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On the Strong Gelfand pairs of hypergroups and admissible vectors related to representations

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ABSTRACT. In this paper, we study the strong Gelfand pairs of hypergroups and we give a characterization of admissible vectors related to the left regular representation of these Gelfand pairs.

1. INTRODUCTION

The shift invaraiant subspace of $L_2(G)$ where G is a locally compact abelian group has been investigated by many authors, specifically by Kamyabi and Toubi in [9]. They have shown that evey shift invariant space can be decomposed as an orthogonal sum of spaces each of which is generated by a single function whose shifts form a Parseval frame. In the case of hypergroups, Tabatabaie and Jokar in [13] have extended the result by giving a characterization of admissible vectors related to the left regular representation of a commutative hypergroup.

Hypergroups generalize locally compact groups where the convolution of two Dirac measures is a Dirac measure. A hypergroup is a locally compact Hausdor space equipped with a convolution product which maps two Dirac measures to a probability measure with compact support.

The goal of this paper is to give a characterization of admissible vectors related to the left regular representation when the hypergroup G has a compact subhypergroup K such that (G, K) is a Gelfand pair. The notion of Gelfand pairs for hypergroups is well-known (see [6, 12, 15]). In the next section, we give background for hypergroups. In section 3, after studying trong Gelfand pairs, we establish a necessary and suffucient condition to a vector in $L_2(G)$ to be Parseval- admissible related to the left regular representation.

2. NOTATIONS AND PRELIMINARIES

Let G be a locally compact Hausdorff space. We keep most of the notations which we used in our previous works :
- $C(G)$ (resp. $M(G)$) the space of continuous complex valued functions (resp. the space of Radon measures) on G,
- $C_b(G)$ (resp. $M_b(G)$) the space of bounded continuous functions (resp. the space of bounded Radon measures) on G,
- $\mathcal{K}(G)$ (resp. $M_c(G)$) the space of continuous functions (resp. the space of Radon measures) with compact support on G,
- $C_0(G)$ the space of elements in $C(G)$ which are zero at infinity,
- δ_x the point measure at $x \in G$,
- $\mathrm{supp}(f)$ the support of the function f,
- $\mathrm{supp}(\mu)$ the support of the measure μ.

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Let us recall the definition of a hypergroup. It is that has been given by Jewet in [8].

Definition 2.1. [6] *Let G be a locally compact Hausdorff space. G is said to be a hypergroup if the following assumptions are satisfied.*

(H1) There is a binary operator $*$ named convolution on $M_b(G)$ under which $M_b(G)$ is an associative algebra such that:
i) the mapping $(\mu, \nu) \longmapsto \mu * \nu$ is continuous from $M_b(G) \times M_b(G)$ in $M_b(G)$.
ii) $\forall x, y \in G$, $\delta_x * \delta_y$ is a measure of probability with compact support.
iii) the mapping: $(x, y) \longmapsto supp(\delta_x * \delta_y)$ is continuous from $G \times G$ in $\mathfrak{C}(G)$.

(H2) There is a unique element e (called neutral element) in G such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x, \forall x \in G$.

(H3) There is an involutive homeomorphism: $x \longmapsto \overline{x}$ from G in G, named involution, such that:
i) $(\delta_x * \delta_y)^- = \delta_{\overline{y}} * \delta_{\overline{x}}, \forall x, y \in G$ with $\mu^-(f) = \mu(f^-)$, where $f^-(x) = f(\overline{x}), \forall f \in C(G)$ and $\mu \in M(G)$.
ii) $\forall x, y, z \in G$, $z \in supp(\delta_x * \delta_y)$ if and only if $x \in supp(\delta_z * \delta_{\overline{y}})$.

The hypergroup G is commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ fot all $x, y \in G$.

Let us provide background and references for hypergroups and harmonic analysis on hypergroups as they relate to our study. This will be useful for those who are just getting into hypergroups.

2.1. Convolution in $M_b(G)$.

Let $x, y \in G$ and for $f \in C(G)$, $(\delta_x * \delta_y)(f)$ will be denoted by $f(x * y)$. Thus,

$$f(x * y) = (\delta_x * \delta_y)(f) = \int_G f(z) d(\delta_x * \delta_y)(z).$$

The convolution of two measures μ, ν in $M_b(G)$ is defined by:

$$(\mu * \nu)(f) = \int_G \int_G (\delta_x * \delta_y)(f) d\mu(x) d\nu(y) = \int_G \int_G f(x * y) d\mu(x) d\nu(y), f \in C(G).$$

For μ in $M_b(G)$, $\mu^* = (\overline{\mu})^-$. So $M_b(G)$ is a Banach *-algebra.
For f in $C(G)$, $f^* = \overline{f^-}$.

2.2. Double coset hypergroup and Gelfand pair.

Let us now consider a hypergroup G provided with a left Haar measure μ_G and K a compact subhypergroup of G with a normalized Haar measure ω_K.

A function $f \in C(G)$ is said to be K-biinvariant if $(\delta_{k_1} * \delta_x * \delta_{k_2})(f) = f(k_1 * x * k_2) = f(x) = \delta_x(f)$ for all $x \in G$ and all $k_1, k_2 \in K$. We denote by $C^K(G)$, (resp. $\mathcal{K}^K(G)$) the space of continuous functions (resp. continuous functions with compact support) which are K-biinvariant. For $f \in C(G)$, one defines the function f^K by

$$f^K(x) = \int_K \int_K f(k_1 * x * k_2) d\omega_K(k_1) d\omega_K(k_2), \forall x \in G.$$

$f^K \in C^K(G)$ and if $f \in \mathcal{K}(G)$, then $f^K \in \mathcal{K}^K(G)$.

For a measure $\mu \in M(G)$, one defines μ^K by

$$\mu^K(f) = \mu(f^K) \text{ for } f \in \mathcal{K}(G).$$

μ is said to be K-biinvariant if $\mu^{K} = \mu$ and we denote by $M^K(G)$ the set of all those measures.

The double coset of x ($\in G$) with respect to K is $K*\{x\}*K = \{k_1 * x * k_2; k_1, k_2 \in K\} = \bigcup_{k_1,k_2\in K} \text{supp}(\delta_{k_1} * \delta_x * \delta_{k_2})$. We write simply KxK for a double coset.
The double cosets space $G//K$ is a locally compact topologycal space ([1], page 53). The mapping $p_K : G \longrightarrow G//K$ defined by:

$$p_K(x) = KxK, x \in G$$

is an open surjective continuous mapping. The following operation

$$\delta_{KxK} * \delta_{KyK} = \int_K \delta_x * \delta_k * \delta_y d\omega_K(k) \text{ (see [12] and [1])}$$

defines a hypergoup structure on $G//K$ where the involution is defined by: $\overline{KxK} = K\overline{x}K$, the neutral element is K and $m = \int_G \delta_{KxK} d\mu_G(x)$ is a left Haar measure on $G//K$. (G, K) is a Gelfand pair if the hypergroup $G//K$ is commutative, that is $M_c^K(G)$ is a commutative subalgebra of $M_c(G)$. Thus, (G, K) is a Gelfand pair if and only if $\mathcal{K}^K(G)$ provided with the convolution is a commutative algebra ([6], theorem 3.2.2). For details on the notion of Gelfand pairs for hypergroups see [6, 12, 15].

In the sequel, the pair (G, K) is assumed to be a Gelfand pair.

2.3. **Fourier and inverse Fourier transforms.**
Let $\widehat{G^K}$ be the set of continuous, bounded and K- biinvariant function ϕ on G such that:
(i) ϕ is K- multiplicative (i. e. $\int_K \phi(x * k * y)_y d\omega_K(k) = \phi(x)\phi(y); \forall x, y \in G$).
(ii) $\phi(e) = 1$,
(iii) $\phi(\overline{x}) = \overline{\phi(x)}$ $\forall x \in G$.
$\widehat{G^K}$ is called the dual space of G with repect to K. $\widehat{G^K}$ is a locally compact Hausdorff space when equipped with the topology of uniform convergence on compact spaces. Note that the function $\mathbf{1} : x \longmapsto 1$ belongs to $\widehat{G^K}$.
When G is commutative, by taking $K = \{e\}$, $\widehat{G} = \widehat{G^{\{e\}}}$ is the dual space of G.
For μ belongs to $M_b(G)$, the Fourier transform of μ, is the mapping

$$\widehat{\mu} : \widehat{G^K} \longrightarrow \mathbb{C} \text{ defined by} : \widehat{\mu}(\phi) = \int_G \phi(\overline{x}) d\mu(x).$$

$\widehat{\mu} \in C_b(\widehat{G^K})$.
The Fourier transform of $f \in \mathcal{K}(G)$ is defined by

$$\widehat{f}(\phi) = \widehat{f\mu_G}(\phi) = \int_G \phi(\overline{x}) f(x) d\mu_G(x).$$

For any f belongs to $\mathcal{K}(G)$, $\widehat{f} \in C_0(\widehat{G^K})$ and $\widehat{f} = \widehat{f^K}$. If $f \in \mathcal{K}(G)$ and $g \in \mathcal{K}^K(G)$ then $\widehat{f * g} = \widehat{f}\widehat{g}$.
The Fourier transform is extended to $L_2(G, \mu_G)$ and $L_1(G, \mu_G)$. There exsits a unique non-negative measure (the Plancherel measure, see [7]) π on $\widehat{G^K}$ such that

$$\int_G |f(x)|^2 d\mu_G(x) = \int_{\widehat{G^K}} \left|\widehat{f}(\phi)\right|^2 d\pi(\phi), \text{ for all } f \in L_2^K(G, \mu_G) \cap L_1(G).$$

Let $\sigma \in M_b(\widehat{G^K})$, the inverse Fourier transform of σ is the mapping

$$\overset{\vee}{\sigma} : G \longrightarrow \mathbb{C} \text{ defined by} : \overset{\vee}{\sigma}(x) = \int_{\widehat{G^K}} \phi(x) d\sigma(\phi).$$

The inverse Fourier transform of $\varphi \in L^1(\widehat{G^K}, \pi)$ is defined by

$$\overset{\vee}{\varphi}(x) = (\varphi\pi)^{\vee}(x) = \int_{\widehat{G^K}} \phi(x)\varphi(\phi) d\pi(\phi),$$

For $\sigma \in M_b(\widehat{G^K})$, $\overset{\vee}{\sigma}$ is K-biinvariant and belongs to $C_b(G)$. It is known that (see [2]):

a) $\left\{\widehat{f} : f \in \mathcal{K}(G)\right\}$ is a sup-norm dense space of $C_0(\widehat{G^K})$.

b) $(\mathcal{K}(\widehat{G^K}))^{\vee}$ is a sup-norm dense subspace of $C_0(G)$.

c) If $f \in L_1^K(G, \mu_G)$ with $\widehat{f} \in L_1(\widehat{G^K}, \pi)$, then $\widehat{f}^{\vee} = f$ and reciprocally if $\varphi \in L_1(\widehat{G}, \pi)$ with $\overset{\vee}{\varphi} \in L_1(G, \mu_G)$, then $\widehat{(\overset{\vee}{\varphi})} = \varphi$.

Thanks to the Plancherel theorem and knowing that $L_1(G, \mu_G) \cap L_2(G, \mu_G)$ is dense in $L_2(G, \mu_G)$, one can extend the Fourier transform to the whole $L_2(G, \mu_G)$ and establish that it is an isometric bijection from $L_2^K(G, \mu_G)$ onto $L_2(\widehat{G^K}, \pi)$ (See, [3]). So if φ belongs to $L_2(\widehat{G^K}, \pi)$, then $\overset{\vee}{\varphi}$ belongs to $L_2^K(G, \mu_G)$ and $\widehat{\overset{\vee}{\varphi}} = \varphi$.

3. Admissible vectors related to representations on hypergroups

3.1. **Strong Gelfand pair.**

Definition 3.2. *The pair (G, K) is called a strong Gelfand pair if*
(i) $\phi \longmapsto \overline{\phi}$ is as involution and
(ii) the pointwise product:

$$\phi.\eta : x \longmapsto \phi(x)\eta(x) = \int_{\widehat{G^K}} \chi(x) d\left(\delta_\phi * \delta_\eta\right)(\chi) \tag{3.1}$$

as convolution defines a hypergroup structure on $\widehat{G^K}$ with the function $\mathbf{1}$ *as the neutral element.*

When the hypergroup G is commutative, G is called a strong hypergroup if $\widehat{G} = \widehat{G^{\{e\}}}$ is a hypergroup with respect to pointwise multiplication.

Let us give some characterizations of a stromg Gelfand pair.

Proposition 3.1. *Let (G, K) be a strong Gelfaid pair. Then*
(i) $(\beta_1 * \beta_2)^{\vee} = \beta_1^{\vee}.\beta_2^{\vee}$ for $\beta_1, \beta_2 \in M_b(\widehat{G^K})$.
(ii) $\widehat{G^K}$ is a commutative hypergroup.

Proof. Let us note that

$$\int_{\widehat{G^K}} \chi(x) d\left(\delta_\phi * \delta_\eta\right)(\chi) = \left(\delta_\phi * \delta_\eta\right)(\widehat{\delta_{\overline{x}}}) \text{ for } \phi, \eta \in \widehat{G^K} \text{ and } x \in G. \tag{3.2}$$

(i) Let $\beta_1, \beta_2 \in M_b(\widehat{G^K})$ and $x \in G$. We have

$$\begin{aligned}
(\beta_1 * \beta_2)^{\vee}(x) &= \int_{\widehat{G^K}} \chi(x) d(\beta_1 * \beta_2)(\chi) \\
&= \int_{\widehat{G^K}} \widehat{\delta_{\overline{x}}}(\chi) d(\beta_1 * \beta_2)(\chi) \\
&= \int_{\widehat{G^K}} \int_{\widehat{G^K}} \left(\int_{\widehat{G^K}} \widehat{\delta_{\overline{x}}}(\chi) d(\delta_\phi * \delta_\eta)(\chi) \right) d\beta_1(\phi) d\beta_2(\eta) \\
&= \int_{\widehat{G^K}} \int_{\widehat{G^K}} \left(\int_{\widehat{G^K}} \chi(x) d(\delta_\phi * \delta_\eta)(\chi) \right) d\beta_1(\phi) d\beta_2(\eta) \\
&= \int_{\widehat{G^K}} \int_{\widehat{G^K}} \phi(x)\eta(x) d\beta_1(\phi) d\beta_2(\eta) \\
&= \int_{\widehat{G^K}} \phi(x) d\beta_1(\phi) \int_{\widehat{G^K}} \eta(x) d\beta_2(\eta) \\
&= \beta_1^{\vee}(\phi).\beta_2^{\vee}(\phi)
\end{aligned}$$

(ii) Let $\phi, \eta \in \widehat{G^K}$. $\delta_\phi * \delta_\eta$ and $\delta_\eta * \delta_\phi$ are belong to $M_b(\widehat{G^K})$. Since $(\delta_\phi * \delta_\eta)^{\vee} = (\delta_\eta * \delta_\phi)^{\vee}$, then $\delta_\phi * \delta_\eta = \delta_\eta * \delta_\phi$ (see [2] th. 3.5 (ii)). Thus, the hypergroup $\widehat{G^K}$ is commutative. □

It is known that, $\widetilde{\phi} \in \widehat{G//K}$ if and only if $\phi = \widetilde{\phi} \circ p_K$ belongs to $\widehat{G^K}$, more $\widetilde{\varphi} \in C_b(\widehat{G//K})$ if and only if $\exists ! \varphi \in C_b(\widehat{G^K})$ such that $\widetilde{\varphi}(\widetilde{\phi}) = \varphi(\phi)$ for $\widetilde{\phi} \in \widehat{G//K}$. Specifically $\varphi \in \mathcal{K}(\widehat{G^K}) \Leftrightarrow \widetilde{\varphi} \in \mathcal{K}(\widehat{G//K})$. For $\beta \in M_b(\widehat{G^K})$, let us define $\widetilde{\beta}$ in $M_b(\widehat{G//K})$ by $\widetilde{\beta}(\widetilde{\varphi}) = \beta(\varphi)$ $, \forall \widetilde{\varphi} \in \mathcal{K}(\widehat{G//K})$. The mapping:

$$\begin{aligned}
M_b(\widehat{G^K}) &\longrightarrow M_b(\widehat{G//K}) \\
\beta &\longmapsto \widetilde{\beta}
\end{aligned}$$

is a linear bijection.

Let (G, K) be a strong Gelfand pair. For any $\widetilde{\beta}$ and $\widetilde{\gamma}$ in $M_b(\widehat{G//K})$, let us put $\widetilde{\beta} * \widetilde{\gamma} = \widetilde{\beta * \gamma}$. This operation defines a convolution on $M_b(\widehat{G//K})$ such as $G//K$ is a strong hypergroup. In this case, the Plancherel measure $\widetilde{\pi}$ is a Haar measure supported on the whole dual space $\widehat{G//K}$ (see [4] p.7, [8] th.12.4A) and the neutral element is the function

$$\begin{aligned}
\widehat{G//K} &\longrightarrow \mathbb{C} \\
\widetilde{\phi} &\longmapsto 1
\end{aligned}$$

Reciprocally, let us suppose that the double coset hypergroup $G//K$ is a strong hypergroup. Then the operation

$$\widetilde{\beta * \gamma} = \widetilde{\beta} * \widetilde{\gamma} \text{ for } \beta, \gamma \text{ in } M_b(\widehat{G^K})$$

defines a convolution on $M_b(\widehat{G^K})$ such as $\widehat{G^K}$ is a hypergroup with respect to pointwise multiplication.

We have also the following characterization.

Proposition 3.2. *Let (G,K) be a strong Gelfand pair. Then the Plancherel measure π is supported on the whole $\widehat{G^K}$ and π is a invariant Haar measure on $\widehat{G^K}$.*

Proof. $\pi(\varphi) = \widetilde{\pi}(\widetilde{\varphi})$ for any φ belongs to $\mathcal{K}(\widehat{G//K})$ (see [7], proof of th.3.1). Since $\widetilde{\pi}$ is a Haar measure supported on the whole dual space $\widehat{G//K}$, then π is a Haar measure supported on the whole $\widehat{G^K}$. Moreover, since $\widehat{G^K}$ is commutative, then π is invariant. □

Proposition 3.3. *Let (G,K) be a strong Gelfand pair, then*

$$\delta_\phi * \delta_\eta(\varphi) = \int_G \phi(x)\eta(x)\overset{\vee}{\varphi}(\overline{x})d\mu_G(x); \text{for } \phi,\eta \in \widehat{G^K} \text{ and } \varphi \in \mathcal{K}(\widehat{G^K}).$$

Proof. Since $\delta_\phi * \delta_\eta = \widehat{\overset{\vee}{\delta_\phi}\overset{\vee}{\delta_\eta}}\pi$, we have

$$\begin{aligned}
\delta_\phi * \delta_\eta(\varphi) &= \int_{\widehat{G^K}} \varphi(\theta)\widehat{\overset{\vee}{\delta_\phi}\overset{\vee}{\delta_\eta}}(\theta)d\pi(\theta) \\
&= \int_{\widehat{G^K}} \varphi(\theta)\left(\int_G \theta(\overline{x})\overset{\vee}{\delta_\phi}(x)\overset{\vee}{\delta_\eta}(x)d\mu_G(x)\right)d\pi(\theta) \\
&= \int_{\widehat{G^K}} \varphi(\theta)\left(\int_G \theta(\overline{x})\phi(x)\eta(x)d\mu_G(x)\right)d\pi(\theta) \\
&= \int_G \phi(x)\eta(x)\left(\int_{\widehat{G^K}} \theta(\overline{x})\varphi(\theta)d\pi(\theta)\right)d\mu_G(x) \\
&= \int_G \phi(x)\eta(x)\overset{\vee}{\varphi}(\overline{x})d\mu_G(x), \forall\varphi \in \mathcal{K}(\widehat{G^K}).
\end{aligned}$$

□

Definition 3.3. *The hypergroup G is called a Pontryagin hypergroup if (G,K) is strong and $\widehat{\widehat{G^K}}$ is a hypergroup with respect to pointwise multiplication (i.e. $\widehat{G^K}$ is a strong hypergroup).*

Remark 3.1. *It is clear that, G is a Pontryagin hypergroup iff the commutative hypergroup $G//K$ is a Pontryagin hypergroup.*

Proposition 3.4. *Let (G,K) be a strong Gelfand pair. Then the following statements hold.*
(i) $\widehat{\delta_x} \in \widehat{\widehat{G^K}}, \forall x \in G$.
(ii) If G is Pontryagin, then $\left\{\widehat{\delta_x} : x \in G\right\} = \widehat{\widehat{G^K}}$.

Proof. Let us note that $\widehat{\delta_x} \in C_b(\widehat{G^K})$ and $\widehat{\delta_x}(\phi) = \phi(\overline{x})$ for $x \in G$ and $\phi \in \widehat{G^K}$.
(i) Let $x \in G$ and $\phi,\eta \in \widehat{G^K}$. We have

$$\begin{aligned}
\widehat{\delta_x}(\phi * \eta) &= \int_{\widehat{G^K}} \widehat{\delta_x}(\theta)d(\delta_\phi * \delta_\eta)(\theta) \\
&= \int_{\widehat{G^K}} \theta(\overline{x})(\delta_\phi * \delta_\eta)(\theta) \\
&= \phi(\overline{x})\eta(\overline{x}) \\
&= \widehat{\delta_x}(\phi)\widehat{\delta_x}(\eta),
\end{aligned}$$

and also

$$\widehat{\delta_x}(\overline{\phi}) = \overline{\phi(\overline{x})} = \overline{\widehat{\delta_x}(\phi)}.$$

Since $\widehat{\delta_x}(\mathbf{1}) = 1$, we conclude that $\widehat{\delta_x} \in \widehat{\widehat{G^K}}$.
(ii) Since $\widehat{G^K}$ is a hypergroup, then $\widehat{G//K}$ is a hypergroup and $\widehat{\widehat{G//K}}$ is isomorphic to $G//K$ via the mapping $KxK \longmapsto \widehat{\delta_{KxK}}$ (see [1], Theorem 2.4.3). Let $\varsigma \in \widehat{\widehat{G^K}}$ and $\widetilde{\varsigma} \in \widehat{\widehat{G//K}}$ such that $\widetilde{\varsigma}(\widetilde{\phi}) = \varsigma(\widetilde{\phi} \circ p_K), \forall \widetilde{\phi} \in \widehat{G//K}$. Then there exists $x \in G$ such that $\widehat{\delta_{KxK}} = \widetilde{\varsigma}$ and for any $\widetilde{\phi} \in \widehat{G//K}$, we have

$$\begin{aligned} \varsigma(\widetilde{\phi} \circ p_K) &= \widehat{\delta_{KxK}}(\widetilde{\phi}) \\ &= \widetilde{\phi}(K\overline{x}K) \\ &= \widetilde{\phi} \circ p_K(\overline{x}) \\ &= \widehat{\delta_x}(\widetilde{\phi} \circ p_K). \end{aligned}$$

Thus, $\varsigma(\phi) = \widehat{\delta_x}(\phi)$ for any $\phi \in \widehat{G^K}$, anf $\varsigma = \widehat{\delta_x}$. □

3.2. **Admissible vectors.**
Let us remind some definitions.

Definition 3.4. *([13]). Let $(U, \mathcal{H}_U)$ be a representation of a hypergroup G in the Hilbert space $\mathcal{H}_U$, H be a subhypergroup of G, and $V \subseteq \mathcal{H}_U$. A vector $h_0 \in \mathcal{H}_U$ is called a (U,V)-admissible vector with respect to H if there are constant numbers $A, B > 0$ such that for every $h \in V$,*

$$A \|h\|^2_{\mathcal{H}U} \leq \int_H |\langle Ux(h_0), h\rangle|^2 \, dm_H(x) \leq B \|h\|^2_{\mathcal{H}U},$$

where m_H is a left Haar measure on H and $Ux = U(\delta_x)$.
If $A = B = 1$, h_0 is called a Parseval (U,V)- admissible vector.

Let $L : M_b(G) \longrightarrow \mathcal{B}(L_2(G, \mu_G))$ be the left regular representation of G.

$$L\mu(f) = \mu * f \ \text{ for } \mu \in M_b(G) \text{ and } f \in L_2(G, \mu_G). \tag{3.3}$$

Proposition 3.5. *Let (G,K) be a Gelfand pair and $\varphi \in L_2^K(G, \mu_G)$. Then the following satements hold.*
(i) For any $x \in G$, $\widehat{Lx(\varphi)} = \widehat{\delta_x}\widehat{\varphi}$.
(ii) $\langle Lx(\varphi), f\rangle = \left\langle (Lx(\varphi))^K, f \right\rangle$ for any $f \in L_2^K(G, \mu_G)$.

Proof. (i) Let $x \in G$. Then $\widehat{Lx(\varphi)} = \widehat{\delta_x * \varphi} = \widehat{\delta_x}\widehat{\varphi}$ since $\varphi \in L_2^K(G, \mu_G)$.
(i) Let $f \in L_2^K(G, \mu_G)$. We have

$$\begin{aligned}
\left\langle (Lx(\varphi))^K, f \right\rangle &= \left\langle (\delta_x * \varphi)^K, f \right\rangle \\
&= \langle \omega_K * \delta_x * \varphi * \omega_K, f \rangle \\
&= \langle \omega_K * \delta_x * \varphi, f \rangle \\
&= \int_G (\omega_K * \delta_x) * \varphi(y) \overline{f(y)} d\mu_G(y) \\
&= \int_G \int_G \varphi(\overline{z} * y) d(\omega_K * \delta_x)(z) \overline{f(y)} d\mu_G(y) \\
&= \int_G \int_K \varphi^-(\overline{y} * k * x) d\omega_K(k) \overline{f(y)} d\mu_G(y) \\
&= \int_K \int_G \varphi(\overline{x} * y) \overline{f(k * y)} d\mu_G(y) d\omega_K(k) \\
&= \int_G \varphi(\overline{x} * y) \overline{f(y)} d\mu_G(y), \text{ since } f \in L_2^K(G, \mu_G) \\
&= \langle \delta_x * \varphi, f \rangle = \langle Lx(\varphi), f \rangle .
\end{aligned}$$

□

Definition 3.5. *For any $\varphi \in L_2(G, \mu_G)$ and H a subhypergroup of G, let us set*

$$\begin{cases}
A_{\varphi,H} &= linear\ span \{Lx(\varphi) : x \in H\} \\
V_{\varphi,H} &= \overline{A_{\varphi,H}}^{L_2(G,\mu_G)} \\
A^K_{\varphi,H} &= linear\ span \left\{ (Lx(\varphi))^K : x \in H \right\} \\
V^K_{\varphi,H} &= \overline{A^K_{\varphi,H}}^{L_2(G,\mu_G)}.
\end{cases}$$

If $H = G$, we write simply A_φ (resp. V_φ) for $A_{\varphi,G}$ (resp. $V_{\varphi,G}$).

Definition 3.6. *Let (G, K) be a Gelfand pair. A complex-valued function of G is called trigonometric polynomial if for some $a_1, ..., a_n \in \mathbb{C}$ and $\phi_1, ..., \phi_n \in \widehat{G^K}$, we have*

$$f = \sum_{i=1}^{n} a_i \phi_i.$$

The set of all trigonometrics polynomial on G is denoted by $Trig(G)$.

We have the following theorems wchich are our main results.

Theorem 3.1. *Let G be a Pontryagin hypergroup and $\varphi \in L_2^K(G)$. Then $f \in V_\varphi^K$ if and only if $f \in L_2^K(G)$ and for some $t \in Trig(\widehat{G^K})$, $\widehat{f} = \widehat{\varphi}t$.*

Proof. Let $f = \sum\limits_{1=1}^{n} a_i (Lx_i(\varphi))^K$ where $a_i \in \mathbb{C}$ and $x_i \in G$ for $i \in \{1, 2, ..., n\} \subset \mathbb{N}$. Then $f = \sum\limits_{1=1}^{n} a_i \delta^K_{x_i} * \varphi \in L_2^K(G)$ and $\widehat{f} = \left(\sum\limits_{1=1}^{n} a_i \widehat{\delta_{x_i}} \right) \widehat{\varphi}$. By Proposition 3.4 (i), $t = \sum\limits_{1=1}^{n} a_i \widehat{\delta_{x_i}} \in \widehat{\widehat{G^K}}$.

Reciprocally, let $f \in L_2^K(G)$ and $\widehat{f} = \widehat{\varphi}t$ where $t \in Trig(\widehat{G^K})$. Then $t = \sum_{i=1}^{n} a_i\varphi_i$ for some $a_1, ..., a_n \in \mathbb{C}$ and $\varphi_1, ..., \varphi_n \in \widehat{\widehat{G^K}}$. By Proposition 3.4 (ii), there exists $x_i \in G$ such that $\varphi_i = \widehat{\delta_{x_i}}$ for any $i \in \{1, 2, ..., n\}$. Thus, $\widehat{f} = \sum_{1=1}^{n} a_i \widehat{\delta_{x_i}}\widehat{\varphi} = \sum_{1=1}^{n} a_i (\widehat{\delta_{x_i} * \varphi})$. Since $f \in L_2^K(G)$ and $\delta_{x_i} * \varphi \in L_2^K(G)$ for any $i \in \{1, 2, ..., n\}$, then
$f = f^K = \sum_{1=1}^{n} a_i (\delta_{x_i} * \varphi)^K \in V_\varphi^K$, and the proof is complete. □

Theorem 3.2. *Let G be a Pontryagin hypergroup and $\varphi \in L_2^K(G) \cap L_1(G)$. φ is a Parseval (L, V_φ^K)- admissible if and only if $|\widehat{\varphi}| = \chi_{\sup p\widehat{\varphi}}$.*

Proof. Let $\varphi \in L_2^K(G) \cap L_1(G)$ and $f \in V_\varphi^K$. Then by Theorem 3.1, $\widehat{f} = \widehat{\varphi}t$ where $t = \sum_{i=1}^{n} a_i\varphi_i$ with $a_i \in \mathbb{C}$ and $\varphi_i \in \widehat{\widehat{G^K}}$ for $i \in \{1, 2, ..., n\}$. For $x \in G$, we have

$$\begin{aligned}
\langle Lx(\varphi), f\rangle &= \left\langle (Lx(\varphi))^K, f\right\rangle \\
&= \int_G (Lx(\varphi))^K(y)\overline{f(y)}d\mu_G(y) \\
&= \int_{\widehat{G^K}} (\widehat{Lx(\varphi)})^K(\phi)\overline{\widehat{f}(\phi)}d\pi(\phi) \\
&= \int_{\widehat{G^K}} (\widehat{Lx(\varphi)})(\phi)\overline{\widehat{f}(\phi)}d\pi(\phi) \\
&= \int_{\widehat{G^K}} \overline{\phi(x)}\widehat{\varphi}(\phi)\overline{\widehat{f}(\phi)}d\pi(\phi) \\
&= \int_{\widehat{G^K}} \overline{\phi(x)}\widehat{\varphi}(\phi)\overline{\widehat{\varphi}(\phi)t(\phi)}d\pi(\phi) \\
&= \int_{\widehat{G^K}} \overline{\phi(x)}\,|\widehat{\varphi}(\phi)|^2\,\overline{t(\phi)}d\pi(\phi) \\
&= \int_{\widehat{G^K}} \overline{\phi(x)}\Lambda(\phi)d\pi(\phi)
\end{aligned}$$

with $\Lambda(\phi) = |\widehat{\varphi}(\phi)|^2\,\overline{t(\phi)}$.
Since $\varphi \in L_1(G)$, then by [3] (Proposition 3.3), $\widehat{\varphi} \in L_\infty(G)$. Thus, knowing that $t \in L_\infty(G)$, we have $|\Lambda(\phi)| \leq \|\widehat{\varphi}\|_\infty^2 \|t\|_\infty$ for any $\phi \in \widehat{G^K}$. That is $\Lambda \in L_\infty(\widehat{G^K})$. Furetheremore,

$$\begin{aligned}
\int_{\widehat{G^K}} |\Lambda(\phi)|\, d\pi(\phi) &= \int_{\widehat{G^K}} |\widehat{\varphi}(\phi)|^2\,|t(\phi)|\, d\pi(\phi) \\
&\leq \|t\|_\infty \int_{\widehat{G^K}} |\widehat{\varphi}(\phi)|^2\, d\pi(\phi) \\
&= \|t\|_\infty\, \|\varphi\|_2^2,
\end{aligned}$$

then $\Lambda \in L_1(\widehat{G^K})$. It follows that $\Lambda \in L_2(\widehat{G^K})$ and $\Lambda^\vee \in L_1(G) \cap L_2(G)$. Thus, we have on a one hand $\widehat{\Lambda^\vee} = \Lambda$ by [2] (Theorem 3.5) and on the other hand $\|\Lambda^\vee\|_2 = \|\Lambda\|_2$. Seeing this we have

$$\begin{aligned}\int_G |\langle Lx(\varphi), f\rangle|^2 \, d\mu_G(x) &= \int_G \left|\int_{\widehat{G^K}} \overline{\phi(x)}\Lambda(\phi)d\pi(\phi)\right|^2 d\mu_G(x)\\ &= \int_G |\Lambda^\vee(\overline{x})|^2 \, d\mu_G(x)\\ &= \int_G |\Lambda^\vee(x)|^2 \, d\mu_G(x)\\ &= \int_{\widehat{G^K}} |\Lambda(\phi)|^2 \, d\pi(\phi)\\ &= \int_{\widehat{G^K}} \left||\widehat{\varphi}(\phi)|^2 \, \overline{t(\phi)}\right|^2 d\pi(\phi)\\ &= \int_{\widehat{G^K}} |\widehat{\varphi}(\phi)|^4 \, |t(\phi)|^2 \, d\pi(\phi).\end{aligned}$$

As $\|f\|_2^2 = \left\|\widehat{f}\right\|_2^2 = \int_{\widehat{G^K}} |\widehat{\varphi}(\phi)|^2 \, |t(\phi)|^2 \, d\pi(\phi)$, it follows that φ is a Parseval (L, V_φ^K)- admissible vector if and only if

$$\int_{\widehat{G^K}} |\widehat{\varphi}(\phi)|^4 \, |t(\phi)|^2 \, d\pi(\phi) = \int_{\widehat{G^K}} |\widehat{\varphi}(\phi)|^2 \, |t(\phi)|^2 \, d\pi(\phi) \Longleftrightarrow \int_{\widehat{G^K}} |\widehat{\varphi}(\phi)|^2 \, |t(\phi)|^2 \, (|\widehat{\varphi}(\phi)|^2 - 1)d\pi(\phi) = 0.$$

This holds if $|\widehat{\varphi}| = \chi_{\mathrm{sup\,p}\widehat{\varphi}}$. Moreover, let us put $E_\varphi^+ = \left\{\phi \in \widehat{G^K} : |\widehat{\varphi}(\phi)|^2 > 1\right\}$ and $E_\varphi^- = \left\{\phi \in \widehat{G^K} : |\widehat{\varphi}(\phi)|^2 < 1\right\}$. If $\int_{\widehat{G^K}} |\widehat{\varphi}(\phi)|^2 \, |t(\phi)|^2 \, (|\widehat{\varphi}(\phi)|^2 - 1)d\pi(\phi) = 0$, then by taking $t = \chi_{E_\varphi^+}$ or $t = \chi_{E_\varphi^-}$, it follows that $E_\varphi^+ = \varnothing$ and $E_\varphi^- = \varnothing$. So $|\widehat{\varphi}| = 1$ on $\mathrm{supp}\widehat{\varphi}$, and the proof is complete. □

4. Conclusion

In this paper we have given a characterization of the admissible vectors related to the left regular representation of a Gelfand pair. This is a generalization of the characterization given by Tabatabaie and Jokar in [13] for commutative hypergroups. Indeed, for any commutative hypergroup G with neutral element e, the pair $(G, \{e\})$ is a Gelfand pair and $\widehat{G^{\{e\}}} = \widehat{G}$.

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