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Rainbow Dominator Coloring of Graphs

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ABSTRACT. Coloring and domination in graphs are two well explored areas of research in graph theory. Blending these notions, the dominator coloring of graphs was introduced in the literature; following which several variants of domination related coloring patterns have been defined and studied, based on different types of coloring and domination in graphs. A vertex coloring of a graph that demands the existence of a path in which every internal vertex between two vertices has a unique color is called a rainbow vertex coloring of the graph. In this article, we investigate the rainbow dominator coloring of graphs; a vertex coloring that combines the concepts of rainbow vertex coloring and dominator coloring of graphs. We discuss some properties of the rainbow dominator coloring of graphs and determine the rainbow dominator chromatic number of certain classes of graphs and their complements.

1. INTRODUCTION

For basic terminology in graph theory, refer to [16], and for concepts pertaining to coloring and theory of domination in graphs, see [1] and [6], respectively.

By G, we always mean a simple, undirected and a finite graph with its vertex set $V(G)$ and edge set $E(G)$. A vertex $v \in V(G)$ in a graph G of order n having degree $n-1$ is called a *universal vertex* of G and a vertex $v \in V(G)$ having degree 0 is called an *isolated vertex* of G. A vertex $v \in V(G)$ with degree 1 is called a *leaf* or a *pendant vertex* in G and the vertex u such that $uv \in E(G)$, where v is a leaf, is called its *support* or a *support vertex* in G. A subset $S \subseteq V(G)$ is called an *independent* set of G if for every pair $u, v \in S$, $uv \notin E(G)$.

Graph coloring is the assignment of colors (labels) to the entities of a graph such as its vertices or edges, according to certain rules and the set of all entities assigned the same color in a coloring c of the graph is called a *color class* with respect to c. A *proper vertex coloring* of a graph G is the assignment of colors to the vertices of G such that each color class with respect to the coloring is an independent set and the minimum number of colors required in a proper vertex coloring of G is called the *chromatic number* of G, denoted by $\chi(G)$. Any chromatic coloring of $V(G)$ with $\chi(G)$ colors is called a χ*-coloring* or a *chromatic coloring* of G.

Beginning with the four color problem that has been modelled in terms of proper vertex coloring of graphs, many variants of graph coloring schemes have been emerging in the literature, in order to meet the modelling requirements of various real-life problems (ref. [1,9,12,15]). One such problem, called the information transfer path problem in networks, has been modelled in terms of *rainbow connections* of graphs (see [2]), based on which the *vertex-rainbow coloring* of graphs has been defined in [8], as follows.

A vertex coloring of a non-trivial connected graph G in which every pair of its vertices are connected by a path whose internal vertices have distinct colors is called a *vertex-rainbow coloring* of G, and the *rainbow vertex-connection number* $rvc(G)$ of G is the minimum

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number of colors that are required to obtain a vertex-rainbow coloring of G. Note that a vertex-rainbow coloring of G need not be proper.

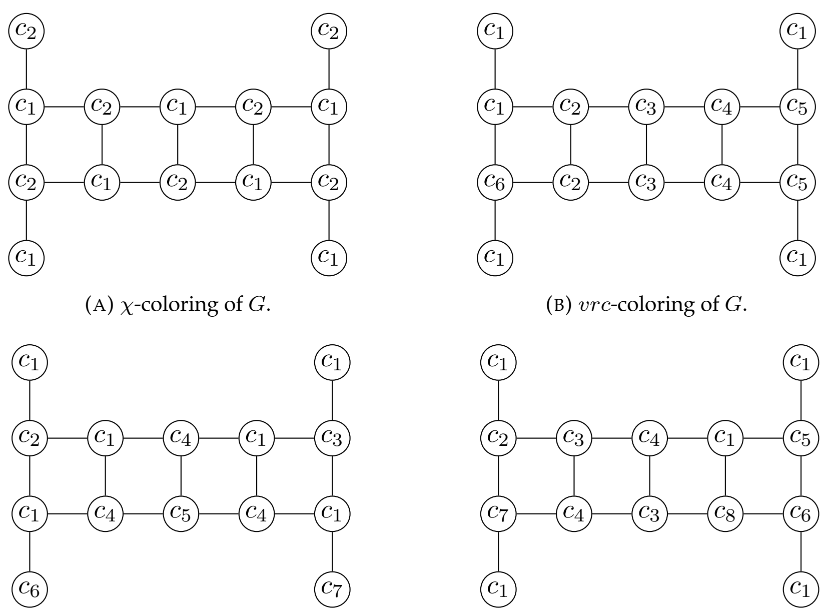
Domination in graphs can be seen as the process of selecting the graph entities; usually vertices, such that an entity of the graph is either selected or is related to the selected entities. In a graph G, if a vertex $v \in V(G)$ is adjacent to all vertices $u \in A$, for some $A \subseteq V(G)$ or $A = \{v\}$, we say that v *dominates* A and A is *dominated* by v. By convention, a vertex v always dominates itself (ref. [11]).

Graph coloring and domination in graphs are two well-known research areas in graph theory, owing to their applications. As the applications of these areas are similar in nature and coincide in many aspects, the notion of *dominator coloring* of graphs was introduced in [5], by blending the concepts of coloring and domination in graphs as a proper vertex coloring of G in which every vertex $v \in V(G)$ dominates at least one color class. The minimum number of colors used to obtain a dominator coloring of G is called the *dominator chromatic number* of G and it is denoted by $\chi_d(G)$.

Following this, several variants of dominator coloring of graphs have been defined and studied, based on different types of coloring and domination in graphs (ref. [3, 4, 10, 11]) and combining the concepts of vertex-rainbow coloring of graphs with the dominator coloring of graphs, the *rainbow dominator coloring* of a graph G has been introduced in [7], as follows.

Definition 1.1. [7] A *rainbow dominator coloring* of a graph G is a proper vertex coloring of G in which every vertex $v \in V(G)$ dominates at least one color class and every pair of its vertices are connected by a path whose internal vertices have distinct colors. The *rainbow dominator chromatic number* of G, denoted by $\chi_{rd}(G)$, is the minimum number of color classes in a rainbow dominator coloring of G.

An illustration of rainbow dominator coloring of a graph$G$ is given in Figure 1, where it can be observed that G has $\chi(G) = 2$, $rvc(G) = 6$, $\chi_d(G) = 7$, and $\chi_{rd}(G) = 8$.



(A) χ-coloring of G.

(B) vrc-coloring of G.

(C) χ_d-coloring of G.

(D) χ_{rd}-coloring of G.

FIGURE 1 An example of a graph G with $\chi(G) < rvc(G) < \chi_d(G) < \chi_{rd}(G)$.

On introducing the notion of rainbow dominator coloring of graphs in [7], the rainbow dominator coloring of certain graphs were discussed in [7,13] and [14], in which we observed that the values of the rainbow dominator chromatic numbers determined for some of these graphs are inaccurate, and not many general results based on the structure of graphs have been obtained, for the coloring. These research gaps in the literature motivate us to study the rainbow dominator coloring of graphs and hence in this article, we obtain certain properties of the the rainbow dominator coloring of graphs, and analyse the coloring pattern of the rainbow dominator coloring of graphs for certain standard graph classes, and their complements.

2. Rainbow Dominator Coloring of Graphs

By Definition 1.1, it can be seen that only connected graphs admit rainbow dominator coloring. Though, the vertex-rainbow coloring of graphs was introduced to model problems in a connected network, dominator coloring plays an important role in modelling problems that arise in disconnected networks (see [5]). Therefore, we modify the definition of the rainbow dominator coloring of graphs as follows, to ensure that disconnected graphs also admit rainbow dominator coloring.

Definition 2.2. A *rainbow dominator coloring* of a graph G is a proper vertex coloring of G in which every vertex $v \in V(G)$ dominates at least one color class and every pair of its vertices are connected by a path whose internal vertices have distinct colors, if such a path exists. The *rainbow dominator chromatic number* of G, denoted by $\chi_{rd}(G)$, is the minimum number of color classes in a rainbow dominator coloring of G.

Based on Definition 2.2, it is immediate that if G is a graph with r components $G_1, G_2, \ldots, G_r$, for any $r \geq 1$; that is, $G \cong G_1 \cup G_2 \cup \ldots \cup G_r$, then

(i) $\max\{diam(G_i) : 1 \leq i \leq r\} - 1 \leq \max\{vrc(G_i) : 1 \leq i \leq r\} \leq \chi_{rd}(G)$.

(ii) $\max\{\chi(G_i) : 1 \leq i \leq r\} \leq \max\{\chi_d(G_i) : 1 \leq i \leq r\} \leq \chi_{rd}(G)$.

(iii) $\max\{\chi(G_i) : 1 \leq i \leq r\} + (r-1) \leq \chi_{rd}(G) \leq \sum\limits_{i=1}^{r} \chi_{rd}(G_i)$.

As a geodesic between two vertices u, v of any graph G with diameter 2 or 3 has at most two internal vertices, which are colored with two different colors in any proper coloring of G, there exists a rainbow path between any two vertices of G, in any of its χ-coloring. However, as such a χ-coloring of G need not necessarily be its dominator coloring, we obtain the following results.

Proposition 2.1. *For any graph G with $diam(G) \leq 3$, $\chi_d(G) = \chi_{rd}(G)$.*

Corollary 2.1. *If G is a graph having a set S of r universal vertices, then $\chi(G) = \chi_d(G) = \chi_{rd}(G) = r + \chi(G[V(G) - S])$.*

The converse of Proposition 2.1 is not true, as the path P_6 has diameter 5, and $\chi_d(P_6) = \chi_{rd}(P_6) = 4$. The converse of Corollary 2.1 is also not true, owing to the fact for the join $G_1 + G_2$ of any two graphs G_1 and G_2, $\chi(G_1 + G_2) = \chi_d(G_1 + G_2) = \chi_{rd}(G_1 + G_2)$. This is because, as every vertex of G_1 (resp. G_2) is made adjacent to every vertex of G_2 (resp. G_1) in $G_1 + G_2$, every vertex of G_1 (resp. G_2) dominates at least $\chi(G_2)$ (resp. $\chi(G_1)$) color classes, in any χ-coloring of $G_1 + G_2$. Also, as $G_1 + G_2$ becomes a graph with diameter 2, irrespective of the values of $diam(G_1)$ and $diam(G_2)$, the following result is obtained, as a consequence of Proposition 2.1, and the fact that $\chi(G_1 + G_2) = \chi(G_1) + \chi(G_2)$.

Proposition 2.2. *For any two graphs G_1 and G_2, $\chi(G_1+G_2) = \chi_d(G_1+G_2) = \chi_{rd}(G_1+G_2) = \chi(G_1) + \chi(G_2)$.*

In any dominator coloring of a graph G with l pendant vertices, either the pendant vertices or the support vertices of G must be given a unique color. Also, in any rainbow coloring of G, all pendant vertices can be assigned the same color, owing to the fact that they are not internal vertices of any path in G. Hence, we have the following result.

Proposition 2.3. *For a graph G of order n with l leaves and s support vertices, $s+1 \leq \chi_{rd}(G) \leq n-l+1$.*

The bounds given in Proposition 2.3 are tight, as it can be observed that for a star $K_{1,s}$ of order $s+1$, for any $s \geq 1$, $\chi_{rd}(K_{1,s}) = 2$, and for a comb graph Cb_t of order $2t$, obtained by attaching a pendant vertex to each vertex of a path P_t has $diam(Cb_t) = t+1$, and $\chi_{rd}(Cb_t) = t$, for all $t \geq 2$.
As we know that $\chi(K_n) = \chi_{rd}(K_n) = n$, we characterise graphs for which a trivial coloring is its optimal rainbow dominator coloring.

Theorem 2.1. *A graph G of order n has $\chi_{rd}(G) = n$ if and only if $G \cong rK_1 \cup K_{n-r}$.*

Proof. If $G \cong rK_1 \cup K_{n-r}$, for some $r \geq 0$, it is clear that $\chi_{rd}(G) = n$, as each of the isolated vertices must be assigned a unique color for them to dominate their own color classes, and all vertices of K_{n-r} are assigned distinct colors in any of its proper coloring. Now, assume that $\chi_{rd}(G) = n$, for some graph G of order n such that $G \ncong rK_1 \cup K_{n-r}$.
Case 1 : If G is connected, and $\chi_{rd}(G) = n$, then $\chi_d(G) < n$, by our assumption. Hence, G must be a graph with diameter 4 or more, having a unique path between every two vertices, whose colors cannot be repeated; implying that G is a tree. However, by Proposition 2.3, if $\chi_{rd}(G) = n$, then G must have exactly one leaf; which is not possible, or $G \cong K_2$; yielding a contradiction. Hence, $G \cong K_n$, for some $n \geq 1$.
Case 2 : Let G be disconnected. If G contains s isolated vertices, and if $S \subseteq V(G)$ is the set of these s isolated vertices in G, then $\chi_{rd}(G') = n-s$, where $G' = G[V(G)-S]$. As the result follows from *Case 1* when G' is connected, G' must be a disconnected graph without isolated vertices. By *Case 1*, each component of G here can be a complete graph; that is, $G \cong K_{t_1} \cup K_{t_2} \cup \ldots \cup K_{t_k}$, for some integer $t_i > 1$; $1 \leq i \leq k$. However, in this case, $\min\{t_i : 1 \leq i \leq k\} - 1$ colors can be given to at least two vertices of G, in any of its optimal rainbow dominator coloring; yielding a contradiction. Hence the result. □

Following this, we discuss the rainbow dominator coloring of certain standard graph classes and determine their rainbow dominator chromatic numbers.

Proposition 2.4. *For $n \geq 5$, $\chi_{rd}(P_n) = n-2$.*

Proof. Let $c : V(P_n) \to \{c_1, c_2, \ldots, c_{n-2}\}$ be a vertex coloring such that

$$c(v_i) = \begin{cases} c_1, & i = 1, 3, n; \\ c_2, & i = 2; \\ c_{i-1}, & 4 \leq i \leq n-1. \end{cases}$$

The coloring c is a rainbow dominator coloring of P_n using $n-2$ colors, as every internal vertex of P_n has a distinct color, and as the vertices v_1, v_2, v_3 dominate the color class $\{v_2\}$, v_n dominates the color class $\{v_{n-1}\}$, and all the remaining vertices v_i; $4 \leq i \leq n-1$, dominate their own color class, in c. As the diameter of a path P_n is $n-1$, it follows that $\chi_{rd}(P_n) = n-2$. □

Based on rainbow dominator coloring of paths and complete graphs obtained above, the following results on the existence of graphs with a given difference between the rainbow dominator chromatic number, and its lower bounds such as the diameter of the graph, chromatic number and dominator chromatic number of the graph are determined.

Theorem 2.2. *For any integer $r \geq 0$, there exists a graph G such that*

(i) $\chi_{rd}(G) - \chi(G) = r$,

(ii) $\chi_{rd}(G) - \chi_d(G) = r$,

(iii) $\chi_{rd}(G) - diam(G) = r$.

Proof. It is immediate that there exists a graph G such that $\chi_{rd}(G) - \chi(G) = r$, Proposition 2.4, it can be seen that for $\chi_{rd}(P_{r+4}) - \chi(P_{r+4}) = r$, for all $r \geq 1$.

For a path P_n; $n \geq 8$, it has been proven that $\chi_d(P_n) = \lceil \frac{n}{3} \rceil + 2$ (see [5]). Therefore, the graph $P_{3(\lfloor \frac{r}{3} \rfloor + 2) - i}$, for $r \equiv i \pmod 2$, has $\chi_{rd}(P_{3(\lfloor \frac{r}{3} \rfloor + 2) - i}) - \chi_d(P_{3(\lfloor \frac{n}{r} \rfloor + 2) - i}) = r$, for all $r \geq 0$, thereby, yielding the required graph.

Construct a graph G_s with $V(G_s) = \{u_i : 1 \leq i \leq s\} \cup \{v_i : 1 \leq i \leq s\} \cup \{w_1, w_2, w_3, w_4\}$, and $E(G_s) = \{v_i v_j : 1 \leq i \neq j \leq s\} \cup \{u_i u_j : 1 \leq i \neq j \leq s\} \cup \{u_1 w_1, w_1 w_2, w_2 w_3, w_3 w_4, w_4 v_1\}$, for $s \geq 5$ (see Figure 2, for illustration). Consider a coloring $c : V(G_s) \to \{c_1, c_2, \ldots, c_{s+2}\}$ defined as follows. For a vertex $v \in V(G_s)$,

$$c(v) = \begin{cases} c_1, & v = u_1; \\ c_{s+1}, & v = v_1; \\ c_i, & v \in \{u_i, v_i : 2 \leq i \leq s\}; \\ c_{s+2}, & v = w_2; \\ c_{i+1}, & v = w_i, i = 1, 3, 4. \end{cases}$$

The coloring c is a dominator coloring of G_s using $s+2$ colors, as v_i; $2 \leq i \leq s$, and w_1 dominate the color class $\{v_1\}$, the vertices u_i; $2 \leq i \leq s$ and w_1 dominate the color class $\{u_1\}$, and the vertices w_2 and w_3, dominate the color class $\{w_2\}$. It is also a rainbow dominator coloring of G_s as any 2 non-adjacent vertices $u_i, v_j \in V(G_s)$, for $2 \leq i, j \leq s$ has a path $u_i - u_1 - w_1 - w_2 - w_3 - w_4 - v_1 - v_j$, which are all colored using six unique colors. Also, as $\chi(G_s) = s$, $\chi_{rd}(G_s) \geq s+1$, as there are two vertex disjoint complete graphs in G_s. If $\chi_{rd}(G_s) = s+1$, then we cannot obtain a dominator coloring of the P_4 induced by the vertices w_1, w_2, w_3, w_4 of G_s. Hence, $\chi_{rd}(G_s) = s+2$.

Here, $diam(G_s) = 7$, for any $s \geq 5$, as the longest path is $u_i - u_1 - w_1 - w_2 - w_3 - w_4 - v_1 - v_j$, for some $u_i, v_j \in V(G)$, where $2 \leq i, j \leq s$. Therefore, for any integer $r \geq 0$, we have $\chi_{rd}(G_{r+5}) - diam(G_{r+5}) = r$; completing the proof. □

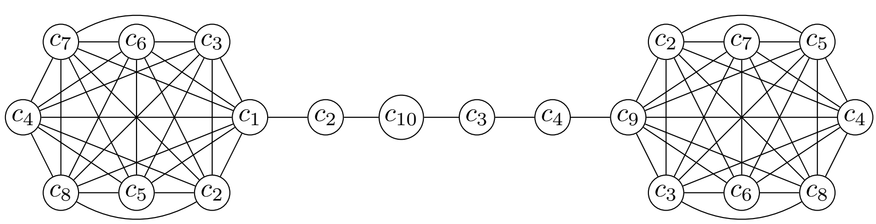


FIGURE 2 Graph G_8 given in Theorem 2.2.

Theorem 2.3. *For $n \geq 3$,* $\chi_{rd}(C_n) = \begin{cases} \chi(C_n), & n \leq 5; \\ \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{6} \rceil, & n \geq 6, \text{ and } n \equiv 1 \pmod 6; \\ \lceil \frac{n}{2} \rceil + \lceil \frac{n}{6} \rceil, & \text{otherwise.} \end{cases}$

Proof. As $diam(C_n) = \lfloor \frac{n}{2} \rfloor$, for a cycle $C_n := v_1 - v_2 - v_3 - \ldots - v_n - v_1$; $n \geq 5$, any $\lfloor \frac{n}{2} \rfloor - 1$ consecutive vertices of C_n, say $v_1, v_2, \ldots, v_{\lfloor \frac{n}{2} \rfloor - 1}$, must be colored using $\lfloor \frac{n}{2} \rfloor - 1$ distinct colors, in any vertex-rainbow coloring c' of C_n. If such a coloring has to be a dominator coloring of C_n, the colors assigned to the vertices v_i; $i \equiv 2 \pmod 3$, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, cannot be used to color the remaining $\lceil \frac{n}{2} \rceil + 1$ vertices $v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor+1}, \ldots, v_n$ of C_n.

However, as the subgraph of C_n induced by the vertices $v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor+1}, \ldots v_n$ is a path $P_{\lceil \frac{n}{2} \rceil+1}$, we need $\left\lceil \frac{\lceil \frac{n}{2} \rceil+1}{3} \right\rceil$ unique colors to obtain a dominator coloring of this subgraph. Hence, we require at least $\lfloor \frac{n}{2} \rfloor - 1 + \left\lceil \frac{\lceil \frac{n}{2} \rceil+1}{3} \right\rceil$ colors to obtain a rainbow dominator coloring of C_n.

Consider a coloring $c : V(C_n) \to \{c_1, c_2, \ldots\}$ such that for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $c(v_i) = c_i$,

$$c(v_{\lfloor \frac{n}{2} \rfloor+i}) = \begin{cases} c_i, & i \equiv 0, 1 \pmod 3; \\ c_{\lfloor \frac{n}{2} \rfloor + \lceil \frac{i}{3} \rceil}, & i \equiv 2 \pmod 3. \end{cases}$$

This coloring of C_n assigns a color to all n vertices, when n is even, and when $n \equiv 2 \pmod 6$, the vertex v_n does not dominate any color class in this coloring. Also, when n is odd, the vertex v_n is left uncolored here. Therefore, we re-define $c(v_n)$, when $n \equiv 2 \pmod 6$, and define $c(v_n)$, when n is odd, as follows.

$$c(v_n) = \begin{cases} c_{\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{6} \rceil}, & \text{when } n \equiv 1, 3 \pmod 6; \\ c_{\lceil \frac{n}{2} \rceil + \lceil \frac{n}{6} \rceil}, & \text{when } n \equiv 2, 5 \pmod 6). \end{cases}$$

The above mentioned coloring is a dominator coloring of C_n as the vertices v_{i-1}, v_i, and v_{i+1} dominate the color class $\{v_i\}$, for all $i \equiv 2 \pmod 3$, and $1 \leq i \leq n-1$. The vertex v_n dominates the color class v_{n-1}, when $n \equiv 0 \pmod 6$, and it dominates its own color class, in all the other cases.

As any consecutive $\lfloor \frac{n}{2} \rfloor$ vertices of C_n are colored using $\lfloor \frac{n}{2} \rfloor$ distinct colors in c, there exists a rainbow path of length $\lfloor \frac{n}{2} \rfloor$ from every v_i to $v_{i+\lfloor \frac{n}{2} \rfloor}$. Hence, c is a rainbow dominator coloring of C_n, yielding $\chi_{rd}(C_n) \leq \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{6} \rceil$, when $n \equiv 0, 2, 4, 5 \pmod 6$, and $\chi_{rd}(C_n) \leq \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{6} \rceil$, when $n \equiv 1, 3 \pmod 6$.

When $n \equiv 0 \pmod 6$, as $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{6} \rceil = \lfloor \frac{n}{2} \rfloor - 1 + \left\lceil \frac{\lceil \frac{n}{2} \rceil+1}{3} \right\rceil$, c is an optimal dominator coloring of C_n. In all the other cases, as $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{6} \rceil = \lfloor \frac{n}{2} \rfloor + \left\lceil \frac{\lceil \frac{n}{2} \rceil+1}{3} \right\rceil$.

If $\chi_{rd}(C_n) = \lfloor \frac{n}{2} \rfloor - 1 + \left\lceil \frac{\lceil \frac{n}{2} \rceil+1}{3} \right\rceil$, when $n \not\equiv 0 \pmod 6$, a color c_i, for some $i \equiv 0, 1 \pmod 3$, will be repeated to two vertices $v_j, v_{j'}$ such that $d(v_j, v_{j'}) \leq \lfloor \frac{n}{2} \rfloor - 1$, prohibiting a rainbow path between a vertex v_{j^*} and some v_{i^*}, where $1 \leq j < j^* < j' \leq n$, and $1 \leq i^* \leq n$; thereby proving the result. □

A *crown graph*, denoted by Cr_t, is a graph obtained by removing the edges v_iu_i, for all $1 \leq i \leq t$, from the complete bipartite graph $K_{t,t}$, where $V(K_{t,t}) = \{v_i : 1 \leq i \leq t\} \cup \{u_i : 1 \leq i \leq t\}$.

Proposition 2.5. *For $t \geq 2$, $\chi_{rd}(Cr_t) = 4$.*

Proof. For a crown graph Cr_t with $V(Cr_t) = \{v_i : 1 \leq i \leq t\} \cup \{u_i : 1 \leq i \leq t\}$, and $E(Cr_t) = \{v_iu_j : 1 \leq i \neq j \leq t\}$, consider a coloring $c : V(C_t) \to \{c_1, c_2, c_3, c_4\}$ such that $c(v_1) = c_1$, $c(u_1) = c_2$, $c(v_i) = c_3$ and $c(u_i) = c_4$, for $2 \leq i \leq t$. The coloring c is a dominator coloring of Cr_t, as the vertices u_1 and v_i; $2 \leq i \leq t$, dominate the color class $\{u_1\}$, and the vertices v_1 and u_i; $2 \leq i \leq t$, dominate the color class $\{v_1\}$. Also, as there exists a path of length 2 between any two u_i's and v_i's, it can be seen that c is a dominator coloring of Cr_t, yielding $\chi_{rd}(Cr_t) \leq 4$.

As any u_i (resp. v_i), is not adjacent to the corresponding v_i (resp. u_i), all u_i's (resp. v_i's) cannot be assigned the same color in any dominator coloring of Cr_t. Hence, as at least four colors are required to obtain a dominator, as well as a rainbow dominator coloring of Cr_t, yielding $\chi_{rd}(Cr_t) = 4$, for any $t \geq 2$. □

For $t \geq 3$, a *wheel graph*, denoted by $W_{1,t}$, is a graph obtained by making a vertex, say v, adjacent to all the vertices of C_t. For $a_i \geq 1$; $1 \leq i \leq r$, a *multi-star* $S_{a_1,a_2,\ldots,a_r}$ is a graph obtained by making the universal vertices of the stars $K_{1,a_1}, K_{1,a_2}, \ldots, K_{1,a_r}$ mutually adjacent.

By the above mentioned definition, it can be observed that a wheel graph $W_{1,t}$ has a universal vertex, a multi-star $S_{a_1,a_2,\ldots,a_r}$, for $r \geq 1$, and $a_i \geq 1$; $1 \leq i \leq r$, contains a K_r. Also, as every vertex in a complete multi-partite graph $K_{a_1,a_2,\ldots,a_r}$ dominates $r-1$ among the r color classes, in any of its χ-coloring, we have the following result.

Proposition 2.6. *For $a_i \geq 1$; $1 \leq i \leq r$, any χ-coloring of the graphs $K_{a_1,a_2,\ldots,a_r}$, $S_{a_1,a_2,\ldots,a_r}$ and $W_{1,t}$; $t \geq 3$, is their rainbow dominator coloring.*

Corollary 2.2. *For $r \geq 1$, $a_i \geq 1$; $1 \leq i \leq r$, and $t \geq 3$,*

(i) $\chi(K_{a_1,a_2,\ldots,a_r}) = \chi_d(K_{a_1,a_2,\ldots,a_r}) = \chi_{rd}(K_{a_1,a_2,\ldots,a_r}) = r.$

(ii) $\chi(S_{a_1,a_2,\ldots,a_r}) = \chi_d(S_{a_1,a_2,\ldots,a_r}) = \chi_{rd}(S_{a_1,a_2,\ldots,a_r}) = r+1.$

(iii) $\chi(W_{1,t}) = \chi_d(W_{1,t}) = \chi_{rd}(W_{1,t}) = \begin{cases} 3; & \textit{when } n \textit{ is even}; \\ 4; & \textit{when } n \textit{ is odd}. \end{cases}$

A *helm graph* of order $n = 2t+1$, denoted by $H_{1,t,t}$, is obtained by attaching a leaf to each vertex of degree 3 in a wheel graph $W_{1,t}$ and a *closed helm graph* $CH_{1,t,t}$ is obtained by making the each leaf of the helm $H_{1,t,t}$ adjacent to the preceding and succeeding pendant vertices in it.

Proposition 2.7. *For $t \geq 3$, $\chi_{rd}(H_{1,t,t}) = t+1$.*

Proof. Let v_i; $1 \leq i \leq t$, be the vertices of degree 4 in the helm $H_{1,t,t}$; $t \geq 3$, u_i; $1 \leq i \leq t$, be the pendant vertices of $H_{1,t,t}$ which are adjacent to the corresponding v_i's, and v be its central vertex of degree $t+1$. By the definition of a helm $H_{1,t,t}$; $t \geq 3$, there are t support vertices and t leaves, and hence by Theorem 2.3, $\chi_{rd}(H_{1,t,t}) \geq t+1$.

A coloring $c : V(H_{1,t,t}) \to \{c_1, c_2, \ldots, c_{t+1}\}$ such that $c(v_i) = c_i$, $c(v) = c(u_i) = c_{t+1}$ is a dominator coloring of $H_{1,t,t}$, as every u_i and v_i dominate the color class $\{v_i\}$, the vertex v dominates the color classes $\{v_i\}$, for all $1 \leq i \leq t$. It is also a rainbow dominator coloring of $H_{1,t,t}$, as there exists a path of length 2, between any two non-adjacent v_i's through v and a path of length at most 3, between any two non-adjacent u_i's though the corresponding v_i's and v, which are all colored with distinct colors in c. Hence, $\chi_{rd}(H_{1,t,t}) = t+1$, for any $t \geq 3$. □

Theorem 2.4. *For $t \geq 5$, $\chi_{rd}(CH_{1,t,t}) = \lceil \frac{t}{3} \rceil + 4$.*

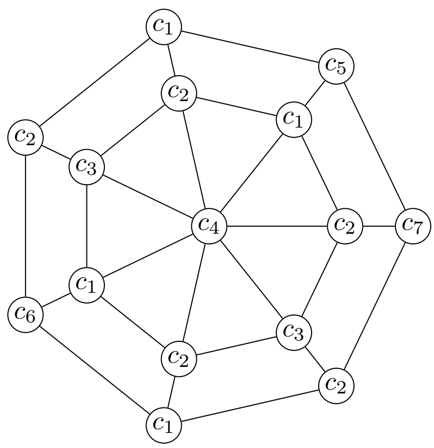
Proof. Let $CH_{1,t,t}$; $t \geq 5$, be a closed helm graph with $V(CH_{1,t,t}) = \{v\} \cup \{v_i : 1 \leq i \leq t\} \cup \{u_i : 1 \leq i \leq t\}$ and $E(CH_{1,t,t}) = \{vv_i : 1 \leq i \leq t\} \cup \{v_iv_{i+1} : 1 \leq i \leq t\} \cup \{u_iu_{i+1} : 1 \leq i \leq t\} \cup \{v_iu_i : 1 \leq i \leq t\}$, where the suffixes are taken modulo t. Consider a coloring $c : V(CH_{1,t,t}) \to \{c_1, c_2, \ldots, c_{4+\lceil \frac{t}{3} \rceil}\}$ such that $c(v_i) = c_j$; $i \equiv j \pmod 3$, for $1 \leq i \leq t-1$, and $j = 1,2,3$, $c(v_t) = c_2$, when $t \equiv 1,2 \pmod 3$, and $c(v_t) = c_3$, when $t \equiv 0 \pmod 3$, $c(v) = c_4$, and for $1 \leq i \leq t$,

$$c(u_i) = \begin{cases} c_{4+\lceil \frac{i}{3} \rceil}, & i \equiv 1 \pmod 3; \\ c_1, & i \equiv 2 \pmod 3; \\ c_2, & i \equiv 0 \pmod 3. \end{cases}$$

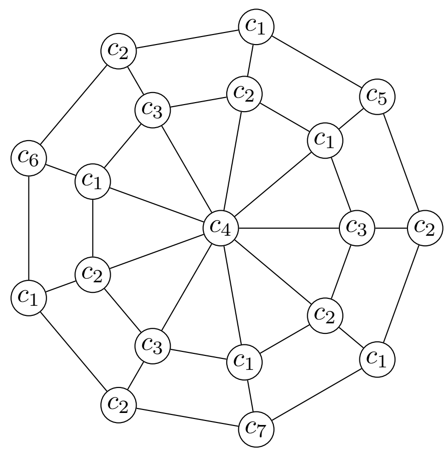
As every v_i; $1 \leq i \leq t$, and v dominate the color class $\{v\}$, and the vertices u_{i-1}, u_i, u_{i+1} dominate the color class $\{u_i\}$, for $i \equiv 1 \pmod 3$, c is a dominator coloring of $CH_{1,t,t}$. In $CH_{1,t,t}$, any two non-adjacent v_i's are at a distance 2, and $d(u_i, v) = 2$, for any $1 \leq i \leq t$, and there exists a path of length 2,3,4, between two non-adjacent u_i's. Therefore, to prove that the above mentioned dominator coloring c of $CH_{1,t,t}$ is its rainbow dominator coloring, we obtain a rainbow path between the non-adjacent vertices u_i, u_j such that $d(u_i, u_j) = 4$, with respect to c.

The path $u_i - v_i - v - v_j - u_j$ is a $u_i - u_j$ rainbow path if $i \pmod 3 \neq j \pmod 3$, as $c(v_i) = c(v_j)$ if and only if $i \pmod 3 = j \pmod 3$, in c. Hence, when $i \pmod 3 = j \pmod 3$, the path $u_i - v_i - v - v_{j-1} - u_{j-1} - u_j$ is a rainbow $u_i - u_j$ rainbow path, because here $c(v_i) \neq c(v_{j-1}) \neq c(u_{j-1})$, for any $1 \leq i \neq j \leq t$. Hence, $\chi_{rd}(CH_{1,t,t}) \leq \lceil \frac{t}{3} \rceil + 4$.

In $CH_{1,t,t}$, the color assigned to v can be assigned only to some of the u_i's. However, if this happens, we must obtain a coloring in which some color class contains only the v_i's for v to dominate that color class. Apart from this, for the v_i's and u_i's to dominate a color class, we must obtain a coloring of $CH_{1,t,t}$ in which the color classes consists the v_j's and $u_{j'}$'s. In such a rainbow dominator coloring of $CH_{1,t,t}$, we need at least $\lceil \frac{t}{3} \rceil + 5$ colors, as $\chi_d(C_t) = \lceil \frac{t}{3} \rceil + 2$. Hence, it follows that $\chi_{rd}(CH_{1,t,t}) = \lceil \frac{t}{3} \rceil + 4$. □



(A) χ_{rd}-coloring of $CH_{1,7,7}$.



(B) χ_{rd}-coloring of $CH_{1,9,9}$.

FIGURE 3 χ_{rd}-colorings of the graph $CH_{1,t,t}$.

3. Rainbow Dominator Coloring of Graph Complements

In this section, the rainbow dominator coloring of the complements of the graphs, for which the rainbow dominator coloring was discussed in Section 2, is investigated. As it is immediate that $\chi_{rd}(nK_1) = n$, we examine the rainbow dominator coloring of the complements of paths and cycles, as follows. Note that as $P_3 \cong K_{1,2}$, $C_3 \cong K_3$, $\overline{P}_4 \cong P_4$, and $\overline{C}_4 \cong 2K_2$, we consider the graphs $\overline{P}_n$, and $\overline{C}_n$, for $n \geq 5$, in the following result.

Theorem 3.5. *For $n \geq 5$, $\chi_{rd}(\overline{P}_n) = \chi_{rd}(\overline{C}_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Let c be a proper vertex coloring of $\overline{P}_n$; $n \geq 5$, where a path $P_n := v_1 - v_2 - \ldots - v_n$, such that $c(v_i) = c(v_{i+1}) = c_i$, for all $1 \leq i \leq n$, and $i \equiv 1 \pmod 2$. As any v_i; $2 \leq i \leq n-1$, is adjacent to all the vertices of $\overline{P}_n$, except v_{i-1}, v_i, v_{i+1}, and v_1 (resp. v_n) is adjacent to all the vertices of $\overline{P}_n$, except v_2 (resp. v_{n-1}), any color can be assigned to at most two vertices of $\overline{P}_n$, in any of its proper coloring. Owing to this, c is a dominator coloring of $\overline{P}_n$. Also, as v_1 or v_n is a common neighbour of any two non-adjacent vertices of $\overline{P}_n$, there exists a rainbow path between them in c, yielding $\chi_{rd}(\overline{P}_n) = \lceil \frac{n}{2} \rceil$, for all $n \geq 5$.

As $\overline{C}_n \cong \overline{P}_n - v_1v_n$, for a cycle $C_n := v_1 - v_2 - \ldots - v_n - v_1$, it can be observed that the above defined rainbow dominator coloring c of $\overline{P}_n$, is also a rainbow dominator coloring of $\overline{C}_n$; yielding the result. □

As a consequence of the fact that $\overline{W}_{1,t} \cong K_1 \cup \overline{C}_t$; $t \geq 3$, and $\overline{W}_{1,3} \cong 4K_1$, the following corollary is immediate.

Corollary 3.3. *For $t \geq 4$, $\chi_{rd}(\overline{W}_{1,t}) = \lceil \frac{t}{2} \rceil + 1$.*

Following this, the rainbow dominator chromatic number of the complement of helm and closed helm are determined in the following results.

Theorem 3.6. *For $t \geq 4$, $\chi_{rd}(\overline{H}_{1,t,t}) = t + 1$.*

Proof. All the t pendant vertices and the vertex of degree t in the helm graph $H_{1,t,t}$ forms a clique of order $t + 1$. Hence, $\chi_{rd}(\overline{H}_{1,t,t}) \geq t + 1$.

Define a coloring $c : V(\overline{H}_{1,t,t}) \to \{c_1, c_2, \ldots, c_{t+1}\}$ of the helm graph $H_{1,t,t}$ with $V(\overline{H}_{1,t,t}) = \{u_i : 1 \leq i \leq t\} \cup \{v_i : 1 \leq i \leq t\} \cup \{v\}$, as described in Proposition 2.7, as $c(u_i) = c(v_i) = c_i$, $c(v) = c_{t+1}$.

The coloring c of $\overline{H}_{1,t,t}$ is its dominator coloring as the vertices u_i; $1 \leq i \leq t$, and v dominate the color class $\{v\}$, and for each $1 \leq i \leq t$, the vertex v_i dominates the color class $\{v_{i+3}, u_{i+3}\}$, owing to the fact that every $v_i \in V(\overline{H}_{1,t,t})$ is adjacent to all the vertices in $V(\overline{H}_{1,t,t}) - \{v, u_i, v_{i-1}, v_{i+1}, v_i\}$, where the suffixes are taken modulo t, and every $u_i \in V(\overline{H}_{1,t,t})$ is adjacent to v. It is also a rainbow dominator coloring of $\overline{H}_{1,t,t}$, as there exists a path of length 2 between any two non-adjacent vertices in the graph. Therefore, $\chi_{rd}(\overline{H}_{1,t,t}) \geq t + 1$, for all $n \geq 4$. □

Theorem 3.7. *For $t \geq 4$, $\chi_{rd}(\overline{CH}_{1,t,t}) = t$.*

Proof. Let $\overline{CH}_{1,t,t}$ be the complement pf the helm graph $H_{1,t,t}$ with $V(\overline{CH}_{1,t,t}) = \{u_i : 1 \leq i \leq t\} \cup \{v_i : 1 \leq i \leq t\} \cup \{v\}$, and $E(\overline{CH}_{1,t,t})$, as described in Theorem 2.4. As the graph $\overline{CH}_{1,t,t}$ has a clique of order t, induced by the vertices v_i, u_{i+1}, for $i \equiv 1 \pmod 2$, $1 \leq i \leq t$, $\chi_{rd}(\overline{CH}_{1,t,t}) = t$.

Define a coloring $c : V(\overline{CH}_{1,t,t}) \to \{c_1, c_2, \ldots, c_t\}$ such that $c(v) = c(v_1) = c(v_2) = c_1$, $c(u_1) = c(u_2) = c_2$, and $c(u_i) = c(v_i) = c_i$, for all $3 \leq i \leq t$. This is a dominator coloring of $\overline{CH}_{1,t,t}$, as each u_i and v_i dominate the color classes $\{u_{i+2}, v_{i+2}\}$ and $\{u_{i-2}, v_{i-2}\}$, for all $1 \leq i \leq t$, and v dominates the color class $\{u_1, u_2\}$. This is also a rainbow dominator coloring of $\overline{CH}_{1,t,t}$, as any two non-adjacent u_i's are adjacent to v, and any two non-adjacent v_i's, say v_{i_1} and v_{i_2} are adjacent to u_{i_3}, $1 \leq i_1 \neq i_2 \neq i_3 \leq t$. Also, there exists a path from v to any v_i through u_j, for some $1 \leq i \neq j \leq t$. Hence, $\chi_{rd}(\overline{CH}_{1,t,t}) = t$, for any $t \geq 4$. □

Proposition 3.8. *For $t \geq 2$, $\chi_{rd}(\overline{Cr}_t) = t$.*

Proof. Let $V(\overline{Cr}_t) = \{u_i : 1 \leq i \leq t\} \cup \{v_i : 1 \leq i \leq t\}$ and $E(\overline{Cr}_t) = \{u_iu_j : 1 \leq i \neq j \leq t\} \cup \{v_iv_j : 1 \leq i \leq j \leq t\} \cup \{u_iv_i : 1 \leq i \leq t\}$. The coloring $c(v_i) = c(u_{i+1}) = c_i$, for $1 \leq i \leq t$, is a dominator coloring of $\overline{Cr}_t$, as every v_i and u_i dominates the color class $\{v_{i-1}, u_{i-1}\}$, with suffixes taken modulo n.

As the graph $\overline{Cr}_t$ contains two vertex disjoint complete graphs K_t induced by the vertices $v_1, v_2, \ldots, v_t$ and $u_1, u_2, \ldots, u_t$, connected by the edges v_iu_i, for $1 \leq i \leq t$, the result follows. □

Theorem 3.8. *For $r \geq 1$, and $a_i \geq 1$; $1 \leq i \leq r$,*

(i) $\chi(\overline{K}_{a_1,a_2,\ldots,a_r}) = \chi_d(\overline{K}_{a_1,a_2,\ldots,a_r}) = \max\{a_i : 1 \leq i \leq r\} + (r-1)$.

(ii) $\chi(\overline{S}_{a_1,a_2,\ldots,a_r}) = \chi_d(\overline{S}_{a_1,a_2,\ldots,a_r}) = \chi_{rd}(\overline{S}_{a_1,a_2,\ldots,a_r}) = \sum\limits_{i=1}^{r} a_i$.

Proof. For any $r \geq 1$, let $1 \leq a_i \leq a_j$, for $1 \leq i < j \leq r$. As $\overline{K}_{a_1,a_2,\ldots,a_r} \cong K_{a_1} \cup K_{a_2} \cup \ldots \cup K_{a_r}$, each K_{a_i} is colored with a_i colors $c_{a_1}, c_{a_2}, \ldots c_{a_i}$, in the χ-coloring of $\overline{K}_{a_1,a_2,\ldots,a_r}$. However, in this coloring no color is exclusive to a K_{a_i}; $1 \leq i \leq r$, for the vertices of each K_{a_i} to dominate. Hence, the color c_{r+i} is assigned to a vertex of K_{a_i}, for each $1 \leq i \leq r-1$; thereby yielding the required rainbow dominator coloring of $\overline{K}_{a_1,a_2,\ldots,a_r}$.

Let $V(\overline{S}_{a_1,a_2,\ldots,a_r}) = \{v_i : 1 \leq i \leq r\} \cup \{u_j^{(i)} : a_1 \leq j \leq a_r; 1 \leq i \leq r\}$, where the v_i's are the universal vertices of the stars K_{1,a_i}, and $u_j^{(i)}$'s are the pendant vertices of K_{1,a_i}, for each $1 \leq i \leq r$. The graph $\overline{S}_{a_1,a_2,\ldots,a_r}$ contains a clique of order $\sum\limits_{i=1}^{r} a_i$ induced by the $u_j^{(i)}$'s pendant vertices of $S_{a_1,a_2,\ldots,a_r}$, and hence, $\chi_{rd}(\overline{S}_{a_1,a_2,\ldots,a_r}) \geq \sum\limits_{i=1}^{r} a_i$. As every v_i; $1 \leq i \leq r$, is adjacent to all the vertices in $V(\overline{S}_{a_1,a_2,\ldots,a_r})$, except itself and $u_j^{(i)}$, for the corresponding i values, there exists a path of length 2 between any two non-adjacent vertices of $\overline{S}_{a_1,a_2,\ldots,a_r}$. Hence, the coloring c of $\overline{S}_{a_1,a_2,\ldots,a_r}$ such that $c(u_j^{(i)}) = c_{j+\sum\limits_{t=1}^{i-1} a_i}$ and $c(v_i) = c(u_1^{(i)})$, for $1 \leq t \leq a_r$, is the required rainbow dominator coloring of $\overline{S}_{a_1,a_2,\ldots,a_r}$, as every vertex v_i and $u_j^{(i)}$ dominates the color class $\{v_{i'}, u_1^{(i')}\}$, for some $1 \leq i \neq i' \leq r$, in c. ☐

4. Conclusion

In this article, we initiated an investigation on the rainbow dominator coloring of graphs, specifically focusing on obtaining the rainbow dominator coloring of certain standard classes of graphs and their complements. As this is just a beginning of the study on this topic, it offers wide avenues for future explorations that includes obtaining tighter bounds for the rainbow dominator chromatic number of the graphs, and determining the rainbow dominator coloring of several classes of graphs and its derived graphs, and addressing several realisation problems.

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