corollary 3.3. *Let* $E=L_p,\ 2\leq p<\infty$ *and* E^* *be its dual space. Let* $A:E\to E^*$ *be a bounded and maximal monotone map such that* $A^{-1}(0)\neq\emptyset$. *Assume that* $J^{-1}A$ *is accretive. Let the sequence* $\{x_n\}$ *be defined as follows: For arbitrary chosen* $x_1\in E$,

$$x_{n+1}=(1-\lambda_n\theta_n)x_n-\lambda_n J^{-1}(Ax_n),\ n\geq 1, \tag{3.23}$$

where $\{\lambda_n\}$ *and* $\{\theta_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim \lambda_n = 0$ *and* $\lim \theta_n = 0$,

(ii) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$, $\lambda_n = o(\theta_n)$,

(iii) $\limsup \dfrac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n\theta_n} \leq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$.

There exists a real positive constant γ_0 *such that if* $\lambda_n \leq \gamma_0\theta_n$ *for all* $n \geq 1$, *the sequence* $\{x_n\}$ *converges strongly to some* $x^* \in A^{-1}(0)$.

Proof. Observing that L_p-spaces, $2 \leq p < \infty$ are 2-uniformly smooth and p-uniformly convex, the result follows from Theorem 3.2. □

4. Applications to maximal monotone mappings and Numerical Simulations

Throughout this paragraph, let $E = L_q([0,1])$ where $q = \frac{3}{2}$ and its conjugate $p = 3$. That is, $E = L_{\frac{3}{2}}([0,1])$ and $E^* = L_3([0,1])$. And let us define

$$\lambda_n = \frac{1}{(n+1)^{\frac{5}{6}}} \quad \text{and} \quad \theta_n = \frac{1}{(n+1)^{\frac{1}{12}}}.$$

Observe that E is q-uniformly smooth. Moreover the sequence $\{\lambda_n\}$ and $\{\theta_n\}$ satisfy the condition in Theorem 3.2.

4.1. **Numerical simulations of the algorithm** (3.17). :

Let the mapping $A : E \to E^*$ be defined by $Ax(t) = \frac{2}{3}x(t)$ for all $t \in [0,1]$. It is clear that the map A is well defined and maximal monotone. Moreover, it is bounded and $J^{-1}A$ is accretive. Therefore, according to theorem 3.2, the sequence $\{x_n\}$ defined by (3.17) converges strongly to $x^* = 0$, the unique solution of $Au = 0$.
Further from Remark 2.2, the corresponding algorithm is the following:

$$x_{n+1} = (1 - \lambda_n\theta_n)x_n - 2\lambda_n\|x_n\|^{2-p}|x_n|^{p-2}x_n, \; n \geq 1. \tag{4.24}$$

The numerical simulations for algorithm (4.24) give the following results:

For $x_1(t) = t^2, \forall\ t \in [0,1]$, we have the following table and graph of the norm $\|x_n\|$

Number of iterations n	$\|x_n\|$ for algorithm (4.24)
36	0.0014343181023861132
37	0.0013663886821424817
38	0.0013032276778647185
39	0.0012443932479560796
40	0.0011894940968733773
41	0.0011381826359605383
42	0.0010901492068283371
43	0.001045117181777406
44	0.0010028387915638297
45	0.0009630915590670635

FIGURE 1. The values of $\|x_n\|$ with respect to n for $x_1(t) = t^2$.

In the graphs below, the y-axis represents the values of $\|x_n\|$ and the x-axis represents the number of iterations n.

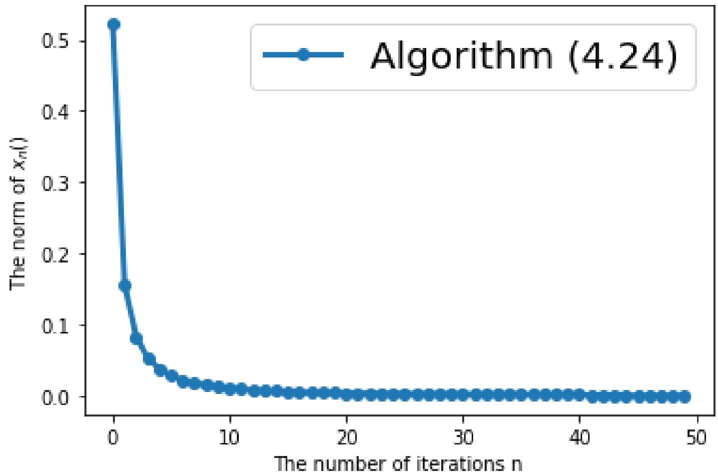

FIGURE 2. The graph of $\|x_n\|$ with respect to n for $x_1(t) = t^2$.

When $x_1(t) = e^{-t}\cos(t),\ \forall\ t \in [0,1]$, we have the following table of some values of the norm of x_n with respect to the iterations.

Number of iterations n	$\|x_n\|$ for algorithm (4.24)
47	0.0013230858360684555
48	0.0012759088502935824
49	0.0012312593464365899
50	0.0011889568373183903
51	0.001148836874022876
52	0.0011107493486950664
53	0.0010745570048359978
54	0.001040134126425833
55	0.0010073653816086259
56	0.0009761448003327562

FIGURE 3. The values of $\|x_n\|$ with respect to n for $x_1(t) = e^{-t}\cos(t)$.

The corresponding graph of the values of the norm of x_n with respect to the iterations is given in the next figure.

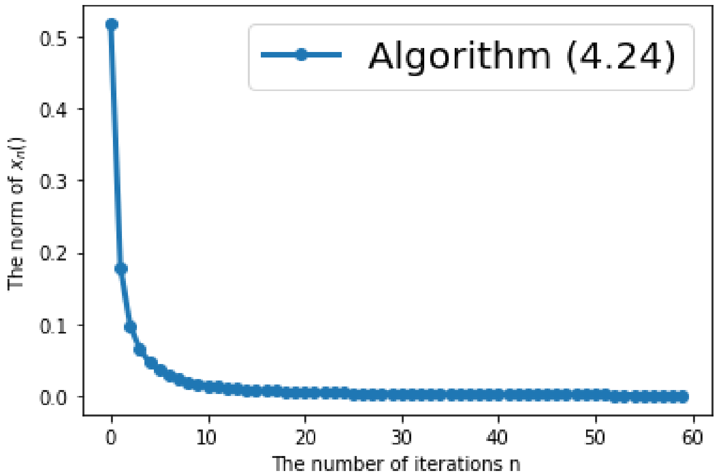

FIGURE 4. The graph of $\|x_n\|$ with respect to n for $x_1(t) = e^{-t}\cos(t)$.

In the tables 1 and 3, we remark that for $\epsilon = 10^{-3}$, taking the initial value as the function $x_1(t) = t^2$, the simulation results show that for each number of iterations $n \geq 45$, $\|x_n\| < \epsilon$ and for $x_1(t) = e^{-t}\cos(t)$, one has $\|x_n\| < \epsilon$ for each $n \geq 56$.
As it can be seen, the figures 2 and 4 show the convergence of the iterative sequence given by (4.24) to $x^* = 0$, the unique solution of $Au = 0$.

4.2. **Numerical simulations of the algorithm** (3.17) **with** $A = J$**.** :
Let $q = \frac{3}{2}$ and its conjugate $p = 3$, define $A : L_q([0,1]) \to L_p([0,1])$ by $(Ax)(t) = Jx(t)$.
It is well-established fact that the normalized duality map J is maximally monotone and uniformly continuous on bounded subsets of L_q.
Keep λ_n and θ_n defined as follows:

$$\lambda_n = \frac{1}{(n+1)^{\frac{5}{6}}} \quad \text{and} \quad \theta_n = \frac{1}{(n+1)^{\frac{1}{12}}}.$$

The corresponding algorithm to (3.17) is the following:

$$x_{n+1} = (1 - \lambda_n\theta_n)x_n - \lambda_n x_n, \; n \geq 1. \tag{4.25}$$

The numerical simulations for algorithm (4.25) give the following results:

For $x_1(t) = t^2$ one has the following table and graph of $\|x_n\|$.

Number of iteration n	$\|x_n\|$ for algorithm (4.25)
6	0.0033711525384336597
7	0.0022740865102903896
8	0.0016062159745367064
9	0.0011758582035416927
10	0.0008859033263804309

FIGURE 5. The values of $\|x_n\|$ with respect to n for $x_1(t) = t^2$.

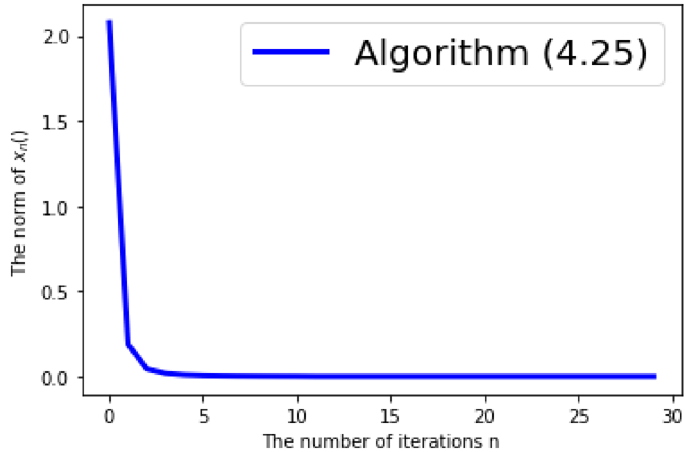

FIGURE 6. The graph of $\|x_n\|$ with respect to n for $x_1(t) = t^2$.

For $x_1(t) = e^{-t}\cos(t)$ one has the following table of some values of and graph of $\|x_n\|$.

Number of iteration n	$\|x_n\|$ for algorithm (4.25)
6	0.004424098904624768
7	0.00298437508374279
8	0.0021079017494830825
9	0.0015431259579549933
10	0.0011626065243740466
11	0.0008968384266678706

FIGURE 7. The values of $\|x_n\|$ with respect to n for $x_1(t) = e^{-t}\cos(t)$.

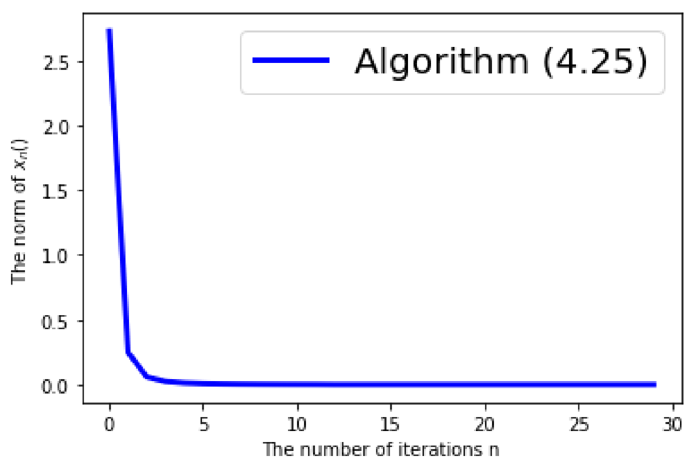

FIGURE 8. The graph of $\|x_n\|$ with respect to n for $x_1(t) = e^{-t}\cos(t)$.

We remark from tables 5 and 7 that for $\epsilon = 10^{-3}$ and $x_1(t) = t^2$, one has

$$\|x_n\| < \epsilon, \ \ \forall \ n \geq 11,$$

and for $x_1(t) = e^{-t}\cos(t)$, one has

$$\|x_n\| < \epsilon, \ \forall \ n \geq 12.$$

Further the figures 6 and 8 show the convergence of the iterative sequence given by (4.25) to zero, unique solution of the equation $Au = 0$.

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