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A Study on Sums and Quotients of c-spaces with Special Reference to Graphical and Non-Graphical c-spaces

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ABSTRACT. In this article, we delved into the concept of sums of c-spaces and explored the relationships between product, quotient, and sum operations on these spaces. Utilizing categorical concepts, we demonstrated that the product of quotients of c-spaces need not be the quotient of their product. For finite graphical c-spaces, we introduced a method for identifying their quotient spaces. Additionally, we investigated a non-graphical attribute known as the C_1 property within this framework.

INTRODUCTION

The paper [18] authored by R. L. Wilder provides a comprehensive exploration of the evolution of the concept of topological connectedness. Connectedness is fundamental in disciplines such as Topology, Digital Topology, and Graph Theory. It is a well-established fact that continuous functions preserve connected sets, but there exist numerous examples in literature where discontinuous functions also map connected sets to connected sets. Some notable examples are

(1) Consider the identity function I on an infinite set X, where the domain has the indiscrete topology and the codomain has the cofinite topology. This function preserves connectedness but is not continuous.
(2) Any discontinuous function defined on a totally disconnected space X preserves connectedness bacause the only connected sets in X are one point sets.
(3) Another example illustrating this phenomenon is the real-valued function $f(x) = \begin{cases} \sin(\frac{1}{x}) & ;\text{ if } x \neq 0 \\ 0 & ;\text{ if } x = 0 \end{cases}$ on $\mathbb{R}$.

In Digital Topology, the focus shifts from continuous functions to functions that maintain connectedness of sets. This distinction is crucial in fields like image processing, where continuous images are analyzed using topological connectivity, while discrete images are better suited to graph theoretical concepts of connectedness. Interestingly, there exist graphs whose connectedness cannot be deduced from topology [1], and conversely, there are topological spaces whose connectedness cannot be inducted from any graphs [11]. Recognizing that continuous images can be seen as a limit of discrete images underscores the necessity of reconciling graph theoretical and topological notions of connectivity.

In 1983, Reinhard Börger [10] successfully axiomatized connectedness with his theory of connectivity classes. This axiomatization has since been pivotal in applied mathematics, particularly in Image Analysis, Signal Processing, and Pattern Recognition [3, 5, 11, 15, 16]. Many studies in these areas are predominantly applied rather than theoretical. This paper continues our previous work [12, 13, 14], which focuses on the foundational structural properties of these spaces.

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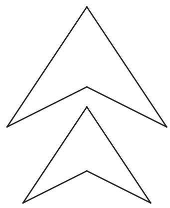

Figure 1: Connected subset of R^2

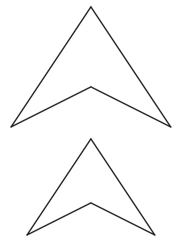

Figure 2: Disconnected subset of R^2

1. Preliminaries

Unless otherwise specified, all terminologies used in this section are derived from [2, 7, 10, 12, 13, 16]. For a nonempty set X, a connective space is a pair $(X, \mathcal{C})$ where $\mathcal{C}$ is a family of subsets of X satisfying the axioms,

(i) Both ϕ and $\{y\}$ for each $y \in X$ are members of $\mathcal{C}$.

(ii) If $\bigcap_{i\in I} C_i \neq \phi$ for a collection $\{C_i : i \in I\}$ from $\mathcal{C}$, then $\bigcup_{i\in I} C_i \in \mathcal{C}$.

(iii) For nonempty sets $C_1,\ C_2 \in \mathcal{C}$ with $C_1 \cup C_2 \in \mathcal{C}$, there exists $z \in C_1 \cup C_2$ such that $\{z\} \cup C_1 \in \mathcal{C}$ and $\{z\} \cup C_2 \in \mathcal{C}$.

(iv) If $A,\ B,\ C_i(i \in I)$ are disjoint members of $\mathcal{C}$ and if $A \cup B \cup \bigcup_{i\in I} F_i \in \mathcal{F}$, then $\exists\ J \subseteq I$ such that $A \cup \bigcup_{j\in J} C_j \in \mathcal{C}$ and $B \cup \bigcup_{i\in I-J} C_i \in \mathcal{C}$.

It is proved that finite connective spaces are precisely simple graphs[7]. If the space $(X, \mathcal{C})$ satisfies the first two axioms only, it is called a c-space and $\mathcal{C}$ is called a c-structure on X. Elements of $\mathcal{C}$ are called *connected sets*. Usually, the space $(X, \mathcal{C})$ is represented by X.

It can be noted that for any set X, the power set $\mathcal{P}(X)$ and $\mathcal{D}_X = \{\phi\} \cup \{\{y\} : y \in X\}$ are c-structures on X and the corresponding c-spaces are called indiscrete and discrete c-spaces in order. For a topological space X, $\mathcal{F} = \{C \subseteq X :\ \text{C is connected in X}\}$ forms a c-structure on X and the corresponding c-space $(X, \mathcal{F})$ is called the associated c-space of X. Similar terminology "*graphical*" can be defined for a graph.

A function $h : X \to Y$ is c-continuous if $h(A)$ is connected for each connected set A in X. If $\{X_i = (X_i, \mathcal{C}_{X_i}) : i \in I\}$ be a family of c-spaces and X a set, $\mathbb{F} = \{f_i : X \to X_i : i \in I\}$ a family of functions, then $\mathcal{S} = \{D \subset X : f_i(D) \in \mathcal{C}_{X_i}$ for each $i \in I\}$ is the strong or largest c-structure[13] on X generated by $\mathbb{F}$. Let $X = \prod_{i\in I} X_i$ and for each i, let $\pi_i : X \to X_i$ be a projection funtion on X. Then X is said to be the product space of $\{X_i : i \in I\}$ if c-structure on X is the largest c-structure generated by $\{\pi_i : i \in I\}$.

To visualize connectedness in the product space, some examples from $\mathbb{R}^2$ is given below.

Cosndier a family $\mathcal{K}$ of subsets of a set X. Then the c-structure generated by $\mathcal{K}$, represented by $\langle\mathcal{K}\rangle$, is the smallest c-structure on X containg $\mathcal{K}$. It is proved in [7] that, for any $K \in \langle\mathcal{K}\rangle$, any two distinct points in K can be joined by a finite chain of connected sets in $\mathcal{K}$ that are contained in K. A c-space $(X, \mathcal{C})$ is said to be 2-generated if $\mathcal{C} = \langle\mathcal{K}\rangle$ where $\mathcal{K} \subseteq \{K \in \mathcal{C} : |K| \leq 2\}$. Note that these are graphical c-spaces and have a crusial part in the study of finite c-spaces[9].

Consider a class of functions $\{g_i : X_i \to Z : i \in J\}$ where Z is any set. Then $\langle\{g_i : i \in J\}\rangle_W$ denotes the weakest(smallest) c-structure on Z that make each g_i c-continuous. In particular, an onto map $h : X \to Z$ is called a quotient map if $\mathcal{C}_Z = \langle\{h\}\rangle_W$. Equivalently, Z is a quotient space of X.

A characterization of a connected set in a generated c-structure is given below.

Theorem 1.1. [13] *Let $(X, \mathcal{C})$ be a c-space with $\mathcal{C} = \langle\mathcal{K}\rangle$, where $\mathcal{K} \subset \mathcal{P}(X)$. A subset B of X is connected if and only if $B = \underset{x\in B}{\cup} E_{x_0x}$, where $x_0 \in B$ and $E_{x_0x} = \overset{n_x}{\underset{i=1}{\cup}} B_{x_i}$ such that $x_0 \in B_{x_1}$, $x \in B_{x_{n_x}}$, $B_{x_i} \in \mathcal{K}$ for each i such that $B_{x_i} \subset B$ and $B_{x_i} \cap B_{x_{i+1}} \neq \phi$ for $1 \leq i \leq n_x - 1$.*

The Tensor Product [2] $X_1 \boxtimes X_2$ of two c-spaces X_i, $(i = 1, 2)$ is the set $X_1 \times X_2$ with the generated c-structure $\langle\{C_1 \times C_2 : C_1 \in \mathcal{C}_{X_1}, C_2 \in \mathcal{C}_{X_2}\}\rangle$ on it. We note that this structure is smaller than the cartesian structure on $X_1 \times X_2$.

Let $A, B \subset X$. The element $x \in X$ is defined to touch A if we can find a nonempty subset K of A such that $K \cup \{x\}$ is connected in X. Moreover A and B is said to touch if we can find y in $A \cup B$ that touches both A and B. Furthermore, points y and z of X said to touch if z touches $\{y\}$. Let $t(A) = \{y \in X : y \text{ touch } A\}$. Then, A is said to be t-closed if $t(A) = A$. The t-closure of a set B, represented by $\overline{B}$, is defined as the smallest t-closed set containing B.

2. Some Properties of Sum or Coproduct of c-spaces

Let $(X_1, \mathcal{C}_{X_1})$ and $(X_2, \mathcal{C}_{X_2})$ be two disjoint c-spaces. Then its sum space [2] is defined to be the c-space $(X, \mathcal{C})$, where $X = X_1 \cup X_2$ and $\mathcal{C} = \mathcal{C}_{X_1} \cup \mathcal{C}_{X_2}$. We can extend the above definition to arbitrary class of c-spaces.

Definition 2.1. *Let $\mathcal{F} = \{(X_i, \mathcal{C}_{X_i}) : i \in I\}$ be a family of c-spaces and $X = \sum_{i\in I} X_i$ be the set theoretical sum of the sets $\{X_i : i \in I\}$. For each $i \in I$, define $\lambda_i : X_i \to X$ by $\lambda_i(x) = (x, i)$. Let $\mathcal{C} = \langle\{\lambda_i : i \in I\}\rangle_W$. Then the c-space $(X, \mathcal{C})$ is defined as the Sum or co-product and is represented by $\sum_{i\in I} X_i$.*

Proposition 2.1. *Let $(X, \mathcal{C})$ be the sum of the c-spaces $\{(X_i, \mathcal{C}_{X_i}) : i \in I\}$. Then $\mathcal{C} = \bigcup_{i\in I}\{C \times \{i\} : C \in \mathcal{C}_{X_i}\}$.*

Proof. Let $K \in \mathcal{C}$. Fix an element $x_0 \in K$. As $\mathcal{C} = \langle\mathcal{B}\rangle_W$, where $\mathcal{B} = \{\lambda_i(C) : C \in \mathcal{C}_{X_i}, i \in I\}$, we have $\mathcal{B} = \{K \times \{i\} : K \in \mathcal{C}_{X_i}, i \in I\}$. By Theorem 1.1, we can write $K = \underset{x\in K}{\cup} E_{x_0x}$, where $E_{x_0x} = \overset{n_x}{\underset{i=1}{\cup}} B_{x_i}$ such that $x_0 \in B_{x_1}$, $x \in B_{x_{n_x}}$, $B_{x_i} \in \mathcal{B}$ for each i with $B_{x_i} \subset K$ and $B_{x_i} \cap B_{x_{i+1}} \neq \phi$ for $1 \leq i \leq n_x - 1$.

Since $B_{x_i} \in \mathcal{B}$ for each i and since $B_{x_i} \cap B_{x_{i+1}} \neq \phi$ for $1 \leq i \leq n_x - 1$, each B_{x_i}, $1 \leq i \leq n_x$, must be of the form $C \times \{k\}$ for some $C \in \mathcal{C}_{X_k}$ and for a fixed $k \in I$. Let $B_{x_1} = C_1 \times \{k\}$ and $B_{x_2} = C_2 \times \{k\}$ for some $C_1, C_2 \in \mathcal{C}_{X_k}$. As $B_{x_1} \cap B_{x_2} \neq \phi$, we have $C_1 \cap C_2 \neq \phi$. Thus $C_1 \cup C_2 \in \mathcal{C}_{X_k}$, so that $(C_1 \cup C_2) \times \{k\} \in \mathcal{B}$ and hence $B_{x_1} \cup B_{x_2} \in \mathcal{B}$. Arguing like this, being a finite union, $E_{x_0x} = \overset{n_x}{\underset{i=1}{\cup}} B_{x_i} \in \mathcal{B}$.

As $E_{x_0x} \in \mathcal{B}$ for each $x \in K$ and since $E_{x_0x} \cap E_{x_0y} \neq \phi$ for each $x, y \in K$, arguing as above, we have $\underset{x\in K}{\cup} E_{x_0x} \in \mathcal{B}$. Thus $K \in \mathcal{B}$, so that $\mathcal{C} \subseteq \mathcal{B}$. As $\mathcal{C} = \langle\mathcal{B}\rangle_W$, we have $\mathcal{B} \subseteq \mathcal{C}$. Hence our result follows. □

Remark 2.1. *If $\{(X_i, \mathcal{C}_{X_i}) : i \in I\}$ is a pairwise disjoint class of c-spaces, then its sum can be identified with the c-space $(X, \mathcal{C})$, with $X = \bigcup_{i\in I} X_i$ and $\mathcal{C} = \bigcup_{i\in I} \mathcal{C}_{X_i}$.*

Since any connected set in X belongs to exactly one component in X[7], any c-space X is same as its sum of components. That is, $X = \sum_{x\in X} C_x$, where C_x is the unique component of X containing x.

We can note that the above observation is not true in the category of topological spaces TOP. For example, consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Here components are singleton sets, so that its topology is discrete. If we write $\mathbb{Q}$ as its sum of its components, topology of $\mathbb{Q}$ becomes discrete, a contradiction.

Proposition 2.2. *The sum of quotients of c-spaces is the quotient of its sums.*

Proof. For each $i \in I$, let Y_i be a quotient space of X_i and the corresponding quotient map be $f_i : X_i \to Y_i$. We will prove that $Y = \sum_{i\in I} Y_i$ is a quotient space of $X = \sum_{i\in I} X_i$. For $x \in X_i$, define $f : X \to Y$ by $f(x) = f_i(x)$. Clearly f is c-continuous.

We will prove that $\mathcal{C}_Y = \langle\{f_i : i \in I\}\rangle_W$. Let $\mathcal{D}_Y = \langle\{f_i : i \in I\}\rangle_W$. Since each f_i is c-continuous, we have $\mathcal{D}_Y \subseteq \mathcal{C}_Y$. Further, let C be any connected set in Y. Then $C \in \mathcal{C}_{Y_i}$ for some $i \in I$. Since Y_i is a quotient space of X_i, we have $\mathcal{C}_{Y_i} = \langle\{f_i\}\rangle_W$, so that $C \in \langle\{f_i\}\rangle_W$ and hence $C \in \langle\{f_i : i \in I\}\rangle_W$. That is, $C \in \mathcal{D}_Y$ and hence $\mathcal{C}_Y \subseteq \mathcal{D}_Y$. Consequently, we have $\mathcal{C}_Y = \mathcal{D}_Y$. That is, $\mathcal{C}_Y = \langle\{f_i : i \in I\}\rangle_W$.

Thus $f : X \to Y$ is quotient map and hence sum of quotients of c-spaces is the quotient of its sums. □

We note that, a family $\mathfrak{C}$ of sets is said to be a cover [4] of a set C if $C \subseteq \bigcup_{K\in\mathfrak{C}} K$.

Theorem 2.2. *Let $\{X_i : i \in I\}$ be a family of c-spaces and Y be any nonempty set. Let $\{f_i : X_i \to Y : i \in I\}$ be a family of functions. Then Y is a quotient space of the sum space $\sum_{i\in I} X_i$ if $\mathcal{C}_Y = \langle\{f_i : i \in I\}\rangle_W$, provided $\{f_i(X_i) : i \in I\}$ covers Y.*

Proof. Without loss of generality, assume that the collection $\{X_i : i \in I\}$ is pairwise disjoint. Let $X = \sum_{i\in I} X_i$. Define $g : X \to Y$ by $g(x) = f_i(x)$ if $x \in X_i$. Since each f_i is c-continuous, g is c-continuous. We claim that g is onto. Let $y \in Y$. Since $Y \subseteq \bigcup_{i\in I} f_i(X_i)$, let $y \in f_i(X_i)$ for some i. Obviously there is an $x \in X_i$ such that $f_i(x) = y$. Thus $g(x) = y$ for some $x \in X$ and hence g is onto.

To prove our claim, it is enough to show that $\mathcal{C}_Y = \langle\{g\}\rangle_W$.

Let $C \in \langle\{g\}\rangle_W$. For c_1, c_2 in C, we can find [7] a finite chain of connected sets $D_1, D_2, ..., D_n$ in X such that $c_1 \in g(D_1)$, $c_2 \in g(D_n)$, $g(D_i) \cap g(D_{i+1}) \neq \phi$ for $i = 1$ to $n-1$ and $g(D_i) \subset C$ for $i = 1$ to n. Clearly for $j = 1$ to n, $D_j \subset X_i$ for some $i \in I$. Rename X_is so that $D_i \subset X_i$ for $i = 1$ to n, so that $g(D_i) = f_i(D_i)$ for $i = 1$ to n. Clearly $C \in \langle\{f_i(K) : K \in \mathcal{C}_{X_i}, i \in I\}\rangle$ and hence $C \in \langle\{f_i : i \in I\}\rangle_W$. Thus $C \in \mathcal{C}_Y$, so that $\langle\{g\}\rangle_W \subseteq \mathcal{C}_Y$.

Conversely let $C \in \mathcal{C}_Y$. Let c_1,c_2 be any two elements of C. Then as above, we can find a finite chain of connected sets K_1, K_2, K_3,... K_m in X such that $c_1 \in f_j(K_1)$ and $c_2 \in f_k(K_m)$, $j, k \in I$ such that $f_q(K_i) \cap f_r(K_{i+1}) \neq \phi$, $q, r \in I$, $1 \leq i \leq m-1$ and $f_s(K_i) \subset C$, for every $1 \leq i \leq m$ and $s \in I$. Since $f_i = g$ for every i, we have $c_1 \in g(K_1)$, $c_2 \in g(K_2)$, $g(K_i) \cap g(K_{i+1}) \neq \phi$ for $i = 1$ to m and $g(K_i) \subset C$ for every i. Thus $C \in \langle\{g(K) : K \in \mathcal{C}_X\}\rangle$ and hence $C \in \langle\{g\}\rangle_W$, so that $\mathcal{C}_Y \subseteq \langle\{g\}\rangle_W$. Consequently, $\mathcal{C}_Y = \langle\{g\}\rangle_W$. Hence the theorem is proved. □

2.1. **Sum and Product Spaces.** In this section, we investigate whether the product of quotients of c-spaces is equivalent to the quotient of their product. It's important to note that in any cartesian closed topological category, this result is established [8]. Here, we demonstrate that the same result holds for the category of c-spaces, despite it not being cartesian closed [2].

Proposition 2.3. *For nonempy c-spaces $\{X_i : i \in I\}$ and a c-space X, $X \times \sum_{i\in I} X_i$ is c-isomorphic to the sum space $\sum_{i\in I}(X \times X_i)$.*

Proof. For conveience, assume that members of the family of c-spaces $\{X_i : i \in I\}$ are pairwise disjoint. Define $f : X \times \sum_{i\in I} X_i \to \sum_{i\in I}(X \times X_i)$ by $f(x, y) = (x, y)$, where $x \in X$, $y \in X_i$ for some $i \in I$. Clearly f is bijective.

Let C be connected in $X \times \sum_{i\in I} X_i$. Then $\pi_1(C)$ is connected in X and $\pi_2(C)$ is connected in $\sum_{i\in I} X_i$, so that $\pi_2(C)$ is connected in X_i for some i. Thus C is connected in $X \times X_i$ for some i and hence it is connected in $\sum_{i\in I}(X \times X_i)$. As $f(C) = C$, f is c-continuous.

Similar arguments shows that f^{-1} is also continuous. Thus f is a c-isomorphism. □

Theorem 2.3. [17] *In a topological construct $\mathcal{T}$, below stated results are equivalent.*

(1) $\mathcal{T}$ is cartesian closed.
(2) For any object T in $\mathcal{T}$, and for any collection of objects $(K_i)_{i\in I}$ from $\mathcal{T}$, below stated results are satisfied.
   i. $T \times \sum_{i\in I} K_i \cong \sum_{i\in I}(T \times K_i)$.
   ii. *If f is a quotient map, then so is $I_T \times f$, where I_T is the identity morphism on T.*
(3) For an object T in $\mathcal{T}$, and for any collection of objects $(K_i)_{i\in I}$ from $\mathcal{T}$, they satisfy the following conditions.
   i. $T \times \sum_{i\in I} K_i \cong \sum_{i\in I}(T \times K_i)$.
   ii. *In $\mathcal{T}$, for the quotient maps g and h, the product $g \times h$ is again a quotient map.*

From this theorem, we can make the following observation.

Remark 2.2. *(1) Finite product of quotients of c-spaces need not be the quotients of its product.*
(2) For a quotient map $f : X \to Y$, the map $I_X \times f : X \times X \to X \times Y$ need not be a quotient map.

Proof. From [10], it can be noted that the category of c-spaces is not a cartesian closed topological category.

(1) By Proposition 2.3, we have $A \times \sum_{i\in I} B_i \cong \sum_{i\in I}(A \times B_i)$. Considering the equivalence of the statements (1) and (2) in the Theorem 2.3, it follows that finite product of quotients of c-spaces need not be the quotient of its products.
(2) Similarly considering the equivalence of the statements (1) and (3) of the Theorem 2.3, we can observe that $I_X \times f : X \times X \to X \times Y$ is not a quotient map.

□

Finding a concrete example to prove the above claim is an open problem. In our forthcoming paper, we will give the condition under which the product of quotients become a quotient of its product. In the next remark we prove that, the sum of c-spaces where each summand is same, can be viewed as product space up to c-isomorphism.

Remark 2.3. *If X is any c-space and if $X_\alpha = X$ for every $\alpha \in I$, then the sum space $\sum_{\alpha\in I} X_\alpha$ is c-isomorphic to the product space $X \times I$, where I is considered to be a discrete c-space.*

Proof. Let $(Z, \mathcal{C}_Z) = \sum_{\alpha\in I} X_\alpha$. Then

$$Z = \underset{i\in I}{\cup}(X_i \times \{i\}) \text{ and } \mathcal{C}_Z = \underset{i\in I}{\cup}\{C \times \{i\} : C \in \mathcal{C}_{X_i}\}$$

Define $f : Z \to X \times I$ by $f(z, k) = (z, k)$, $z \in X_k$ and $k \in I$. We can verify that f is a c-isomorphism. □

2.2. **Quotient Space of a Graphical c-space.** A c-space X is called graphical[9] if connected sets of X are exactly the connected subgraphs of G, for some graph G. Let $\mathcal{P}$ be a partition of a vertex set $V(G)$ of a graph G. Then $\mathcal{P}$ is called a connected partition if every member of $\mathcal{P}$ is connected as a subgraph of G. In this section, we generalize the following theorem.

Theorem 2.4. [13] *X be the associated c-space of a finite connected graph G. Let G^* be a quotient graph of G corresponding to a connected partition of the vertex set of G. If X^* is the assocaited c-space of G^*, then X^* is a quotient space of X.*

Here we claim that the above theorem is valid even if graph is not connected.

Proof. Let $G = \sum_{i\in I} C_i$, where C_is are the components of the given graph G. Consider a connected partition $\mathcal{P}_i$ of the vertices of C_i. Then $\{P_i : P_i \in \mathcal{P}_i, i \in I\}$ is a partition of the vertex set of G, which is connected. For each $i \in I$, let X_i be the associated c-space of C_i and X_i^* be the associated c-space of the graph G_i, which is the quotient graph of C_i with respect to the partition $\mathcal{P}_i$ of C_i.

By the above Theorem 2.4, X_i^* is a quotient space of X_i. Then by Proposition 2.2, $X^* = \sum_{i\in I} X_i^*$ is a quotient space of $\sum_{i\in I} X_i$. But obviously $X = \sum_{i\in I} X_i$. Hence X^* is a quotient space of X. □

2.3. **Product and Sum of Topologizable and Graphical c-spaces.** A c-space $(X, \mathcal{C})$ is called topological or topologizable [9], if we can find a topology τ on X such that connected sets of (X, τ) are precisely that of $(X, \mathcal{C})$. That is, $(X, \mathcal{C})$ is the the associated c-space of (X, τ). This section deals with whether the product and the sum of topologizable(resp. graphical) c-spaces are topologizable(resp. graphical) or not. A partial settlement of the problem is given.

Remark 2.4. *We note the following:*

a. *Let T_i, $i = 1, 2$ be two topological spaces such that X_i, $i = 1, 2$ is their associated c-spaces in order. Then $X_1 \times X_2$ need not be the associated c-space of $T_1 \times T_2$. For example, $\mathbb{R}^2$ as a c-space has more connected sets than $\mathbb{R}^2$ as a topological space. For clarity, We may refer to the figure given in the preliminary section. Hence, product of topologizable c-spaces is not topologizable in the product topology of the corresponding topological spaces.*
b. *Let G_i, $i = 1, 2$ be two graphs such that X_i, $i = 1, 2$ be their associated c-spaces in order. Then $X_1 \times X_2$ need not be the associated c-space of the graph products (Cartesian Product, Tensor Product and Lexicographic Product) of G_1 and G_2. This can be shown by taking suitable graphs G_1 and G_2. Readers are requested to exercise the same and can try for other graph products too.*

Proposition 2.4. *The sum of topologizable c-spaces are topologizable. The same statement holds for graphical c-spaces.*

Proof. Let $\{X_i : i \in I\}$ be a family of pairwise disjoint topologizable c-spaces. Then there exists a family $\{(X_i, \tau_i) : i \in I\}$ of topological spaces such that X_i is the associated c-space of (X_i, τ_i) for each $i \in I$.

Then as a c-space, the connected sets of $X = \sum_{i \in I} X_i$ are precisely the connected sets of each X_i. As a topological space, in $X = \sum_{i \in I} X_i$, it can be easily shown that connected sets of X are precisely the connected sets of each X_i. From the above two facts, it follows that sum of topologizable c-spaces is topologizable.

In the case of graphical c-spaces, proof is straight forward. □

3. Nongraphical c-space: C_1 c-property

The concept of C_1 axiom can be found in [7] related to connective sapces. Except the definition, no further studies were found in the literature. The definition can be carried over to the class of c-spaces too. Hence the c-space X is said to be C_1 if distinct points of X are disconnected [7].

Proposition 3.5. [7] *Let X be a c-space. Then,*

(1) If distinct points of X do not touch, then X is C_1. That is, for every $x \in X$, $\overline{\{x\}} = \{x\}$.
(2) X have no 2 element connected sets if and only if X is C_1.

Consequently, a 2- generated the c-space is C_1 if and only if it is a discrete c-space. From this it is clear that if a space X is C_1, it is nongraphical.

Theorem 3.5. *A c-space X is C_1 if and only if $\overline{\{x\}} \neq \overline{\{y\}}$ for every $x, y \in X$ with $x \neq y$.*

Proof. Let X be a C_1 c-space. Then $\overline{\{x\}} = \{x\}$ for every $x \in X$. Let x, y be any two distinct elements of X. Then

$$\overline{\{x\}} = \{x\} \neq \{y\} = \overline{\{y\}}$$

Conversely let $\overline{\{x\}} \neq \overline{\{y\}}$ for any two distinct points x and y in X.

First we claim that x touches $\{y\}$ implies $\overline{\{x\}} = \overline{\{y\}}$.

Suppose x touches $\{y\}$. Then $x \in t(\{y\})$. We know that for any subset A of X, $\overline{A} = t^{\gamma}(A)$ where γ is the least ordinal such that $t^{\gamma}(A) = t^{\gamma+1}(A)$. Thus let $\overline{\{x\}} = t^{\gamma_1}(\{x\})$ with $t^{\gamma}(\overline{\{x\}}) = \overline{\{x\}}$ for every $\gamma \geq \gamma_1$ and $\overline{\{y\}} = t^{\gamma_2}(\{y\})$ with $t^{\gamma}(\overline{\{y\}}) = \overline{\{y\}}$ for every $\gamma \geq \gamma_2$.

With out loss of generality we let, $\gamma_1 \leq \gamma_2$. As $x \in t(\{y\})$, applying t-closure, we have $t(\{x\}) \subseteq t^2(\{y\})$. By repeating the same process we obtain the expression $t^{\gamma_1}(\{x\}) \subseteq t^{\gamma_1+1}(\{y\})$. That is, $\overline{\{x\}} \subseteq t^{\gamma_1+1}(\{y\})$. If $\gamma_2 \neq \gamma_1 + 1$, further applying t-closure operator on both sides of the above expression, to the desired stage, we have $\overline{\{x\}} \subseteq t^{\gamma_2}(\{y\})$, so that $\overline{\{x\}} \subseteq \overline{\{y\}}$.

Since x touches $\{y\}$, y also touches $\{x\}$, so that $y \in t(\{x\})$. Proceeding as above, we have $\overline{\{y\}} \subseteq \overline{\{x\}}$. From the above two subset relations, the required result $\overline{\{x\}} = \overline{\{y\}}$ follows.

Let x be any point of X. Let y be a touching point of $\{x\}$. If $y \neq x$, by our assumption $\overline{\{x\}} \neq \overline{\{y\}}$. Then by our claim above, x will not touch y . So we have $\overline{\{x\}} = \{x\}$ and it follows that X is C_1. □

Proposition 3.6. *The associated c-space of a T_1 topological space is C_1.*

Proof. We know that in a T_1 topological space X, finite subsets are closed. Since every closed set is t-closed in its associated c-space[7], one point sets are t-closed in the associated c-space of X. That is, $\overline{\{x\}} = \{x\}$ for every $x \in X$. Hence the associated c-space of X is C_1. □

Before moving to the next proposition, let us recall the deinition of a $\check{C}$ech closure operator. A function $F : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is said to be a $\check{C}$ech closure operator (or a closure operator) on X if, $F(\phi) = \phi$, $B \subseteq F(B)$ for every $B \in \mathcal{P}(X)$ and $F(B \cup C) = F(B) \cup F(C)$ for every $B, C \in \mathcal{P}(X)$. The space (X, F) is called a closure space. A c-space $(X, \mathcal{C})$ is said to be *induced by a closure operator* [9] F on X if the collection of all connected subsets of (X, F) is $\mathcal{C}$.

Proposition 3.7. *Let X be a non discrete finite C_1 space. Then*

(1) X cannot be a connective space.
(2) X cannot be induced from a closure operator.

Proof. Assume that X is a non discrete finite C_1 space.

(1) We know that finite connective spaces are precisely finite 2-generated c-spaces[9]. By Proposition 3.5, non discrete 2- generated c-spaces are not C_1. From these two facts, Part 1 follows.
(2) We know that finite c-space induced by a closure operator is 2-generated[9]. By Proposition 3.5, non discrete 2-generated c-spaces are not C_1. From these two facts, Part 2 follows.

□

Remark 3.5. *From the Proposition 3.5, it follows that property of being C_1 is* hereditary *and that sum of a family of C_1 c-spaces is C_1. We may further note that C_1 is* not a divisible property.

For example, consider $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ where $X = \{a, b, c\}$, $\mathcal{C}_X = \mathcal{D}_X \cup \{X\}$, $Y = \{d, e\}$ and $\mathcal{C}_Y = \mathcal{D}_Y \cup \{Y\}$. Define $h : X \rightarrow Y$ as $h(a) = d, h(b) = d, h(c) = e$. Clearly h is c-continuous since Y is equipped with the weak c-structure. Hence Y is a quotient space of X. Note that X is C_1 and Y is not C_1. Thus quotient space of a C_1 space need not be C_1.

Proposition 3.8. *For each $i \in I$, let X_i be a nonempty c-space and $X = \prod_{i \in I} X_i$ be their product c-space. Then X is C_1 if and only if each X_i is C_1.*

Proof. For each $i \in I$, let X_i be C_1. If possible let X is not C_1. Then we can find $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ in X with $x \neq y$ such that $B = \{x, y\}$ is connected in X. Since $x \neq y$, there exists $i \in I$ such that $x_i \neq y_i$. Consider the projection function $\pi_i : X \rightarrow X_i$. As π_i is c-continuous, $\pi_i(B) = \{x_i, y_i\}$ is connected in X_i, a contradiction. Hence X is C_1.

For the converse, let, $X = \prod_{i \in I} X_i$ be C_1. Embedd each X_i in X. C_1 being hereditary, it follows that each X_i is C_1. □

Proof of the following proposition follows from the above proposition using the fact that each $\pi_i : \boxtimes_{i \in I} X_i \rightarrow X_i$ is c-continuous.

Proposition 3.9. *Let $X = \boxtimes_{i \in I} X_i$. Then X is C_1 if and only if each X_i is C_1.*

Theorem 3.6. *Let $h_i : X \rightarrow X_i$, $i \in I$ be a collection of functions from a set X to a family X_i of C_1 c-spaces for each i in I such that least one of them is injective. Then the strong c-structure generated by $\{h_i : i \in I\}$ on X is C_1.*

Proof. Without loss of generality, let $h_k : X \rightarrow X_k$, $k \in I$ be an injective function.

Let $\mathcal{C}$ be the strong c-structure on X generated by $\{h_i : i \in I\}$. Then $A \in \mathcal{C}$ if and only if $h_i(A) \in \mathcal{C}_{X_i}$ for every $i \in I$. Assume for contrary that $A = \{x, y\}$ is connected in X. Then for each $i \in I$, $h_i(A)$ is connected in X_i. Since h_k is injective, $|h_k(A)| = 2$, which is not possible as X_k is C_1. Hence X cannot contain a two element connected set. Then by Proposition 3.5, X is C_1. □

3.1. **Function Spaces and C_1 c-spaces.** For the c-spaces X and Y, let $\mathcal{C}(X,Y)$ denotes the set of all c-continuous functions from X to Y. In [2], a subset M of $\mathcal{C}(X,Y)$ is said to be connected if for every $K \in \mathcal{C}_X$, $\langle M, K\rangle \in \mathcal{C}_Y$, where $\langle M, K\rangle = \bigcup_{f\in M} f(K)$. The collection of this connected sets constitute the standard c-structure on $\mathcal{C}(X,Y)$. Unless otherwise specified, from here onwards, $\mathcal{C}(X,Y)$ is considered as a c-space with the standard c-structure.

In [2], it is also proved that, M is connected in $\mathcal{C}(X,Y)$ if and only if for all $x \in X$, $\langle M, \{x\}\rangle \in \mathcal{C}_Y$. Let us discuss how the function space of c-continuous functions and C_1 c-spaces are related.

Theorem 3.7. *For the c-spaces X and Y, Y is C_1 if and only if $\mathcal{C}(X,Y)$ is C_1.*

Proof. Let Y be C_1. Assume for contrary that $\mathcal{C}(X,Y)$ is not C_1. Then by Proposition 3.5, $\mathcal{C}(X,Y)$ has a connected set $C = \{f_1, f_2\}$, where $f_1 \neq f_2$. Then for every $x \in X$,

$$\begin{aligned} &\langle C, \{x\}\rangle \text{ is connected in } Y \\ &\iff \{f_1(x), f_2(x)\} \text{ is connected in } Y \\ &\iff f_1(x) = f_2(x), \text{ since } Y \text{ is } C_1 \\ &\iff f_1 = f_2, \text{ a contradiction.} \end{aligned}$$

Hence $\mathcal{C}(X,Y)$ is C_1.

Conversely, let $\mathcal{C}(X,Y)$ be C_1. If possible, let $\{b,c\} \in \mathcal{C}_Y$.

Let $\leq$ be a well-order on X. Fix an element $k \in X$. Let g and h be two functions from X to Y defined by

$$h(y) = \begin{cases} b & ; y \leq k \\ c & ; y \geq k \end{cases}$$

and

$$g(y) = \begin{cases} c & ; y \leq k \\ b & ; y \geq k \end{cases}$$

Obviously g and h are c-continuous and hence $g,\ h \in \mathcal{C}(X,Y)$.

Let $M = \{g,h\}$. For each x in X,

$$\begin{aligned} \langle M, \{x\}\rangle &= \{g(x), h(x)\} \\ &= \{b, c\}, \text{ a connected set in } Y. \end{aligned}$$

Hence $M \in \mathcal{C}(X,Y)$, a contradiction. By Proposition 3.5, Y is C_1. □

4. Conclusion

In this paper, our aim was to shed light on certain structural properties of c-spaces, including their sum, product, and quotient. We also introduced and examined an additional non-graphical property known as the C_1 property. Additionally, we studied function spaces of c-continuous functions within this framework. We also presented an open problem concerning the product of quotient spaces (see the Remark 2.2).

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