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Variations on Pascal's Hexagon Theorem

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ABSTRACT. Pascal's hexagon theorem and its dual, Brianchon's theorem, are anchor points in projective geometry. We consider a hexagon circumscribed around a conic C and the points of contact of its sides, which in turn form a hexagon inscribed in C. This configuration carries nice incidences between sides, diagonals and off-diagonals of the two hexagons. We show that in general there are exactly 19 points where three of these lines meet. These 19 points lie in threes on a total of 11 new straight lines, 3 of which are again concurrent. The proofs are based on the theorems of Pascal, Brianchon, and Kirkman. We explore some additional nice features of this geometric configuration. In addition we propose a flexible and systematic algebraic method which allows to detect and prove such incidence results.

1. INTRODUCTION

Pascal's hexagon theorem is one of the most beautiful and important theorems in projective geometry. Also known as *the hexagrammum mysticum theorem*, it triggered a whole chain of incidence results, like its dual formulation, the theorem of Brianchon, or the famous results by Steiner and Kirkman, see [1] for a most elegant presentation of the corresponding results. In the present article we want to explore some variations in the neighbourhood of Pascal's theorem and establish incidence results in conic hexagons. More precisely, we consider a hexagon $P_1 \dots P_6$ circumscribed around a conic C with points of tangency $A_1, \dots, A_6$. The set of twelve points $P := \{P_1, \dots, P_6, A_1, \dots, A_6\}$ determines a set L of 54 different lines passing through pairs of those points. We present an approach which allows to detect systematically all concurrent triples among these 54 lines. Apart from the points in P we identify 19 such points of concurrency. These 19 points lie in threes on a total of 11 new straight lines, 3 of which are again concurrent. The proofs are based on the theorems of Pascal, Brianchon, and Kirkman. In addition we propose a flexible and systematic algebraic method which allows to detect and prove such incidence results.

The article is organized as follows. In Section 2 we consider an algebraic formulation of the geometric setting that yields a discrete model using integer projective coordinates. This discrete model allows to detect all triples of concurrent lines in the set L. This part makes sure that we dot miss any concurrencies and thus obtain a complete list. In addition, the approach yields also a method to systematically prove the incidence results algebraically. Section 3 is dedicated to the geometric proofs of the incidence results. In Section 4 we present some additional geometric features of the given configuration.

2. AN ALGEBRAIC MODEL

We will work in the standard model of the real projective plane. The set of points $\mathbb{P}$ is given by $\mathbb{R}\mathrm{P}^2 = \mathbb{R}^3 \setminus \{0\}/\sim$, where $X \sim Y \in \mathbb{R}^3 \setminus \{0\}$ are equivalent if $X = \lambda Y$ for some $\lambda \in \mathbb{R}$. Similarly, the set of lines $\mathbb{B}$ is also $\mathbb{R}^3 \setminus \{0\}/\sim$, where again $g \sim h \in \mathbb{R}^3 \setminus \{0\}$ are

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equivalent, if $g = \lambda h$ for some $\lambda \in \mathbb{R}$. A point $[X]$ and a line $[g]$ are incident if $\langle X, g\rangle = 0$, where we denoted equivalence classes by square brackets and the standard inner product in $\mathbb{R}^3$ by $\langle\cdot,\cdot\rangle$. Since we mostly work with representatives we will omit the square brackets in the notation of equivalence classes. We consider the Euclidean plane $\mathbb{R}^2$ as embedded in $\mathbb{RP}^2$ by

$$(x_1, x_2) \mapsto (x_1, x_2, 1).$$

A non-degenerate conic in $\mathbb{RP}^2$ is given by the equation $\langle X, CX\rangle = 0$ where C is a regular, real, symmetric 3×3 matrix which has eigenvalues of both signs. By abuse of notation we will denote both, the conic and the matrix, with the same letter C. The tangent in a point P of C is given by CP. Vice versa, the contact point of a tangent p at C is given by $C^{-1}p$. The intersection of two lines g and h can be computed by $g \times h$, where $\times$ is the cross product in $\mathbb{R}^3$. Similarly, the line passing through the points X and Y is $X \times Y$. Hence, three lines f, g, h are concurrent iff $\langle f \times g, h\rangle = \det(f, g, h) = 0$, and three points X, Y, Z are collinear iff $\langle X \times Y, Z\rangle = \det(X, Y, Z) = 0$. See, e.g., [6] for more information or a general introduction to projective geometry.

Throughout this text we consider a hexagon $P_1 \dots P_6$ in the projective plane *circumscribed* around a conic C. The points of contact $A_1, \dots, A_6$ of its sides form a hexagon which is *inscribed* in C (see Figure 1). Let $P := \{P_1, \dots, P_6, A_1, \dots, A_6\}$. The $\binom{12}{2} = 66$ pairs of points in P define $66 - 12 = 54$ different lines (observe that the 6 sides of the hexagon $P_1 \dots P_6$ are counted three times). The goal is to find all triples of concurrent lines among these 54 lines.

2.1. **Algebraic proofs.** By a suitable projective transformation, we may assume that C is given by the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We parametrize $C \setminus \{(-1, 0, 1)\}$ by

$$\varphi : \mathbb{R} \to \mathbb{RP}^3, \quad \xi \mapsto (1 - \xi^2, 2\xi, 1 + \xi^2)$$

and set $A_i = \varphi(\xi_i)$. The group of projective transformations of the plane which leaves the conic C invariant acts three times transitively on C. This means we can prescribe the position of three points, say A_4, A_5 and A_6, on C (see, e.g., [4, Lemma 2.11]). In particular we may assume that

$$\xi_4 = 1,\ \xi_5 = 0,\ \xi_6 = -1, \tag{2.1}$$

without loss of generality. This helps to reduce the size of expressions. The point is that every incidence that we will encounter can then be expressed by a polynomial in the variables ξ_1, ξ_2, ξ_3, which must be shown to vanish identically. This is done by simply expanding the polynomial (either by hand or by computer) and verifying that all coefficients are zero. This algebraic approach has the advantage that it is an algorithm that works in every case. As an example we give an algebraic proof of Theorem 3.1 using this method (see Section 3.1 below). However, algebraic proofs usually fail to give a deeper geometric insight. For this reason we will give geometric proofs for all other incidence result that we encounter. A similar technique to prove and detect such incidence results have been used in [5].

2.2. **Incidence detection.** Before we can start proving incidence results, we now describe how to find systematically all triples of concurrent lines in our setup. We first consider a sufficiently general special case which indicates the possible candidates of concurrent triples among the 54 lines. This is done in the following way. Choose six different integers, e.g.,

$$(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) = (3, 7, 11, 17, 23, 29). \tag{2.2}$$

Then compute the points $A_i = \varphi(\xi_i)$, thereby the points $P_i = CA_i \times CA_{i+1}$, and finally, again using the cross product, the 54 straight lines which the points define. The projective coordinates of these points and lines are integer numbers. Therefore one can see if three of these lines, say f, g, h, are concurrent by checking if $\det(f, g, h) = 0$. It is easy to write a program on a computer algebra system which calculates exactly with integers and which does this test for all $\binom{54}{3}$ possible triples of lines. The program returns (apart from the 12 points in P) exactly 19 new points where three of the 54 lines intersect. Of course, the incidences found in this way can have come about by chance due to the special choice of points A_i. However in the next section we prove that these incidence results hold in general. But notice that in general there are certainly not more than 19 points where triples of the straight lines meet.

3. Incidence results

In Section 3.1 we will prove that there are exactly 19 triples of concurrent lines among the 54 lines determined by the points in $P = \{P_1, \ldots, P_6, A_1, \ldots, A_6\}$. We denote the set of the 19 points where the triples meet by Q. In Section 3.2 we will show that the 19 points in Q lie in threes on a total of 11 new lines, and that 3 of these lines are again concurrent.

3.1. **Triples of concurrent lines.** A first incidence point which is detected in the discrete case in Section 2.2 is Brianchon's point.

Theorem 3.1. *Let $P_1 \ldots P_6$ be a hexagon circumscribed around a conic C. Then the diagonals $d_i := P_iP_{i+3}$ for $i = 1, 2, 3$ are concurrent (see Figure 1).*

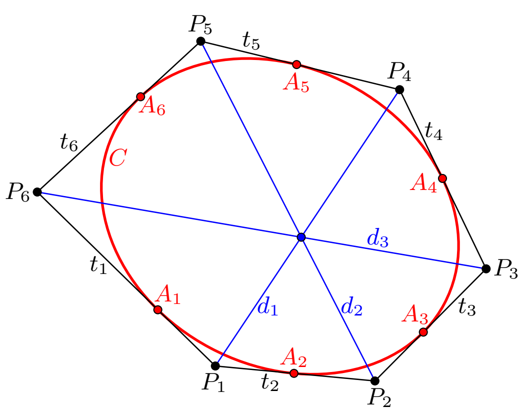

FIGURE 1. Brianchon's Theorem.

Proof. The points $A_1, \ldots, A_6$ lie on C. The tangent t_i in A_i is given by

$$t_i = CA_i = (1 - \xi_i^2, 2\xi_i, -\xi_i^2 - 1).$$

The intersection P_i of the tangents t_i and t_{i+1} (indices are read cyclically) is

$$P_i = t_i \times t_{i+1} = \big((\xi_i^2+1)\xi_{i+1} - (\xi_{i+1}^2+1)\xi_i, \xi_{i+1}^2 - \xi_i^2, (\xi_{i+1}^2-1)\xi_i + (1-\xi_i^2)\xi_{i+1}\big)\,.$$

For the diagonal d_i joining P_i and P_{i+3} we find

$$d_i = P_i \times P_{i+3} = \begin{pmatrix} (\xi_{i+3}\xi_{i+4}+1)(\xi_i+\xi_{i+1}) - (\xi_i\xi_{i+1}+1)(\xi_{i+3}+\xi_{i+4}) \\ 2(\xi_i\xi_{i+1} - \xi_{i+3}\xi_{i+4}) \\ (\xi_{i+3}\xi_{i+4}-1)(\xi_i+\xi_{i+1}) - (\xi_i\xi_{i+1}-1)(\xi_{i+3}+\xi_{i+4}) \end{pmatrix}. \tag{3.3}$$

We need to show that the diagonals d_1, d_2, d_3 in the hexagon $P_1 \ldots P_6$ are concurrent. This can be verified by checking that $\det(d_1, d_2, d_2) = 0$. Indeed, under the assumption (2.1) this can even be computed by hand, because the matrix (d_1, d_2, d_3) turns out to be quite simple:

$$(d_1, d_2, d_2) = \begin{pmatrix} (1-\xi_1)(\xi_2-1) & 2\xi_1\xi_2 & \xi_2+\xi_1(\xi_2+1)-1 \\ (\xi_2+1)(\xi_3+1) & 2\xi_2\xi_3 & \xi_3+\xi_2(1-\xi_3)+1 \\ (1-\xi_1)(\xi_3+1) & \xi_1+\xi_3 & \xi_1\xi_3+1 \end{pmatrix}.$$

□

Before we come to the discussion of the remaining 18 incidence points, we recall the following result for quadrangles which are circumscribed around a conic.

Lemma 3.1. *Let $Q_1Q_2Q_3Q_4$ be a quadrangle circumscribed around a conic C with points of tangency B_1, B_2, B_3, B_4 (see Figure 2). Let U, V, W denote the diagonal points in the complete quadrangle $Q_1Q_2Q_3Q_4$, i.e., U is the intersection of the lines $Q_1 \times Q_2$ and $Q_3 \times Q_4$, V is the intersection of the lines $Q_2 \times Q_3$ and $Q_4 \times Q_1$, and W is the intersection of the lines $Q_1 \times Q_3$ and $Q_2 \times Q_4$. Let X, Y, Z be the diagonal points of the complete quadrangle $B_1B_2B_3B_4$, i.e., X is the intersection of the lines $B_1 \times B_2$ and $B_3 \times B_4$, Y is the intersection of the lines $B_2 \times B_3$ and $B_4 \times B_1$, and Z is the intersection of the lines $B_1 \times B_3$ and $B_2 \times B_4$. Then, the diagonal points W and Z agree, the points Q_1, Q_3, Y, Z are collinear, the points Q_2, Q_4, X, Z are collinear, and the points X, Y, U, V are collinear.*

Proof. The projective maps operate four times transitively on the projective plane. In particular, there is a projective map φ with maps the quadrangle $B_1B_2B_3B_3$ to a rectangle $B_1'B_2'B_3'B_3'$. The image $C' = \varphi(C)$ is a conic which lies symmetrical to the symmetry axes of this rectangle. To see this recall that five points in general position define a unique conic. Then, by symmetry, the images of the points A_1, A_2, A_3, A_4 lie on the symmetry axes of the rectangle. It follows that the image of the points W and Z are the symmetry center of the rectangle, and hence $W = Z$. Similarly, the lines $B_1 \times B_2$ and $B_3 \times B_4$, and the lines $B_2 \times B_3$ and $B_4 \times B_1$ are parallel to the symmetry axes through Q_2, Q_4 and Q_1, Q_3, respectively. This implies that the lines $B_1 \times B_2$, $B_3 \times B_4$ and $Q_2 \times Q_4$ meet in X, and the lines $B_2 \times B_3$, $B_4 \times B_1$ and $Q_1 \times Q_3$ meet in Y. Also by symmetry, we have that the lines $Q_1 \times Q_2$ and $Q_3 \times Q_4$ are parallel, as well as the images of the lines $Q_2 \times Q_3$ and $Q_4 \times Q_1$. Hence the images of the points Y, X, U, V lie on the ideal line. □

As a side remark, notice that the above proof also reveals that the polar line of the point X is the line through the points Q_2, Q_4, X, Z, the polar line of the point Y is the line through the points Q_1, Q_3, Y, Z, and the polar line of the point Z is the line through the points X, Y, U, V.

Now we come to a first result, detected in the discrete case in Section 2.2, which connects the diagonals of the circumscribed hexagon $P_1 \ldots P_6$ with the diagonals of the inscribed hexagon $A_1 \ldots A_6$.

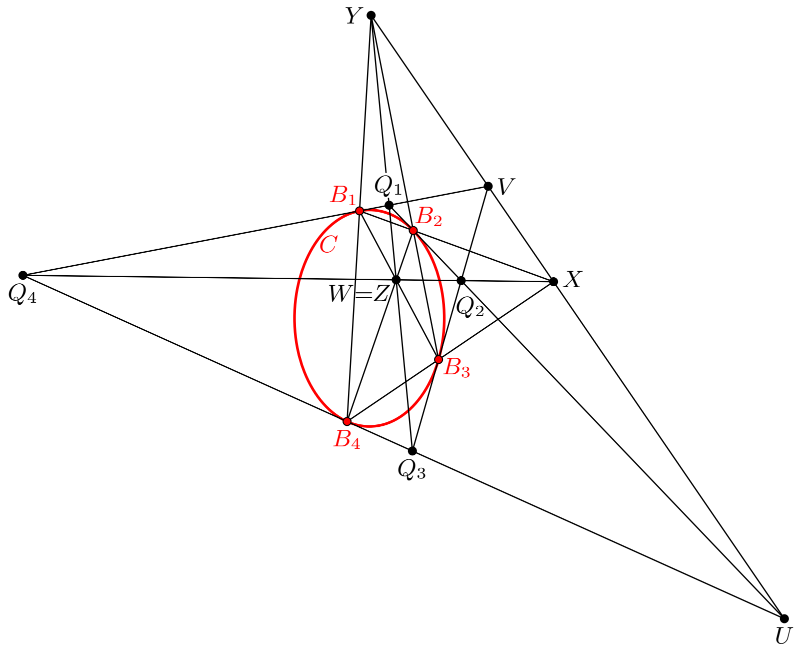

FIGURE 2. Incidences for a quadrilateral $Q_1Q_2Q_3Q_4$ that is circumscribed around a conic C.

Theorem 3.2. *Let $P_1 \dots P_6$ be a hexagon circumscribed around a conic C, and $A_1, \dots, A_6$ the contact points (see Figure 3). Let d_i be the diagonals joining P_i and P_{i+3}, and e_i the diagonals of the inscribed hexagon joining the points A_i and A_{i+3}. Then the lines e_i, e_{i+1} and d_i for $i = 1, 2, 3$ are concurrent.*

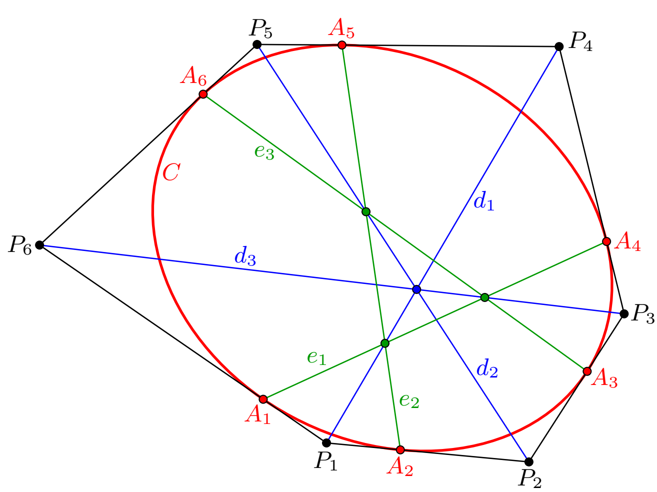

FIGURE 3. Illustration for Theorem 3.2.

Proof. We apply Lemma 3.1 separately to the quadrangle built from the tangents in the points A_1, A_2, A_4, A_5, then the quadrangle built from the tangents in A_3, A_4, A_6, A_1, and finally the quadrangle built from the tangents in $A_5A_6A_2A_3$. The point where the lines e_i, e_{i+1} and d_i meet corresponds to the point $W = Z$ in the lemma. □

The next theorem involves the off-diagonals of the circumscribed hexagon $P_1 \dots P_6$ and the off-diagonals of the inscribed hexagon $A_1 \dots A_6$.

Theorem 3.3. *Let $P_1 \dots P_6$ be a hexagon circumscribed around a conic C, and $A_1, \dots, A_6$ the contact points (see Figure 4). Consider the off-diagonals g_i joining the vertices P_i and P_{i+2} in the circumscribed hexagon, and the off-diagonals f_i of the inscribed hexagon joining the points A_i and A_{i+2}, $i = 1, \dots, 6$. Then the lines g_i, f_i, f_{i+1} are concurrent.*

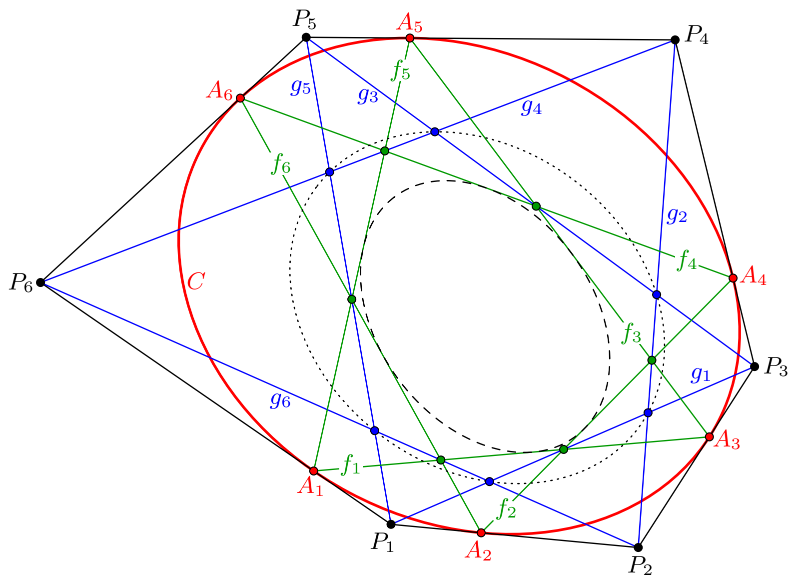

FIGURE 4. Illustration for Theorems 3.3, 4.10 and 4.11.

Proof. We apply Lemma 3.1 separately to the six quadrangles built from the tangents in the points $A_i, A_{i+1}, A_{i+2}, A_{i+3}$, for $i = 1, 2, \dots, 6$. The point where the lines g_i, f_i and f_{i+1} meet corresponds to the point $W = Z$ in the lemma. □

Figure 4 shows some further interesting incidences, which we will discuss in Section 4. But we first continue with the systematic list of concurrent triples of lines.

Theorem 3.4. *The main diagonals d_i of the hexagon $P_1 \dots P_6$ and the off-diagonals f_{i+1} and f_{i+4} of the hexagon $A_1 \dots A_6$ are concurrent (see Figure 5).*

Proof. This time, we apply Lemma 3.1 separately to the quadrangle built from the tangents in the points A_1, A_2, A_4, A_5, then the quadrangle built from the tangents in A_3, A_4, A_6, A_1, and finally the quadrangle built from the tangents in $A_5A_6A_2A_3$. The point where the lines e_i, e_{i+1} and d_i meet corresponds to the point Y in the lemma. □

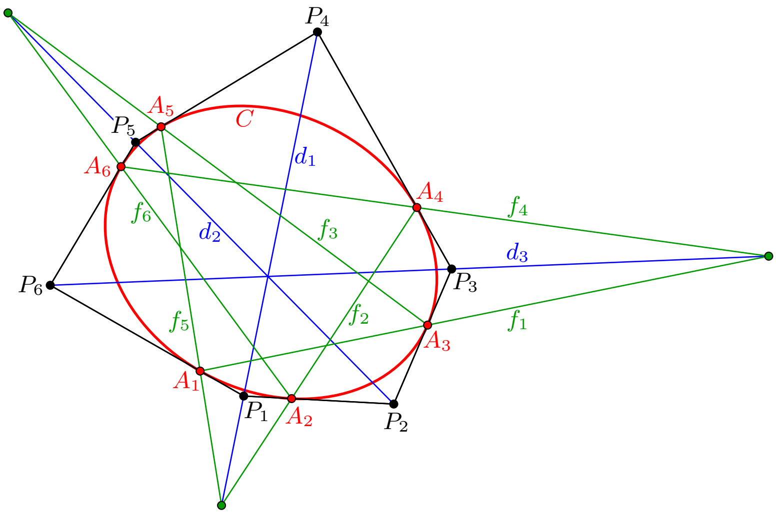

FIGURE 5. Illustration for Theorem 3.4.

Theorem 3.5. *Let $P_1, \ldots, P_6$ be a hexagon circumscribed around a conic C, and $A_1, \ldots, A_6$ the contact points. Let s_i be the side A_iA_{i+1} of the polygon $A_1, \ldots, A_6$. Then, for $i = 1, \ldots, 6$, the diagonal e_i of the polygon $A_1, \ldots, A_6$, its side s_{i+4} and the off-diagonal g_{i+3} of the polygon $P_1, \ldots, P_6$ are concurrent.*

Proof. Here, we apply Lemma 3.1 separately to the quadrangles which are built from the tangents in the points $A_i, A_{i+3}, A_{i+4}, A_{i+5}$ for $i = 1, 2, \ldots, 6$. The point where the lines e_i, s_{i+4} and g_{i+3} meet corresponds to the point Y in the lemma. □

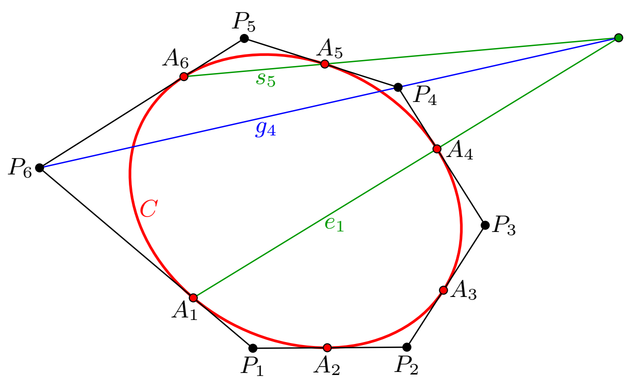

FIGURE 6. Illustration for Theorem 3.5. The diagonal e_i of the polygon $A_1, \ldots, A_6$, its side s_{i+4} and the off-diagonal g_{i+3} of the polygon $P_1, \ldots, P_6$ are concurrent. Only the situation for $i = 1$ is shown.

This finishes the list of concurrent lines. To summarize, we have 1 point of concurrency from Theorem 3.1, 3 points from Theorem 3.2, 6 points from Theorem 3.3, and 6 points from Theorem 3.5. Hence, together with the consideration of the discrete case in Section 2.2, we obtain the following.

Corollary 3.1. *Among the 54 lines which connect pairs of points from the set $\{P_1, \ldots, P_6, A_1, \ldots, A_6\}$ there are in general exactly 19 triples of concurrent lines.*

3.2. **Triples of collinear points in the set Q.** Again with the set in (2.2) we can explicitly compute the integer projective coordinates of the 19 points in the set Q where triples of lines meet. To see if three of these points Q_i, Q_j, Q_k are collinear one has to check if $\det(Q_i, Q_j, Q_k) = 0$. A computer program which tests all triples of points in Q returns 11 triples of collinear points. We now show that this holds not only by chance for the special choice (2.2), but in general.

Theorem 3.6. *For $i = 1, \ldots, 6$, the three points*

$$(A_i \times A_{i+2}) \times (A_{i+1} \times A_{i-1}), \quad (A_i \times A_{i+3}) \times (A_{i-2} \times A_{i-1}), \quad (A_{i+1} \times A_{i+3}) \times (A_{i+2} \times A_{i-2})$$

from the set Q are collinear.

Proof. This follows directly from Pascal's theorem applied to the hexagon

$$A_i A_{i+2} A_{i-2} A_{i-1} A_{i+1} A_{i+3}.$$

□

Theorem 3.7. *For $i = 1, \ldots, 3$, the three points*

$$(A_i \times A_{i+2}) \times (A_{i+1} \times A_{i-1}), \quad (A_i \times A_{i+3}) \times (A_{i+1} \times A_{i-2}), \quad (A_{i-1} \times A_{i+3}) \times (A_{i+2} \times A_{i-2})$$

from the set Q are collinear.

Proof. This follows directly from Pascal's theorem applied to the hexagon

$$A_i A_{i+2} A_{i-2} A_{i+1} A_{i-1} A_{i+3}.$$

□

Theorem 3.8. *For $i = 1, \ldots, 2$, the three points*

$$(A_i \times A_{i+3}) \times (A_{i+1} \times A_{i+2}), \quad (A_i \times A_{i-1}) \times (A_{i+1} \times A_{i-2}), \quad (A_{i-1} \times A_{i+2}) \times (A_{i+3} \times A_{i-2})$$

from the set Q are collinear.

Proof. This follows directly from Pascal's theorem applied to the hexagon

$$A_i A_{i+3} A_{i-2} A_{i+1} A_{i+2} A_{i-1}.$$

□

From the Theorems 3.6, 3.7 and 3.8 we have $6 + 3 + 2 = 11$ Pascal lines on which the points from the set Q lie in threes. Three of these 11 Pascal lines are concurrent. Again, by using the set (2.2) we can check, that only three of the 11 lines can be concurrent.

Theorem 3.9. *The three Pascal lines from Theorem 3.7 are concurrent.*

Proof. The hexagon $A_1A_4A_6A_2A_5A_3$ gives one of the three Pascal lines. The other two come from the hexagon with indices shifted by 2 and 4, i.e., from the hexagon $A_3A_6A_2A_4$ A_1A_5 and $A_5A_2A_4A_6A_3A_1$. Therefore the claim follows from Kirkman's theorem. □

4. Additional incidence results

We continue to consider the hexagon $P_1 \ldots P_6$ circumscribed around a conic C with points of tangency $A_1, \ldots, A_6$. In this section we explore some additional geometric incidences of this configuration. We start with the following simple observation which has some nice consequences.

Proposition 4.1. *The six points*

$$(P_1 \times P_2) \times (P_5 \times P_6), \quad (P_2 \times P_3) \times (P_6 \times P_1), \quad (P_3 \times P_4) \times (P_1 \times P_2),$$
$$(P_4 \times P_5) \times (P_2 \times P_3), \quad (P_5 \times P_6) \times (P_3 \times P_4), \quad (P_6 \times P_1) \times (P_4 \times P_5)$$

lie on a conic D.

Proof. This follows immediately from Poncelet's porism (see, e.g., [2]) applied to the triangles

$$(P_1 \times P_2) \times (P_5 \times P_6), \quad (P_3 \times P_4) \times (P_1 \times P_2), \quad (P_5 \times P_6) \times (P_3 \times P_4)$$

and

$$(P_2 \times P_3) \times (P_6 \times P_1), \quad (P_4 \times P_5) \times (P_2 \times P_3), \quad (P_6 \times P_1) \times (P_4 \times P_5).$$

Indeed, five of these six points define a conic D, and by Poncelet's theorem the sixth point must also lie on D. □

Theorem 4.10. *The lines $f_1, \ldots, f_6$ from Theorem 3.3 form a hexagon which is circumscribed around a conic E (see the dashed conic in Figure 4).*

Proof. Observe that the line $f_1 = A_1 \times A_3$ is the polar line of the point

$$(P_1 \times P_6) \times (P_2 \times P_3).$$

By shifting the indices each time by one, we obtain the corresponding result for the lines $f_2, \ldots, f_6$. Hence the conic E is simply the conjugate conic of D from Propostion 4.1 with respect to C (see [3, Theorem 1.5]). □

The configuration described in Theorem 3.3 also contains an inscribed conic.

Theorem 4.11. *The lines $g_1, \ldots, g_6$ from Theorem 3.3 form a hexagon which is inscribed in a conic (see the dotted conic in Figure 4).*

However, this result is buried a little deeper and we need a lemma to prove it.

Lemma 4.2. *Let $A_1 \ldots A_6$ be a hexagon inscribed in a conic C, and R_i the intersection of the sides $A_i \times A_{i+1}$ and $A_{i+2} \times A_{i+3}$. Then, the hexagon $R_1 \ldots R_6$ is cirumscribed around a conic D (see Figure 7).*

Proof. By Brianchon's theorem, we need to show that the diagonals $R_i \times R_{i+3}$ for $i = 1, 2, 3$ are concurrent (see Figure 7). To see this, observe that the Pascal line of the hexagon $R_1R_2R_5R_4R_3R_6$ is the line through the points R_1 and R_4. By shifting the index by 2, we get the Pascal line $R_3R_4R_1R_6R_5R_2$ through the points R_3 and R_6, and by another shift by 2, we get the Pascal line $R_5R_6R_3R_2R_1R_4$ through the points R_5 and R_2. By Kirkman's theorem, the three Pascal lines are concurrent and we are done. □

Proof of Theorem 4.11. Theorem 4.11 is just the dual statement of Lemma 4.2. □

Next, we consider the intersections of the off-diagonals g_i of the circumscribed hexagon $P_1 \ldots P_6$. We find:

Theorem 4.12. *Let again g_i be the off-diagonals joining the points P_i and P_{i+2}, and Q_i the intersection of the lines g_i and g_{i+1}. In the hexagon $Q_1 \ldots Q_6$ we consider the diagonals k_i joining the points Q_i and Q_{i+3}. Let d_i still denote the diagonals in the hexagon $P_1 \ldots P_6$. Then the lines d_i, k_{i+1} and k_{i+2} for $i = 1, 2, 3$ are concurrent (see Figure 8).*

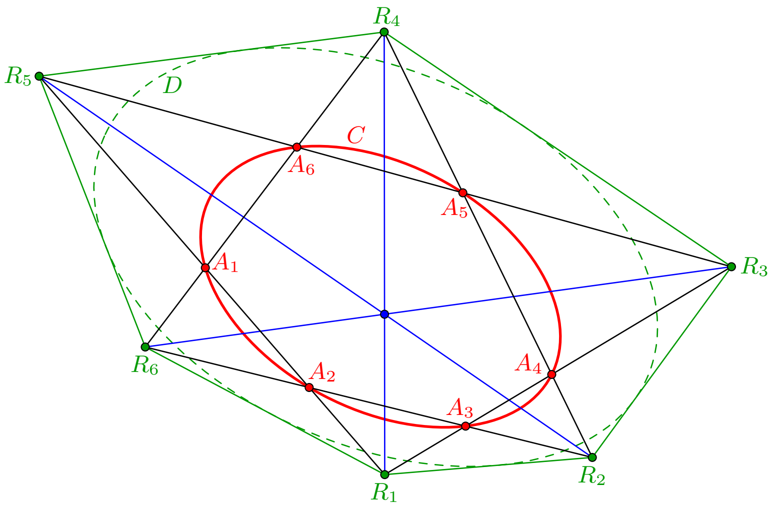

FIGURE 7. Proof of Lemma 4.2.

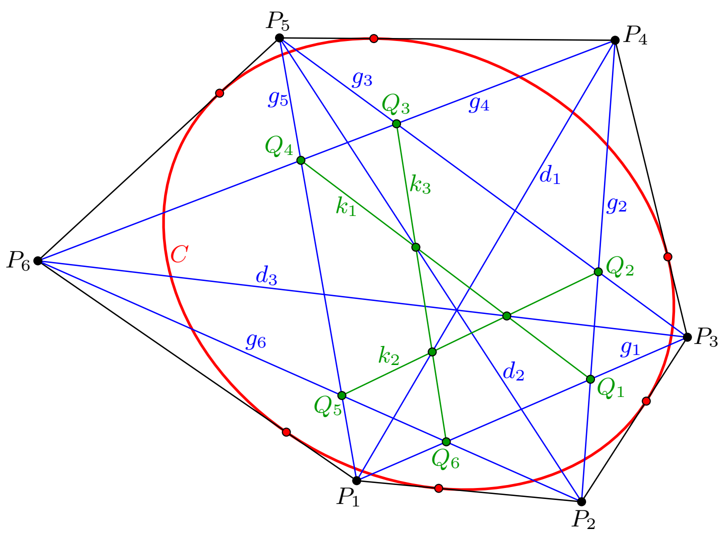

FIGURE 8. Illustration for Theorem 4.12.

Proof. By Theorem 4.11 we have that the points $Q_1, \ldots, Q_6$ lie on a conic. Then the line $P_1 \times P_4$ is the Pascal line for the conic hexagon $Q_1Q_2Q_5Q_4Q_3Q_6$. Shifting the indices by 2 we get that $P_3 \times P_6$ is the Pascal line for the conic hexagon $Q_3Q_4Q_1Q_6Q_5Q_2$. Another shift by 2 yields that $P_5 \times P_2$ is the Pascal line for the conic hexagon $Q_5Q_6Q_3Q_2Q_1Q_4$. We conclude that the lines d_i, k_{i+1}, k_{i+2} are concurrent. □

A similar result occurs, if we consider the intersections of the off-diagonals f_i of the inscribed hexagon $A_1, \ldots, A_6$.

Theorem 4.13. *Let again f_i be the off-diagonals joining the points A_i and A_{i+2}, and R_i the intersection of the lines f_i and f_{i+1}. In the hexagon $R_1 \ldots R_6$ we consider the diagonals h_i joining* *the points R_i and R_{i+3}. Let e_i still denote the diagonals in the hexagon $A_1 \dots A_6$. Then the lines h_i, e_{i+1} and e_{i+2} for $i = 1, 2, 3$ are concurrent (see Figure 9).*

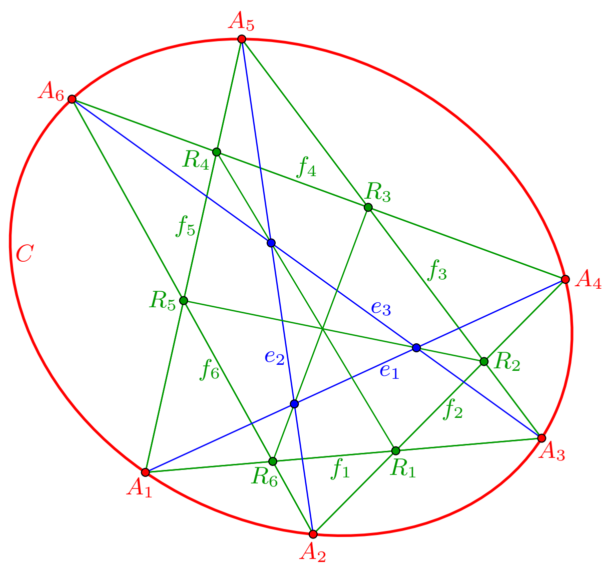

FIGURE 9. Illustration for Theorem 4.13.

Proof. Observe that the line $h_1 = R_1 \times R_4$ is the Pascal line in the hexagon $A_1A_3A_6A_4A_2A_5$. Shifting the indices by 2, we see that $h_3 = R_3 \times R_6$ is the Pascal line in the hexagon $A_3A_5A_2A_6A_4A_1$. Another shift by 2 yields the Pascal line $h_2 = R_5 \times R_2$ in the hexagon $A_5A_1A_4A_2A_6A_3$. In particular, the lines h_i, e_{i+1}, e_{i+2} are concurrent, which proves Theorem 4.13. □

Notice that the argument used above yields an alternative proof of Theorem 4.10. Indeed, by Kirkman's Theorem we have that the three Pascal lines h_1, h_2, h_3 are concurrent. This shows that the diagonals in the hexagon $R_1 \dots R_6$ are concurrent, and the inverse of Brianchon's theorem implies that the hexagon is circumscribed around a conic, as stated in Theorem 4.10.

5. CONCLUSIONS

We have considered a hexagon $P_1 \dots P_6$ circumscribed around a conic C and the points of contact of its sides, which in turn form a hexagon $A_1 \dots A_6$ inscribed in C. We have shown that among the 54 lines which connect pairs of the points $P_1, \dots, P_6, A_1, \dots, A_6$ there are in general exactly 19 concurrent triples. The 19 points of concurrency lie in threes on a total of 11 new straight lines, 3 of which are again concurrent. We have offered a flexible and systematic algebraic method which allows to detect and to prove such incidence results. In addition we provided purely geometric proofs based on the theorems of Pascal, Brianchon, and Kirkman. Moreover, we have explored some additional geometric properties of the two hexagons. This also opens up the door to further research.

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