CREAT. MATH. INFORM.
Volume **34** (2025), No. 2,
Pages 299-306

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2025.02.12

On medial Bd-algebras

DANIEL ABRAHAM ROMANO[1]

ABSTRACT. The concept of Bd-algebras was introduced in 2022 by T. Bantaojai et al. At the same time, the authors, in addition to analyzing the basic properties of this class of algebras, also introduced the concept of ideal in these algebras. In this paper, introducing the concept of pre-ideals in (medial) Bd-algebras, we analyze the connections between pre-ideals, ideals and congruences in (medial) Bd-algebras.The concept of pre-filters in (medial) Bd-algebras is also discussed here.

1. INTRODUCTION AND PRELIMINARIES

In 1966, Y. Imai and K. Iséki ([11, 12]) introduced BCK/BCI-algebras. In 1983, Q. P. Hu and X. Li ([10]) introduced BCH-algebras. An algebra $\mathfrak{A} =: (A, *, 0)$ of type $(2, 0)$ (i.e., a non-empty set A with a binary operation $'*'$ and a constant 0), is called a BCH-algebra if it satisfies the following axioms:

(Re) $(\forall x \in A)(x * x = 0)$,

(Ex) $(\forall x, y, z \in A)((x * y) * z = (x * z) * y)$,

(An) $(\forall x, y \in A)((x * y = 0 \wedge y * x = 0) \implies x = y)$.

It is well known that for any BCH-algebra $\mathfrak{A}$ the following holds

(M) $(\forall x \in A)(x * 0 = x)$.

A BCH-algebra $\mathfrak{A} =: (A, *, 0)$ is said to be a BCI-algebra if it, additionally, satisfies the identity

(BCI) $(\forall x, y, z \in A)(((x * y) * (x * z)) * (z * y) = 0)$.

(In the text [12], K. Iséki literally wrote "In this note, we shall consider a new algebra induced by the BCI-system of propositional calculus by C. A, Meredith quoted into A. N. Prior, Formal Logic ([16], p. 316).") A BCK-algebra is a BCI-algebra $\mathfrak{A} =: (A, *, 0)$ satisfying the law

(L) $(\forall x \in A)(0 * x = 0)$.

(BCK-algebra is a generalization of the concepts of set-theoretic difference and propositional calculi.) The concept of BH-algebras, as a generalization of BCK/BCI/BCH-algebras, was introduced in 1998 ([13]) by Y. B. Jun, E. H. Roh and H. S. Kim. An algebra $\mathfrak{A} =: (A, *, 0)$ of type $(2, 0)$ with the following axioms (Re), (M) and (An) is called a BH-algebra. The notion of B-algebras was introduced by J. Neggers and H. S. Kim ([15]). They defined a B-algebra as an algebra $\mathfrak{A} =: (A, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(Re) $(\forall x \in A)(x * x = 0)$

(M) $(\forall x \in A)(x * 0 = x)$

Received: 04.08.2024. In revised form: 17.12.2025. Accepted: 29.05.2025
2020 *Mathematics Subject Classification.* 03G25; 06F35.
Key words and phrases. *Bd-algebra, (pre-) ideal in Bd-algebra, homomorphism between Bd-algebras, pre-filter in Bd-algebras.*

(B) $(\forall x, y, z \in A)((x * y) * z = x * (z * (0 * y)))$.

It is known that several generalizations of B-algebras have been extensively studied by many researchers and the internal architectures of such algebras have been studied in detail. One such generalization is the BI-algebra, introduced in 2017 ([8]) by A. Borumand Saeid, H. S. Kim and A. Rezaei. According to the creators of the BI-algebra concept, the notion of BI-algebras comes from the (dual) implication algebra ([8], pp.180). An algebra $\mathfrak{A} =: (A, *, 0)$ is a dual implication algebra ([8], Definition 2.3) if it satisfies (Ex) and the following two axioms

(Im) $(\forall x, y \in A)(x * (y * x) = x)$, and

(Com) $(\forall x, y \in A)(x * (x * y) = y * (y * x))$.

Definition 1.1 ([8], Definition 3.1). *An algebra* $\mathfrak{A} =: (A, *, 0)$ *of type* $(2, 0)$ *is called a BI-algebra if the following holds:*

(Re) $(\forall x \in A)(x * x = 0)$.

(Im) $(\forall x, y \in A)(x * (y * x) = x)$.

The BI-algebra $\mathfrak{A} =: (A, *, 0)$ is said to be right distributive ([8], Definition 3.8) if it holds

(DR) $(\forall x, y, z \in A)((x * y) * z = (x * z) * (y * z))$.

The BI-algebra determined in this way should not be confused with the basic implicative algebra, determined by Definition 12 in [20], which is often also denoted by the prefix BI in the literature.

The properties of (right distributive) BI-algebras were also the focus of this author in [17].

Some of the important properties of this class of logical algebras are given by the following proposition:

Proposition 1.1 ([8], Proposition 3.7). *Let* $\mathfrak{A} =: (A, *, 0)$ *be a BI-algebra. Then:*

(M) $(\forall x \in A)(x * 0 = x)$,

(L) $(\forall x \in A)(0 * x = 0)$,

(iii) $(\forall x, y \in A)(x * y = (x * y) * y)$,

(vi) $(\forall x, y, z \in A)(x * y = z \implies (z * y = z \wedge y * z = y))$.

The concept of ideal in BI-algebras is determined by the following definition:

Definition 1.2 ([8], Definition 4.1). *A subset* J *of a BI-algebra* $\mathfrak{A} =: (A, *, 0)$ *is called an ideal of* $\mathfrak{A}$ *if the following holds:*

(J0) $0 \in J$,

(J1) $(\forall x, y \in A)((x * y \in J \wedge y \in J) \implies x \in J)$.

A common practice, practiced by many researchers of logical algebras, is to, in addition to investigating the internal architecture of the observed logical algebra and its connection with other logical algebras, also observe the substructures (such as, for example, subalgebras and ideals in them) in such an algebra.

Ideals in BI-algebras are discussed in [3, 8, 18]. In this sense, while in [3] the properties of normal ideals in BI-algebras were considered, the focus of the paper [18] is on some classes of implicative ideals in BI-algebras.

Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra. We introduce a relation $\preccurlyeq$ on the set A by

$$(\forall x, y \in A)(x \preccurlyeq y \iff x * y = 0).$$

For an ideal J in a BI-algebra $\mathfrak{A}$ holds ([8], Proposition 4.5)

(J2) $(\forall x, y \in A)((x \preccurlyeq y \wedge y \in J) \implies x \in J)$.

S. A. Bhatti [5] (see also [4] and [7]) introduced the notion of strong ideals in BCI-algebras and obtained some results about it. This class of ideals in BCI-algebras was the subject of study in [9] by S. M. Hong, Y. B. Jun and J. Meng and in [1] by H. A. Abujabal and J. Meng. In [5], the concept of strong ideal in a BCI-algebra is determined as follows: An ideal J of a BCI-algebra $\mathfrak{A} =: (A, *, 0)$ is a strong ideal in $\mathfrak{A}$ if the following holds

(SJ) $(\forall x, y \in A)((x \in J \wedge y \in A \setminus J) \implies x * y \in A \setminus J)$.

The concept of regular ideals in BCI-algebras was introduced in 1986 by D. Meng ([14]): An ideal J of a BCI-algebra $\mathfrak{A}$ is called regular if the following holds

(JR) $(\forall x, y \in A)((x * y \in J \wedge x \in J) \implies y \in J)$.

S. H. Bhatti proved ([6]) that the concept of regular ideals and the concept of strong ideals in BCI-algebras are coincident.

The concept of strong ideals in BH-algebras was discussed in [2] by S. S. Ahn and J. H. Lee: a non-empty subset J of a BH-algebra $\mathfrak{A} = (A, *, 0)$ is called a strong ideal in $\mathfrak{A}$ if it satisfies (J0) and

(StJ) $(\forall x, y, z \in A)(((x * y) * z \in J \wedge y \in J) \implies x * z \in J)$.

In doing so, it is shown that a sub-algebra S in a BH-algebra $\mathfrak{A} = (A, *, 0)$ is a strong ideal in $\mathfrak{A}$ if and only if the following holds

$(\forall x, y, z \in A)((x \in S \wedge y * z \in A \setminus S) \implies (y * x) * z \in A \setminus S)$.

The concept of strong ideals in (pseudo-)BCH-algebras, determined as an ideal that satisfies the condition (SJ), was the subject of interest in [19] written by A. Walendziak.

In this paper, we introduce the concepts of strong ideals in BI-algebras and study their properties. It has been proven that every ideal in a right distributive BI-algebra is a strong ideal in such an algebra. In addition, we introduce the concept of weak ideals in BI-algebras and establish some of its important basic properties. Also, it was shown that these two concepts are mutually independent. Finally, it is proven that these two concepts coincide in the right distributive BI-algebra.

2. The main results: Strong and weak ideals in BI-algebras

In what follows, we deal with the creation of the direct product BI-algebras. Let $\{(A_i, *_i, 0_i) : i \in I\}$ be a family of BI-algebras. If on the set

$$\prod_{i \in I} A_i =: \{f : I \longrightarrow \cup_{i \in I} A_i \mid (\forall i \in I)(f(i) \in A_i)\},$$

we define the operation $\odot$ as follows

$$(\forall f, g \in \prod_{i \in I} A_i)(\forall \in I)((f \odot g)(i) =: f(i) *_i g(i)),$$

we created the structure $(\prod_{i \in I} A_i, \odot, f_0)$, where f_0 was chosen as follows

$$(\forall i \in I)(f_0(i) =: 0_i).$$

Before we start working with direct products of BI-algebras, we say that the operation determined in this way is well-defined. If a priori we accept conditions that ensure the existence of non-empty direct product, we can prove the following theorem.

Theorem 2.1. *The direct product of any family of BI-algebras, determined as above, is a BI-algebra.*

Proof. By direct verification, it can be proved that this structure satisfies the axioms of BI-algebra:

Let $f, g \in \prod_{i\in I} A_i$ be arbitrary elements and $i \in I$. Then, we have:

(Re) $(f \odot f)(i) = f(i) *_i f(i) = 0_i$.

(BI) Considering that

$((f \odot (g \odot f))(i) = f(i) *_i (g(i) *_i f(i)) = f(i)$,

we have that (BI) is a valid formula for the observed structure.

Therefore, the structure $(\prod_{i\in I} A_i, \odot, f_0)$ is a BI-algebra. □

2.1. **Strong ideals.** The design of the concept of strong ideals in BI-algebras introduces the following concept of strong ideals in BH-algebras ([2]).

Definition 2.3. *A non-empty subset J of a BI-algebra $\mathfrak{A} =: (A, *, 0)$ is called a strong ideal in $\mathfrak{A}$ if it satisfies (J0) and the following condition:*

(StJ) $(\forall x, y, z \in A)(((x * y) * z \in J \wedge y \in J) \implies x * z \in J)$.

Proposition 2.2. *Any strong ideal in a BI-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.*

Proof. Putting $z = 0$ in (StJ), we obtain (J1). □

Proposition 2.3. *In every BI-algebra $\mathfrak{A} =: (A, *, 0)$, the subset $\{0\}$ is a strong ideal in $\mathfrak{A}$.*

Proof. Let $x, y, z \in A$ be arbitrary elements such that $(x * y) * z = 0$ and $y = 0$. Then $x * z = (x * 0) * z = 0$ in accordance with (M). So, the subset $\{0\}$ is a strong ideal in $\mathfrak{A}$. □

The concept of sub-algebras in a BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is introduced by a standard way. A nonempty subset S of A is a sub-algebra in $\mathfrak{A}$ if it satisfies the condition

(S1) $(\forall x, y \in A)((x \in S \wedge y \in S) \implies x * y \in S)$.

It can immediately be concluded that the sub-algebra S in a BI-algebra $\mathfrak{A}$ satisfies the condition

(S0) $0 \in S$.

Example 2.1. *Let $A = \{0, a, b, c\}$ be a set with the operation given with the table*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

Then $\mathfrak{A} =: (A, \cdot, 0)$ is a BI-algebra ([8], *Example 3.3).*

Subsets $S_0 = \{0\}$, $S_1 = \{0, a\}$, $S_2 = \{0, b\}$, $S_3 = \{0, c\}$, $S_4 = \{0, a, b\}$, and $S_6 = \{0, b, c\}$ are sub-algebras in $\mathfrak{A}$. Sub-set $K = \{0, a, c\}$ is not a sub-algebra in $\mathfrak{A}$, because, for example, we have $a \in K$ and $c \in K$ but $a * c = b \notin K$.

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_2 = \{0, b\}$, $J_3 = \{0, c\}$, $J_5 = \{0, a, c\}$ are ideals in $\mathfrak{A}$. Subset $S_4 = \{0, a, b\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $a \in S_4$ and $c * a = b \in S_4$ but $c \notin S_4$. Also, subset $S_6 = \{0, b, c\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $c \in S_6$ and $a * c = b \in J_6$ but $a \notin J_6$.

The ideals J_0 and J_2 are strong ideals in $\mathfrak{A}$.

The ideal J_1 is not a strong ideal in $\mathfrak{A}$ because, for example, we have $(c * a) * b = b * b = 0 \in J_1$ and $a \in J_1$ but $c * b = c \notin J_1$. The ideal J_3 is not a strong ideal in $\mathfrak{A}$ because, for example, we have $(a * c) * b = b * b = 0 \in J_3$ and $c \in J_3$ but $a * b = a \notin J_3$. The ideal J_5 is not a strong ideal in $\mathfrak{A}$ because, for example, we have $(a * a) * c = 0 * c = 0 \in J_5$ and $a \in J_5$ but $a * c = b \notin J_5$. □

Remark 2.1. *As shown in the previous example, a sub-algebra, that is, an ideal in a BI-algebra, does not have to be a strong ideal in that algebra.*

Theorem 2.2. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism of BI-algebras. If C is a strong ideal of $\mathfrak{B}$, then $f^{-1}(C)$ is a strong ideal in $\mathfrak{A}$.*

Proof. Since $f(0) = 0$, we have $0 \in f^{-1}(C)$. Let $x, y, z \in A$ be such that $(x*y)*z \in f^{-1}(C)$ and $y \in f^{-1}(C)$. Then $(f(x) * f(y)) * f(z) = f((x * y) * z) \in C$ and $f(y) \in C$. Since C is a strong ideal in $\mathfrak{B}$, it follows from (StJ) that $f(x * z) = f(x) * f(z) \in C$. So that $x * z \in f^{-1}(C)$. Hence $f^{-1}(C)$ is a strong ideal in $\mathfrak{A}$. □

Corollary 2.1. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism of BI-algebras. Then $Kerf := \{x \in A : f(x) = 0\}$ is a strong ideal of $\mathfrak{A}$.*

Proof. Since the subset $\{0\}$ is a strong ideal in the BI-algebra $\mathfrak{B}$, by Proposition 2.3, we have that the kernel $Kerf = f^{-1}(\{0\})$ of the homomorphism f is a strong ideal in $\mathfrak{A}$ in accordance with the previous theorem. □

The following theorem gives another determination of the concept of strong ideal in BI-algebras.

Theorem 2.3. *Let J be an ideal in a BI-algebra $\mathfrak{A} = (A, *, 0)$. Then J is a strong ideal in $\mathfrak{A}$ if and only if the following holds*

(StJ1) $(\forall x, y, z \in A)((x * z \in A \setminus J \ \wedge \ y \in J) \implies (x * y) * z \in A \setminus J)$.

Proof. Let J be a strong ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be arbitrary elements such that $x * z \notin J$ and $y \in J$. If we assume that $(x * y) * z \in J$, then it would be $x * z \in J$ since J is a strong ideal in $\mathfrak{A}$. We got a contradiction. So, it must be $(x * y) * z \notin J$.

Conversely, let (StJ1) be a valid formula for the ideal J and let $x, y, z \in A$ be such that $(x * y) * z \in J$ and $y \in J$. Assume that $x * z \notin J$. Then, according to (StJ1), there would be $(x * y) * z \notin J$. We got a contradiction. Therefore, $x * z \in J$. This proves that J is a strong ideal in $\mathfrak{A}$. □

Analogously to the previous one, the validity of the following theorem can be proven:

Theorem 2.4. *Let J be an ideal in a BI-algebra $\mathfrak{A} = (A, *, 0)$. Then J is a strong ideal in $\mathfrak{A}$ if and only if the following holds*

(StJ2) $(\forall x, y, z \in A)(((x * y) * z \in J \ \wedge \ x * z \in A \setminus J) \implies y \in A \setminus J)$.

On the other hand, if a BI-algebra is right distributive, we have:

Theorem 2.5. *Every ideal in a right distributive BI-algebra is a strong ideal in it.*

Proof. Let J be an ideal in a right distributive BI-algebra $\mathfrak{A} = (A, *, 0)$ and let $x, y, z \in A$ be such that $(x*y)*z \in J$ and $y \in J$. First, by Theorem 3.1 in [17], we have $y \in J \implies y*z \in J$ for arbitrary $z \in A$. On the other hand, from $(x*y)*z \in J$ and $y*z \in J$ it follows $x*z \in J$ according to Theorem 3.5 in [18] since $\mathfrak{A}$ is a right distributive BI-algebra. □

Remark 2.2. *Let $\mathfrak{A} = (A, *, 0)$ as in Example 2.1. The concept of positive implicative ideals in BI-algebra was introduced in* [18], *Definition 3.3. As shown in* [18], *Example 3.6, the ideal $J_5 = \{0, a, c\}$ is a positive implicative ideal in the BI-algebra $\mathfrak{A}$ but it is not a strong ideal in $\mathfrak{A}$ as shown in Example 2.1.*

Further on, we have:

Theorem 2.6. *Let $\{(A_i, *_i, 0_i) : i \in I\}$ be a family of BI-algebras, K be a subset of I and let J_i be a strong ideal in $(A_i, *_i, 0_i)$ for each $i \in K$. Then $\prod_{i \in I} T_i$, where $T_i = J_i$ for $i \in K$ and $T_i = A_i$ for $i \in I \setminus K$, is a strong ideal in the BI-algebra $\prod_{i \in I} A_i$.*

Proof. First, it is clear that $f_0 \in \prod_{i\in I} T_i$.

If $K = \emptyset$, then $\prod_{i\in I} T_i = \prod_{i\in I} A_i$, so $\prod_{i\in I} T_i$ is certainly an ideal in $\prod_{i\in I} A_i$. Assume, therefore, that $K \neq \emptyset$.

Let $x, y, z \in \prod_{i\in I} A_i$ be such that $(x \odot y) \odot z \in \prod_{i\in I} T_i$ and $y \in \prod_{i\in I} T_i$. This means $(x(i) *_i y(i)) *_i z(i) \in J_i$ and $y(i) \in J_i$ for each $i \in K$. Then $(x \odot z)(i) = x(i) *_i z(i) \in J_i$ since J_i is a strong ideal in $(A_i, *_i, 0_i)$ for each $i \in K$. Hence $x \odot z \in \prod_{i\in I} T_i$.

As shown, $\prod_{i\in I} T_i$ is a strong ideal in $\prod_{i\in I} A_i$. □

Example 2.2. *Let $\mathfrak{A} = (A, *, 0)$ be a BI-algebra as in Example 2.1. The subset $J_2 = \{0, b\}$ is a strong ideal in $\mathfrak{A}$ as shown in Example 2.1. Then according to the Theorem 2.1, $\mathfrak{A} \times \mathfrak{A} =: (A \times A, \otimes, (0, 0))$ is a BI-algebra also, where where the operation $\otimes$ is defined as follows*

$$(\forall x, y, u, v \in A)((x, y) \otimes (u, v) =: (x * u, y * v)).$$

The subsets $J_2 \times A$, $A \times J_2$ and $J_2 \times J_2$ are strong ideals in $\mathfrak{A} \times \mathfrak{A}$ according to the Theorem 2.6. □

At the end of this subsection, let us to prove:

Theorem 2.7. *The family $\mathfrak{J}_s(A)$ of all strong ideals in a BI-algebra $\mathfrak{A} =: (A, *, 0)$ is a complete lattice.*

Proof. Let $\{J_i : i \in I\}$ be a family of ideals in a BI-algebra $\mathfrak{A}$. Since it is obvious that $0 \in \cap_{i\in I} J_i$ holds, it remains to prove the validity of (StJ) for the set $\cap_{i\in I} J_i$. Let $x, y, z \in A$ be such that $(x * y) * z \in \cap_{i\in I} J_i$ and $y \in \cap_{i\in I} J_i$. Then $(x * y) * z \in J_i$ and $y \in J_i$ for each $i \in I$. Thus $x * z \in J_i$ for each $i \in I$ by (StJ). Hence, $x \cdot z \in \cap_{i\in I} J_i$.

If we denote by $\mathcal{Z}$ the family of all ideals of the algebra $\mathfrak{A}$ that contain the set $\cup_{i\in I} S_i$, then $\cap\mathcal{Z}$ is an ideal in $\mathfrak{A}$ according to the first part of this proof.

If we put $\sqcap_{i\in I} J_i = \cap_{i\in I} J_i$ and $\sqcup_{i\in I} J_i = \cap\mathcal{Z}$, then $(\mathfrak{J}_s(A), \sqcap, \sqcup)$ is a complete lattice. □

Corollary 2.2. *Let $\mathfrak{A} = (A, *, 0)$ be a BI-algebra. For each $x \in A$ there is a minimal strong ideal J_x in $\mathfrak{A}$ that contains x.*

Proof. This, according to what has been proven, is the intersection J_x of all strong ideals in the BI-algebra $\mathfrak{A}$ that contain the element x. Indeed, if J is a strong ideal in $\mathfrak{A}$ containing x, then $J_x \subseteq J$. Therefore, J_x is a minimal ideal in $\mathfrak{A}$ that contains x. □

2.2. **Weak ideals.** We introduce the concept of weak ideals in BI-algebras by the following definition:

Definition 2.4. *A non-empty subset J of a BI-algebra $\mathfrak{A} =: (A, *, 0)$ is called a weak ideal in $\mathfrak{A}$ if it satisfies the following condition:*

(WJ) $(\forall x, y, z \in A)((x * (y * z) \in J \wedge y \in J) \implies x * z \in J)$.

Let us prove that every weak ideal J in a BI-algebra $\mathfrak{A}$ satisfies the condition (J0).

Proposition 2.4. *Let J be a weak ideal in a BI-algebra $\mathfrak{A} = (A, *, 0)$. Then the formula (J0) is valid.*

Proof. Let J be a weak ideal in $\mathfrak{A}$. Since J is a nonempty subset in A, there exists at least one $x \in A$ such that $x \in J$. Then $J \ni x = x * 0 = x * (x * x)$ and $x \in J$ in accordance with (M) and (Re). This $0 = x * x \in J$ according to (WJ). □

Proposition 2.5. *Every weak ideal in a BI-algebra $\mathfrak{A} = (A, *, 0)$ is a sub-algebra in $\mathfrak{A}$.*

Proof. Let J be a weak ideal in $\mathfrak{A}$ and let $x, y \in A$ be such that $x \in J$ and $y \in J$. Then $J \ni x = x * 0 = x * (y * y)$ and $y \in J$ with respect to (M) and (Re). Thus $x * y \in J$ by (WJ). □

Proposition 2.6. *Every weak ideal in a BI-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.*

Proof. If we put $z = 0$ in (WJ), we get (J1) due to the respect to (M). □

Example 2.3. *Let $\mathfrak{A} = (A, *, 0)$ be a BI-algebra as in the Example 2.1. The ideals J_0, J_1, J_2 are weak ideals in $\mathfrak{A}$.*

The ideal $J_3 = \{0, c\}$ is not a weak ideal in $\mathfrak{A}$ because, for example, for $x = c$, $y = 0$ and $z = a$, we have $c * (0 * a) = c * 0 = c \in J_3$, $0 \in J_3$ but $c * a = b \notin J_3$.

The ideal $J_5 = \{0, a, c\}$ is not a weak ideal in $\mathfrak{A}$ because, for example, for $x = a$, $y = a$ and $z = c$, we have $a * (a * c) = a * b = a \in J_5$ and $a \in J_5$ but $a * c = b \notin J_5$. □

A weak ideal in a BI-algebra does not have to be a strong ideal as shown in examples 2.1 and 2.3: The ideal J_1 is a weak ideal in a BI-algebra $\mathfrak{A}$ (Example 2.3) but it is not a strong ideal in that algebra (Example 2.1).

For the family $\mathfrak{J}_w(A)$ of all weak ideals in a BI-algebra $\mathfrak{A}$, it can be proved analogously to the proof of Theorem 2.7 that:

Theorem 2.8. *$\mathfrak{J}_w(A)$ it forms a complete lattice.*

The following theorem gives another determination of the concept of weak ideal in BI-algebras.

Theorem 2.9. *Let J be an ideal in a BI-algebra $\mathfrak{A} = (A, *, 0)$. Then J is a weak ideal in $\mathfrak{A}$ if and only if the following holds*

(WJ1) $(\forall x, y, z \in A)((x * z \in A \setminus J \wedge y \in J) \implies x * (y * z) \in A \setminus J)$.

Proof. Let J be a weak ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be arbitrary elements such that $x*z \notin J$ and $y \in J$. If we assume that $x * (y * z) \in J$, then it would be $x * z \in J$ since J is a weak ideal in $\mathfrak{A}$. We got a contradiction. So, it must be $x * (y * z) \notin J$.

Conversely, let (WJ1) be a valid formula for the ideal J and let $x, y, z \in A$ be such that $x * (y * z) \in J$ and $y \in J$. Assume that $x * z \notin J$. Then, according to (WJ1), there would be $x * (y * z) \notin J$. We got a contradiction. Therefore, $x * z \in J$. This proves that J is a weak ideal in $\mathfrak{A}$. □

Also, analogously to the proof of the previous theorem, it can be proved that it holds:

Theorem 2.10. *Let J be an ideal in a BI-algebra $\mathfrak{A} = (A, *, 0)$. Then J is a weak ideal in $\mathfrak{A}$ if and only if the following holds*

(WJ2) $(\forall x, y, z \in A)((x * (y * z) \in J \wedge x * z \in A \setminus J) \implies y \in A \setminus J)$.

However, we have:

Theorem 2.11. *If $\mathfrak{A} = (A, *, 0)$ is a right distributive BI-algebra, then every strong ideal in $\mathfrak{A}$ is a weak ideal in $\mathfrak{A}$.*

Proof. Let J be a strong ideal in a right distributive BI-algebra $\mathfrak{A}$ and let $x, y, z \in A$ be such that $x * (y * z) \in J$ and $y \in J$. Since by Proposition 3.12(v) in [8], we have $(x * y) * z \preccurlyeq x * (y * z)$, we conclude that $(x * y) * z \in J$ due to the presence of the valid formula (J2). Then $x * z \in J$ according to (StJ). □

Corollary 2.3. *Every ideal in a right distributive BI-algebra is a weak ideal in it.*

Proof. Every ideal in the right distributive BI-algebra is a strong ideal in it, according to Theorem 2.5. This means that every ideal in a right distributive BI-algebra is a weak ideal in it according to Theorem 2.11 □

Remark 2.3. *Let $\mathfrak{A} = (A, *, 0)$ be a BI-algebra as in Example 2.1. The ideal $J_1 = \{0, a\}$ is not a positive implicative ideal in $\mathfrak{A}$, as shown in* [18]*, Example 3.3, but it is a weak ideal in $\mathfrak{A}$ as shown in Example 2.3.*

3. CONCLUSIONS

The concept of BI-algebras was introduced in 2017 by A. Borumand Saeid, H. S. Kim and A. Rezaei. Ideals in such logical algebras were discussed in [3, 8, 18]. This paper is a continuation, in the literal sense, of previous research on ideals in BI-algebras. Here, in this paper, two new ideals in BI-algebras are designed: strong and weak ideals. It is shown that these two concepts in BI-algebras are different from each other.

As a continuation of such research on ideals in BI-algebras, one could, for example, create the concept of p-ideals in BI-algebras as follows:

Definition 3.5. *Let $\mathfrak{A} = (A, *, 0)$ be a BI-algebra. A nonempty subset J in A is a p-ideal in $\mathfrak{A}$ if in addition to (J0) it also satisfies the following condition*

(pJ) $(\forall x, y, z \in A)(((x * z) * (y * z) \in J \ \wedge \ y \in J) \implies x \in J)$.

It can be immediately seen that every p-ideal in a BI-algebra $\mathfrak{A} = (A, *, 0)$ is an ideal in $\mathfrak{A}$. Indeed, if we put $z = 0$ in (pJ), we get (J1).

One could, of course, try to establish a connection between the notion of p-ideal determined in this way and the notion of a strong/weak ideal in BI-algebras.

REFERENCES

[1] Abujabal, H. A. and Meng, J. On (*)-ideals and positive implicative ideals in BCI-algebras. *Proyecciones*, **18** (1999), no. 2, 195–205. DOI: 10.22199/ S07160917.1999.0002.00007

[2] Ahn, S. S. and Lee, J. H. Rough strong ideals in BH-algebras. *Honam Math. J.* **32** (2010), no. 2, 203–215.

[3] Ahn, S. S., Ko, J. M. and Borumand Saeid, A. On ideals of BI-algebras. *J. Indones. Math. Soc.* **25** (2019), no. 1, 24–34. DOI: 10.22342/jims.25.1.735.24-34

[4] Bhatti, S. A., Chaudhry, M. A. and Ahmad, B. Obstinate ideals in BCI-algebras. *J. Nat. Sci. Math.* **30** (1990), no. 1, 21–31.

[5] Bhatti, S. A. Closed-ness and open-ness of ideals in BCI-algebras. *Math. Japon.*, **36** (1991), no. 5, 915–921.

[6] Bhatti, S. A. Equivalence of strong and regular ideals in BCI-algebras. *J. Math., Punjab Univ.* **24** (1991), 81–83.

[7] Bhatti, S. A. and Zhang, X. H. Strong ideals, associative ideals and p-ideals in BCI-algebras. *J. Math., Punjab Univ.* **27** (1994). 113–120.

[8] Borumand Saeid, A., Kim, H. S. and Razaei, A. On BI-algebras. *An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat.* **25** (2017), no. 1, 177–194. https://doi.org /10.1515/auom-2017-0014

[9] Hong, S. M., Jun, Y. B. and Meng, J. On strong ideals and p-ideals in BCI-algebras. *Sci. Math.* **1** (1998), no. 2, 177–180.

[10] Hu, Q. P. and Li, X. On BCH-algebras. *Math. Semin. Notes, Kobe Univ.* **11** (1983), 2, 313–320.

[11] Imai, Y. and Iséki, K. On axiom systems of propositional calculi. XIV. *Proc. Japan Acad.* **42** (1966), 19–22.

[12] Iséki, K. An algebra related with a propositional calculus. *Proc. Japan Acad.* **42** (1966), 26–29.

[13] Jun, Y. B., Roh, E. H. and Kim, H. S. On BH-algebras. *Sci. Math.* **1** (1998), no. 3, 347–354.

[14] Meng, D. On regular ideals of BCI-algebras. *Northeast. Math. J.* **2** (1986), no. 4, 499–504 (Chinese. English summary).

[15] Neggers, J. and Kim, H. S. On B-algebras. *Mat. Vesn.* **54** (2002), no. 1, 21–29.

[16] Prior, A. N. *Formal Logic*. 2nd ed. Oxford University Press (1962).

[17] Romano, D. A. Right distributive BI-algebras. *Ann. Commun. Math.* **7** (2024), no. 3, 254-263. DOI: 10.6202/acm.2024.070304

[18] Romano, D. A. Some classes of implicative ideals in BI-algebras. *Kragujevac J. Math.* **51** (2027), no. 1, 23-—33.

[19] Walendziak, A. Strong ideals and horizontal ideals in pseudo-BCH-algebras. *Ann. Univ. Paedagog. Crac., Stud. Math.* **15** (2016), 15–25. DOI: 10.1515/ aupcsm-2016-0002

[20] Zhang, X., Ali Borzooei, R. and Jun, Y. B. Q-filters of quantum B-algebras and basic implication algebras. *Symmetry*, **2018**, 10, 573; doi:10.3390/sym10110573

[1] INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE,
KORDUNAŠKA STREET 6, 78000 BANJA LUKA, BOSNIA AND HERZEGOVINA
ORCID: 0000-0003-1148-3258
Email address: daniel.a.romano@hotmail.com