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Rainbow Dominator Coloring of Some Cycle Related Graphs

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ABSTRACT. The concept of dominator coloring of graphs emerged as a combination of the two prominent structural aspects of graphs, namely coloring and domination in graphs. The vertex coloring that demands the existence of a rainbow path between any two vertices of a graph; that is, a path in which every internal vertex has a unique color, is called a rainbow vertex coloring of a graph. Melding the concepts of rainbow vertex coloring and dominator coloring of graphs, the rainbow dominator coloring of graphs has been studied, in the literature. In this article, we investigate the rainbow dominator coloring of some cycle related graphs, and their complements.

1. INTRODUCTION

For basic terminology in graph theory, refer to [17], and for concepts pertaining to coloring and theory of domination in graphs, see [1] and [6], respectively.

By G, we always mean a simple, undirected and a finite graph with its vertex set $V(G)$ and edge set $E(G)$. A vertex $v \in V(G)$ in a graph G of order n degree 1 is called a *pendant vertex* in G, and the vertex u such that $uv \in E(G)$ is called its *support* or a *support vertex* in G. A subset $S \subseteq V(G)$ is called an *independent* set of G if for every pair $u, v \in S$, $uv \notin E(G)$.

Graph coloring is the assignment of colors (labels) to the entities of a graph such as its vertices or edges, according to certain rules and the set of all entities assigned the same color in a coloring c of the graph is called a *color class* with respect to c. A *proper vertex coloring* of a graph G is the assignment of colors to the vertices of G such that each color class with respect to the coloring is an independent set of G, and the minimum number of colors required in a proper vertex coloring of G is called the *chromatic number* of G, denoted by $\chi(G)$. Any proper coloring of $V(G)$ with $\chi(G)$ colors is called a *χ-coloring* of G.

Beginning with the proper vertex coloring of graphs, several variants of graph coloring schemes are emerging in the literature, in order to meet the modelling requirements of various real-life problems (ref. [1, 9, 12, 16]). The *vertex-rainbow coloring* of graphs, which is defined in [8], as given below, is one such coloring that has been used to model the information transfer path problem in networks (see [2]).

A vertex coloring of a non-trivial connected graph G in which every pair of its vertices are connected by a path whose internal vertices have distinct colors is called a *vertex-rainbow coloring* of G, and the *rainbow vertex-connection number* $rvc(G)$ of G is the minimum number of colors used to obtain such a coloring of G. Note that a vertex-rainbow coloring of G need not be proper (see [8]).

Domination in graphs can be seen as the process of selecting the graph entities; usually vertices, such that an entity of the graph is either selected or is related to the selected

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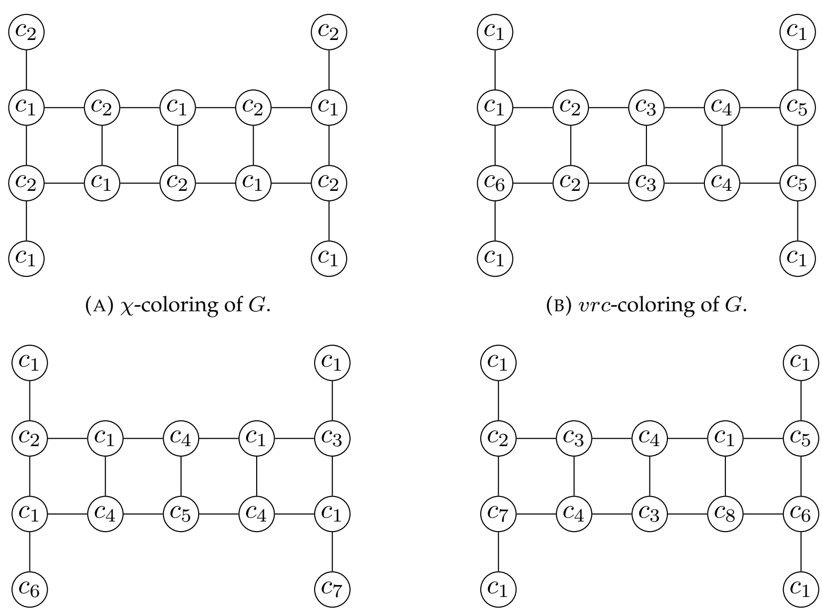
entities. In a graph G, if a vertex $v \in V(G)$ is adjacent to all vertices $u \in A$, for some $A \subseteq V(G)$ or $A = \{v\}$, we say that v *dominates* A and A is *dominated* by v. By convention, a vertex v always dominates itself (ref. [11]).

Graph coloring and domination in graphs are two well-known research areas in graph theory, and as the applications of these areas similar in nature and coincide in many aspects, the notion of *dominator coloring* of graphs was introduced in [5], by blending the concepts of coloring and domination in graphs as a proper vertex coloring of a graph G in which every vertex $v \in V(G)$ dominates at least one color class. The minimum number of colors used to obtain a dominator coloring of G is called the *dominator chromatic number* of G, denoted by $\chi_d(G)$.

Following this, several variants of dominator coloring of graphs have been defined and studied, based on different types of coloring and domination in graphs (ref. [3,4,10,11]). Combining the concepts of vertex-rainbow coloring and dominator coloring of graphs, the *rainbow dominator coloring* of a graph G was introduced in [7]. However, as that definition was not suitable to model problems in a disconnected network, the rainbow dominator coloring of graphs was modified in [13], as follows.

Definition 1.1. [13] A *rainbow dominator coloring* of a graph G is a proper vertex coloring of G in which every vertex $v \in V(G)$ dominates at least one color class and every pair of its vertices are connected by a path whose internal vertices have distinct colors, if such a path exists. The *rainbow dominator chromatic number* of G, denoted by $\chi_{rd}(G)$, is the minimum number of color classes in a rainbow dominator coloring of G.

An illustration of rainbow dominator coloring of a graph G is given in Figure 1, where it can be seen that G has $\chi(G) = 2$, $rvc(G) = 6$, $\chi_d(G) = 7$, and $\chi_{rd}(G) = 8$.



(A) χ-coloring of G. (B) vrc-coloring of G.

(C) χ_d-coloring of G. (D) χ_{rd}-coloring of G.

FIGURE 1 An example of a graph G with $\chi(G) < rvc(G) < \chi_d(G) < \chi_{rd}(G)$.

On re-defining the notion of rainbow dominator coloring of graphs in [13], the rainbow dominator coloring of certain standard graphs, and their complements were discussed in

[13]. In this article, we investigate the rainbow dominator coloring of certain cycle related graphs and their complements, by analysing their coloring patterns and determining their rainbow dominator chromatic numbers.

2. Results and Discussions

In this section, we determine the rainbow dominator chromatic number of certain cycle related graphs, such as gear graph, sunflower graph, closed sunflower graph, prism graphs, etc., and their complements, by analysing their structures and obtaining their rainbow dominator chromatic numbers.

A *gear graph* $G_{1,t,t}$; $t \geq 2$, of order $n = 2t + 1$ is obtained by making a vertex, say v, adjacent to the vertices v_i; $i \equiv 0 \pmod 2$, of the cycle C_{2t}.

Theorem 2.1. *For* $t \geq 2$, $\chi_{rd}(G_{1,t,t}) = \begin{cases} \lceil \frac{2t}{3} \rceil + 2, & 2 \leq t \leq 4; \\ \lceil \frac{2t}{3} \rceil + 3, & t \geq 5. \end{cases}$

Proof. Let $G_{1,t,t}$; $t \geq 2$, be a gear graph with $V(G_{1,t,t}) = \{v\} \cup \{v_i : 1 \leq i \leq 2t\}$ and $E(G_{1,t,t}) = \{v_i v_{i+1}, : 1 \leq i \leq 2t\} \cup \{vv_j : j \equiv 1 \pmod 2, 1 \leq j \leq 2t\}$, where the suffixes are taken modulo $2t$.

When $t = 2$, a χ-coloring c' of $G_{1,2,2}$ such that $c'(v_1) = c'(v_3) = c_1$ and $c'(v) = c'(v_2) = c'(v_4) = c_2$, is also its rainbow dominator coloring. When $t = 3$, consider a coloring c^* of $G_{1,3,3}$ such that $c^*(v) = c_2$, $c^*(v_1) = c^*(v_3) = c_1$, $c^*(v_4) = c^*(v_6) = c_2$, $c^*(v_i) = c_{2+\lceil \frac{i}{3} \rceil}$, otherwise, and when $t = 4$, extend the coloring c^* of $G_{1,3,3}$ to $G_{1,4,4}$ by assigning $c^*(v_7) = c_1$.

The coloring c^* is a dominator coloring of $G_{1,3,3}$ and $G_{1,4,4}$, as the vertex v dominates the color class $\{v_5\}$, the vertices v_{i-1}, v_i, v_{i+1}, for $1 \equiv 2 \pmod 3$, and $1 \leq i \leq 2t$, dominate the color class $\{v_i\}$, where $t + 1 = 1$. It can be verified that there exists a rainbow path of length 3 (resp. length 4) between any two vertices at a distance of $diam(G_{1,3,3})$ (resp. $diam(G_{1,4,4})$), in the coloring c^*. Hence, $\chi_{rd}(G_{1,t,t}) \leq \lceil \frac{2t}{3} \rceil + 2$.

For the vertex v_i; $i \equiv 2 \mod 6$, in $G_{1,t,t}$, $t = 3, 4$, to dominate a color class in any of its dominator coloring, either it must be assigned a unique color or only the vertices v_i, v_{i+1} must be assigned a specific color. Therefore, as $\chi_d(C_{2t}) = \lceil \frac{2t}{3} \rceil + 2$, the optimality of c^* follows.

When $t \geq 5$, consider a coloring c of $G_{1,t,t}$ such that $c(v) = c_3$, and

$$c^*(v_i) = \begin{cases} c_1, & i \equiv 1 \pmod 3, 1 \leq i \leq 2t - 1; \\ c_2, & i \equiv 3 \pmod 6, 1 \leq i \leq 2t; \\ c_3, & i \equiv 0 \pmod 6, 1 \leq i \leq 2t; \\ c_{\lceil \frac{i}{3} \rceil + 3}, & i \equiv 2 \pmod 3, 1 \leq i \leq 2t; \\ c_{\lceil \frac{2t}{3} \rceil + 3}, & i = 2t, 2t \equiv 1 \pmod 3). \end{cases}$$

This coloring c of $G_{1,t,t}$; $t \geq 5$, is its dominator coloring, as the vertex v dominates the color class $\{v_5\}$, the vertices v_{i-1}, v_i, v_{i+1}, for $1 \leq i \leq 2t$, and $i \equiv 2 \pmod 3$, dominate the color class $\{v_i\}$, when $2t \equiv 0, 2 \pmod 3$, and when $2t \equiv 1 \pmod 3$, v_{i-1}, v_i, v_{i+1}, for $1 \leq i \leq 2t - 1$, and $i \equiv 2 \pmod 3$, dominate the color class $\{v_i\}$, and v_{2t} dominates itself.

In $G_{1,t,t}$, as $d(v, v_i) \leq 2$, for all $1 \leq i \leq 2t$, $d(v_i, v_j) = 2$, when $i, j \equiv 1 \pmod 2$, and $d(v_i, v_j) = 3$, when $i \equiv 1 \pmod 2$, and $j \equiv 1 \pmod 2$, for any $1 \leq i \neq j \leq 2t$, the path between them is always colored using distinct colors. Hence, to prove that c is a rainbow dominator coloring of $G_{1,t,t}$, we must obtain a rainbow path with respect to c only between the vertices v_i, v_j such that $i, j \equiv 0 \pmod 2$, for any $1 \leq i \neq j \leq 2t$, as $d(v_i, v_j) = 4$, in this case.

If $i, j \equiv 2 \pmod 6$, then the path $v_i - v_{i-1} - v - v_{j+1} - v_j$ is a rainbow path between them, as $c(v_{i-1}) = c_1$ and $c(v_{j+1}) = c_2$. If either $i \equiv 0 \pmod 6$, or $j \equiv 0 \pmod 6$, then $v_i - v_{i-1} - v - v_{j-1} - v_j$ is a (v_i, v_j)-rainbow path as $c(v_{i-1}) \neq c(v_{j-1})$, in this case owing to the fact that $c(v_i)$ is unique for all $i \equiv 2 \pmod 3$. Similarly, if either $i \equiv 4 \pmod 6$, or $j \equiv 4 \pmod 6$, then $v_i - v_{i+1} - v - v_{j+1} - v_j$ is a (v_i, v_j)-rainbow path as $c(v_{i+1}) \neq c(v_{j+1})$, in this case, owing to the same reason. Hence, $\chi_{rd}(G_{1,t,t}) \leq \lceil \frac{2t}{3} \rceil + 3$.

As every v_i; $i \equiv 0 \pmod 2$, in the cycle C_{2t} of $G_{1,t,t}$ is adjacent only to the vertices v_{i-1}, and v_{i+1}, it can dominate a color class with respect to any of its proper coloring if and only if the color class is either $\{v_i\}$ or a subset of v_{i-1}, v_{i+1}. As this domination property of the vertices v_i; $i \equiv 1 \pmod 2$, of $G_{1,t,t}$ are same as the ones exhibited by the vertices of any cycle, we require at least $\chi_d(C_{2t})$ colors to obtain a dominator coloring of $G_{1,t,t}$. However, in any dominator coloring $\bar{c}$ of C_{2t}, $\bar{c}(v_{i-1}) = \bar{c}(v_{i+1})$, for all the vertices v_i; $i \equiv 2 \pmod 3$, or $\bar{c}(v_{i-1}) = \bar{c}(v_{j-1})$ and $\bar{c}(v_{i+1}) = \bar{c}(v_{j+1})$, for v_i, v_j; $1 \leq i \neq j \leq 2t$, such that $i, j \equiv 2 \pmod 3$.

In the first case, we cannot obtain a rainbow path between two vertices v_i and v_j such that $i, j \equiv 2 \pmod 6$, and in the second case, the vertex v cannot be assigned any of the two colors that are assigned to more than one vertex of C_{2t}. Hence, in both the cases, we require at least one color in addition to the number of colors used in any minimum dominator coloring of $G_{1,t,t}$ to obtain its minimum rainbow dominator coloring. As it has been proved in [5] $\chi_d(C_{2t}) = \lceil \frac{2t}{3} \rceil + 2$, for all $t \geq 3$, the result follows. □

Proposition 2.1. *For $t \geq 2$, $\chi_{rd}(\overline{G}_{1,t,t}) = t + 1$.*

Proof. In the complement $\overline{G}_{1,t,t}$ of a gear graph $G_{1,t,t}$ as described in Theorem 2.1, any v_i; $1 \leq i \leq 2t$, is adjacent to all the vertices of $\overline{G}_{1,t,t}$, except v_{i-1}, v_i, v_{i+1}, where the suffixes are taken modulo $2t$ and the vertex v is adjacent to all the vertices v_i; $1 \leq i \leq 2t$, for $i \equiv 0 \pmod 2$. Hence, the vertices v, v_i; $1 \leq i \leq 2t$, for $i \equiv 0 \pmod 2$, induce a clique of order $t + 1$ in $\overline{G}_{1,t,t}$, yielding $\chi_{rd}(\overline{G}_{1,t,t}) \geq t + 1$.

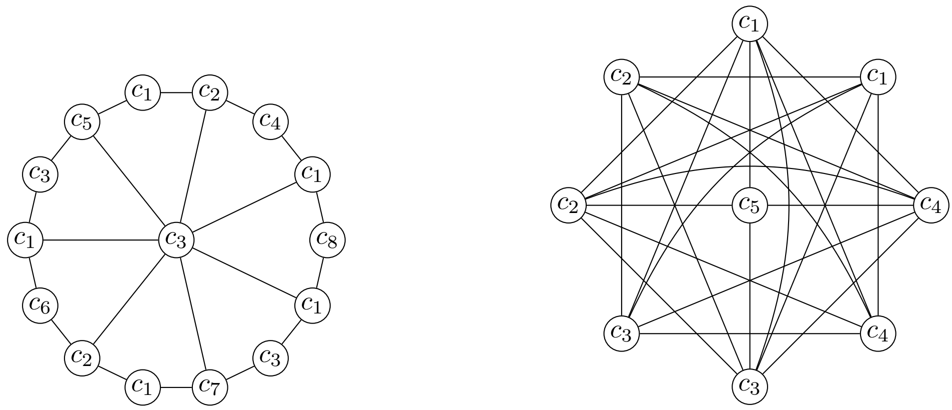
The coloring $c : V(\overline{G}_{1,t,t}) \to \{c_1, c_2, \ldots, c_{t+1}\}$ such that $c(v_i) = c(v_{i+1}) = c_{\lceil \frac{i}{2} \rceil}$, for $1 \leq i \leq 2t$, when $i \equiv 0 \pmod 2$, and $c(v) = c_{t+1}$ is a dominator coloring of $\overline{G}_{1,t,t}$, as each v_i; $i \equiv 0 \pmod 2$, and v dominate the color class $\{v\}$, and v_i; $i \equiv 1 \pmod 2$, dominate the color class $\{v_{i+2}, v_{i+3}\}$, for $1 \leq i \leq 2t$, where the suffixes are taken modulo $2t$, with respect to c.

For any two vertices $v_i, v_j \in V(\overline{G}_{1,t,t})$, $1 \leq i \neq j \leq 2t$, $d(v_i, v_j) = 1$, when both i and j are of the same parity; otherwise, $d(v_i, v_j) = 2$, as there exists a path $v_i - v_{i+2} - v_j$ or $v_i - v_{i+2} - v_j$ of length 2. Also, as $d(v, v_j) = 1$, when j is even and $d(v, v_j) = 2$, when j is odd, there exists a rainbow path between any two non-adjacent vertices of $\overline{G}_{1,t,t}$ in c. Therefore, $\chi_{rd}(\overline{G}_{1,t,t}) = t + 1$. □

A *sunflower graph* $SF_{1,t,t}$ of order $2t + 1$ is a graph obtained by replacing every edge v_iv_{i+1}; $1 \leq i \leq t$, by a triangle $v_i - u_i - v_{i+1} - v_i$, in a wheel graph $W_{1,t} = C_t + K_1$, where the vertices v_i; $1 \leq i \leq t$, are vertices of degree 3 in $W_{1,t}$, and the suffixes are taken modulo t.

Theorem 2.2. *For $t \geq 4$, $\chi_{rd}(SF_{1,t,t}) = \lceil \frac{t}{2} \rceil + 2$.*

Proof. Let $SF_{1,t,t}$; $t \geq 4$, be a sunflower graph with $V(SF_{1,t,t}) = \{v\} \cup \{v_i : 1 \leq i \leq t\} \cup \{u_i : 1 \leq i \leq t\}$ and $E(SF_{1,t,t}) = \{vv_i : 1 \leq i \leq t\} \cup \{v_iv_{i+1} : 1 \leq i \leq t\} \cup \{v_iu_i : 1 \leq i \leq t\} \cup \{u_iv_{i+1} : 1 \leq i \leq t\}$, where the suffixes are taken modulo t. Consider a coloring



(A) χ_{rd}-coloring of $G_{1,7,7}$. (B) χ_{rd}-coloring of $\overline{G}_{1,4,4}$.

FIGURE 2 χ_{rd}-coloring of gear graph and its complement.

$c : V(SF_{1,t,t}) \to \{c_r : 1 \leq r \leq \lceil \frac{t}{2} \rceil + 2\}$ such that for any vertex $w \in V(SF_{1,t,t})$,

$$c(w) = \begin{cases} c_{\lceil \frac{i}{2} \rceil}, & w \in \{v_i : 1 \leq i \leq t,\ i \equiv 1 \pmod 2\}; \\ c_{\lceil \frac{t}{2} \rceil + 1}, & w \in \{v_i : 1 \leq i \leq t,\ i \equiv 0 \pmod 2\}; \\ c_{\lceil \frac{t}{2} \rceil + 2}, & w \in \{v\} \cup \{u_i : 1 \leq i \leq t\}. \end{cases}$$

With respect to c, every u_i, v_i; $i \equiv 1 \pmod 2$, dominate the color class $\{v_i\}$, and when $i \equiv 0 \pmod 2$, the vertices u_i dominate the color class $\{v_{i+1}\}$ and v_i dominate the color classes $\{v_{i-1}\}$ and $\{v_{i+1}\}$, for all $1 \leq i \leq t$. Also, as the vertex v dominates the color class $\{v_i\}$, for all $i \equiv 1 \pmod 2$, in c, it is a dominator coloring of $SF_{1,t,t}$.

In $SF_{1,t,t}$, $d(v_i, v_j) = 2$, $d(v, v_i) = 1$, $d(v, u_i) = 2$, for all $1 \leq i \neq j \leq t$, and $d(u_i, u_j) = 4$, for all $1 \leq i \leq t$, and $j \geq i + 4$, where the suffixes are taken modulo t. The path $u_i - v_i - v - v_j - u_j$ is a rainbow path between any two vertices u_i and u_j, with respect to c, where $i \equiv 1 \pmod 2$, irrespective of the parity of j. Also, the path $u_i - v_{i+1} - v - v_{j+1} - u_j$ is a rainbow path between any two vertices u_i and u_j, with respect to c, where $i, j \equiv 0 \pmod 2$. Hence, $\chi_{rd}(SF_{1,t,t}) \leq \lceil \frac{t}{2} \rceil + 2$, for all $t \geq 4$.

In $SF_{1,t,t}$, a vertex u_i; $1 \leq i \leq t$, dominates a color class with respect to any of its dominator coloring if it is $\{u_i\}$ or $\{v_{i+1}\}$ or $\{u_i\}$. If every u_i dominates a color class that contains one distinct vertex, then we need t unique colors, in such a coloring of $SF_{1,t,t}$. In c, as every u_i, u_{i+1}, for $1 \leq i \leq t$, and $i \equiv 1 \pmod 2$, dominates the color class v_i, it gives an optimal rainbow dominator coloring of $SF_{1,t,t}$. If $\chi_{rd}(SF_{1,t,t}) < \lceil \frac{t}{2} \rceil + 2$, then at least one singleton color class has been removed from c, which leads to the vertex u_i, for some $1 \leq i \leq t$, not dominating any color class, proving the result. □

Proposition 2.2. . *For $t \geq 4$, $\chi_{rd}(\overline{SF}_{1,t,t}) = t + 1$.*

Proof. For the complement $\overline{SF}_{1,t,t}$ of a sunflower graph $SF_{1,t,t}$ constructed as given in Theorem 2.2, the coloring $c : V(\overline{SF}_{1,t,t}) \to \{c_1, c_2, \ldots, c_{t+1}\}$ such that $c(v_i) = c(u_i) = c_i$, for $1 \leq i \leq t$, and $c(v) = c_{t+1}$ is its rainbow dominator coloring using $t + 1$ colors. This is because the vertices v, u_i; $1 \leq i \leq t$, dominate the color class $\{v\}$, and the vertices v_i; $1 \leq i \leq t$, dominate the color class $\{v_{i+3}, u_{i+3}\}$, where $t + j = j$, for any $1 \leq j \leq t$, and $d(v, v_i) = 2$, $d(v_i, u_i) = 2$, and $d(v_i, v_j) = 2$, for all $1 \leq i \neq j \leq t$; enusring the existence rainbow paths $v - u_{i+2} - v_i$, $v_i - u_{j-2} - u_j$, and $v_i - u_{i+2} - v_j$, between the respective pairs of non-adjacent vertices, with respect to c. As the vertices $v, u_i;\ 1 \leq i \leq t$, induce a clique of order $t+1$ in $\overline{SF}_{1,t,t}$, for any $t \geq 4$, the result follows. □

A *closed sunflower graph* $CSF_{1,t,t};\ t \geq 3$, is obtained from the sunflower graph $SF_{1,t,t}$ by making each $u_i;\ 1 \leq i \leq t$, adjacent to u_{i+1} and u_{i-1}, where the suffixes are taken modulo t.

Theorem 2.3. *For* $t \geq 4$, $\chi_{rd}(CSF_{1,t,t}) = \lceil \frac{t}{2} \rceil + 3$.

Proof. Let $CSF_{1,t,t};\ t \geq 4$, be a closed sunflower graph with $V(CSF_{1,t,t}) = \{v\} \cup \{v_i : 1 \leq i \leq t\} \cup \{u_i : 1 \leq i \leq t\}$ and $E(CSF_{1,t,t}) = \{vv_i : 1 \leq i \leq t\} \cup \{v_iv_{i+1} : 1 \leq i \leq t\} \cup \{v_iu_i : 1 \leq i \leq t\} \cup \{u_iv_{i+1} : 1 \leq i \leq t\} \cup \{u_iu_{i+1} : 1 \leq i \leq t\}$, where the suffixes are taken modulo t. Consider a coloring $c : V(CSF_{1,t,t}) \to \{c_r : 1 \leq r \leq \lceil \frac{t}{2} \rceil + 3\}$ such that $c(v) = c_2$,

$$c(v_i) = \begin{cases} c_{\lceil \frac{i}{2} \rceil + 3}, & i \equiv 1 \pmod 2,\ 1 \leq i \leq t-1; \\ c_3, & i = t, \text{ when } t \equiv 1 \pmod 2; \\ c_1, & i \equiv 0 \pmod 2,\ 1 \leq i \leq t. \end{cases}$$

and

$$c(u_i) = \begin{cases} c_2, & i \equiv 1 \pmod 2,\ 1 \leq i \leq t-1; \\ c_3, & i \equiv 0 \pmod 2,\ 1 \leq i \leq t-2; \\ c_{\lceil \frac{t}{2} \rceil + 3}, & i = t-1, \text{ when } t \equiv 1 \pmod 2; \\ c_2, & i = t-1, \text{ when } t \equiv 0 \pmod 2; \\ c_1, & i = t, \text{ when } t \equiv 1 \pmod 2; \\ c_3, & i = t, \text{ when } t \equiv 0 \pmod 2. \end{cases}$$

When t is even, every $v_i, v_{i+1}, u_i; i \equiv 1 \pmod 2$, dominate the color class $\{v_i\}$, and the vertices $u_i; i \equiv 0 \pmod 2$, dominate the color class $\{v_{i+1}\}$, for all $1 \leq i \leq t$, where we take $t + j = j$, with respect to c. When t is odd, every v_i, v_{i+1}, and $u_i; i \equiv 1 \pmod 2$, dominate the color class $\{v_i\}$, for all $1 \leq i \leq t-2$. The vertices $v_i, u_i; i = t-1, t$, dominate the color class $\{u_{t-1}\}$, with respect to c.

With respect to c, there exists a rainbow path between any two non-adjacent vertices v_i, v_j, v, u_i of $CSF_{1,t,t}$, owing to the fact that $d(v_i, v_j) \leq 2$, $d(v, u_i) = 2$, for all $1 \leq i \neq j \leq t$. As $d(u_i, u_j) = 4$, for all $j \geq i+4$, and $1 \leq i \leq t$, where the suffixes are taken modulo t, the path $u_i - v_i - v - v_j - u_j$, when $i \equiv 1 \pmod 2$, and the path $u_i - v_{i+1} - v - v_j - u_j$, when $i \equiv 0 \pmod 2$, are rainbow paths of length 4 between u_i, u_j, for any $1 \leq i \neq j \leq t$, with respect to c. Hence, $\chi_{rd}(CSF_{1,t,t}) \leq \lceil \frac{t}{2} \rceil + 3$, for all $t \geq 4$.

In the closed sunflower graph $CSF_{1,t,t}$, any u_i can dominate a color class of cardinality at most 2; that is, every u_i can dominate a color class only when it is one of the following forms : $\{u_i\}$, $\{v_i\}$,$\{v_{i+1}\}$, $\{u_{i-1}, u_{i+1}\}$, $\{u_{i-1}, v_{i+1}\}$, or $\{v_{i-1}, u_{i+1}\}$. If every u_i must dominate a color class of cardinality 2, with respect to some dominator coloring of $CSF_{1,t,t}$, we require at least t colors in such a dominator coloring, as such a color class can be either $\{u_{i-1}, v_{i+1}\}$, or $\{v_{i-1}, u_{i+1}\}$.

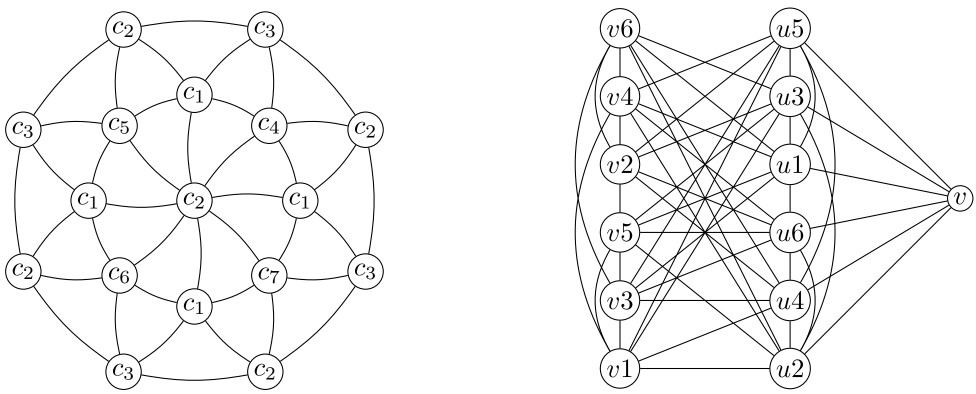
If each u_i dominates a color class of cardinality 1; the color class is either $\{u_i\}$, $\{v_i\}$ or $\{v_{i+1}\}$. However, if each u_i dominates a distinct color class of cardinality 1 in a dominator coloring of $CSF_{1,t,t}$, we use at least t colors in such a coloring. Hence, a dominator coloring such that u_i and u_{i+1}, for each $i \equiv 1 \pmod 2$, dominating the same color class is an optimal one. However, as these $\lceil \frac{t}{2} \rceil$ colors cannot be given to any other v_i and v, we need at least two colors in addition to it. However, the $\lceil \frac{t}{2} \rceil + 1$ assigned to v_i's cannot be assigned to u_i's, it can be seen that we need at least $\lceil \frac{t}{2} \rceil + 3$ colors in any dominator coloring of $CSF_{1,t,t}$, completing the proof. □

Proposition 2.3. *For $t \geq 4$, $\chi_{rd}(\overline{CSF}_{1,t,t}) = t+1$.*

Proof. Let $\overline{CSF}_{1,t,t}$ be the complement of a closed sunflower graph $CSF_{1,t,t}$ constructed as given in Theorem 2.3. The coloring $c : V(\overline{CSF}_{1,t,t}) \to \{c_1, c_2, \ldots, c_{t+1}\}$ such that $c(v_i) = c(u_i) = c_i$, for $1 \leq i \leq t$, and $c(v) = c_{t+1}$ is its rainbow dominator coloring using $t+1$ colors, owing to the same arguments mentioned in Proposition 2.2, and the fact that any two non-adjacent u_i's have a path of length 2 through v in $\overline{CSF}_{1,t,t}$.

Based on the definition of a closed sunflower graph $CSF_{1,t,t}$, it can be seen that the graph $\overline{CSF}_{1,t,t}$ contains a clique of order $\lfloor \frac{t}{2} \rfloor + 1$, induced by the vertices v and u_i; $i \equiv 0 \pmod 2$, $1 \leq i \leq t$. For every $1 \leq i \leq t$, as each u_i is adjacent to the vertices $v_i, v_{i+1}, u_{i+1}, u_{i-1}$ in $CSF_{1,t,t}$, any color assigned to a vertex u_i in a proper coloring of $\overline{CSF}_{1,t,t}$ can be assigned only to v_i or to v_{i+1}, as v_{i+1} and u_{i-1} are adjacent in $\overline{CSF}_{1,t,t}$, and v_i is adjacent to v in $\overline{CSF}_{1,t,t}$.

In addition to this, as the v_i's and u_i's induce a cycle in $CSF_{1,t,t}$, it can be observed that any color can be assigned to at most two vertices of $\overline{CSF}_{1,t,t}$, in any of its proper coloring; thereby proving that the coloring c of $\overline{CSF}_{1,t,t}$ given above with $t+1$ is the required optimal rainbow coloring of $\overline{CSF}_{1,t,t}$, for any $t \geq 4$. □



(A) χ_{rd}-coloring of $CSF_{1,8,8}$.   (B) χ_{rd}-coloring of $\overline{CSF}_{1,4,4}$.

FIGURE 3 χ_{rd}-coloring of a closed sunflower graph and its complement.

A *prism graph* of order $2t$, denoted by Y_t, is obtained by making every vertex v_i of a cycle $C_t = v_1 - v_2 - \ldots - v_t - v_1$ adjacent to the vertex u_i of a cycle $C'_t = u_1 - u_2 - \ldots - u_t - u_1$.

Theorem 2.4. *For $t \geq 4$, $\chi_{rd}(Y_t) = t$.*

Proof. Let Y_t; $t \geq 4$, be a prism graph with $V(Y_t) = \{v_i, u_i : 1 \leq i \leq t\}$ and $E(Y_t) = \{v_iu_i : 1 \leq i \leq t\} \cup \{v_iv_{i+1} : 1 \leq i \leq t\} \cup \{u_iu_{i+1} : 1 \leq i \leq t\}$, where the suffixes are taken modulo t. As a rainbow path between any two vertices of Y_t is same as the rainbow path between them in a cycle C_t, we need at least $\chi_{rd}(C_t)$ colors in a rainbow dominator coloring of Y_t. Hence, the vertices of one of the two cycles of order t, say $v_1 - v_2 - \ldots - v_t - v_1$, are colored with $\chi_{rd}(C_t)$ colors. Now, to obtain a dominator coloring of the other cycle of order t in Y_t, we need at least $\lceil \frac{t}{3} \rceil$ colors in addition to the $\chi_{rd}(C_t)$ colors used to color the vertices v_i; $1 \leq i \leq t$, of Y_t. Hence, $\chi_{rd}(Y_t) \geq \chi_{rd}(C_t) + \lceil \frac{t}{3} \rceil$.

As it has been proved in [13] that $\chi_{rd}(C_t) = \chi(C_t)$, when $t \leq 5$, $\chi_{rd}(C_t) = \lfloor \frac{t}{2} \rfloor + \lceil \frac{t}{6} \rceil$, when $t \geq 5$, $t \equiv 1 \pmod 6$, and $\chi_{rd}(C_t) = \lceil \frac{t}{2} \rceil + \lceil \frac{t}{6} \rceil$, when $t \geq 5$, $t \equiv 0, 2, 3, 4, 5 \pmod 6$, it can be observed that $\chi_{rd}(Y_t) \geq t$, for all $t \geq 3$.

To prove the result, we obtain a coloring pattern of Y_t in th following cases, based on the value t.

Case 1: When $t \equiv 0 \pmod 6$, consider the coloring $c : V(Y_t) \to \{c_1, c_2, \dots, c_t\}$ such that for $1 \le i \le \lfloor \frac{t}{2} \rfloor$, $c(v_i) = c_i$,

$$c(v_{\lfloor \frac{t}{2} \rfloor + i}) = \begin{cases} c_i, & i \equiv 0, 1 \pmod 3; \\ c_{\lfloor \frac{t}{2} \rfloor + \lceil \frac{i}{3} \rceil}, & i \equiv 2 \pmod 3. \end{cases}$$

and for $1 \le i \le t$,

$$c(u_i) = \begin{cases} c(v_{i+2}), & i \equiv 1 \pmod 3; \\ c(v_{i+1}), & i \equiv 0 \pmod 3; \\ c_{\lceil \frac{t}{2} \rceil + \lceil \frac{t}{6} \rceil + \lceil \frac{i}{3} \rceil}, & i \equiv 2 \pmod 3. \end{cases}$$

This is a dominator coloring of Y_t; $t \equiv 0 \pmod 6$, as the vertices u_{i-1}, u_i, u_{i+1} and v_{i-1}, v_i, v_{i+1}, for $1 \le i \le t$, when $i \equiv 2 \pmod 3$, dominate the color class $\{u_i\}$ and $\{v_i\}$, respectively.

Case 2: When $t \equiv 1 \pmod 6$, let $c' : V(Y_t) \to \{c_1, c_2, \dots, c_t\}$ be a coloring such that $c'(v_i) = c(v_i)$, for all $1 \le i \le t-1$, $c'(v_t) = c_{\lfloor \frac{t}{2} \rfloor + \lceil \frac{t}{3} \rceil}$, and for $1 \le i \le t-2$, $c'(u_i) = c(u_i)$, $c'(u_{t-1}) = c_1$, and $c'(u_t) = c_4$, where c is a dominator coloring of Y_t; $t \equiv 0 \pmod 6$, as defined in Case 1.

The coloring c' is a dominator coloring of Y_t; $t \equiv 1 \pmod 6$, as c is a dominator coloring of Y_t; $t \equiv 0 \pmod 6$, where the vertices u_{i-1}, u_i, u_{i+1} and v_{i-1}, v_i, v_{i+1}, for $1 \le i \le t-2$, when $i \equiv 2 \pmod 3$, dominate the color class $\{u_i\}$ and $\{v_i\}$, respectively, and the vertices u_t, v_t dominate the color class $\{v_t\}$.

Case 3: When $t \equiv 2 \pmod 6$, consider the coloring $c^* : V(Y_t) \to \{c_1, c_2, \dots, c_t\}$ such that $c^*(v_t) = c_{\lceil \frac{t}{2} \rceil + \lceil \frac{t}{3} \rceil}$, for $1 \le i \le \lfloor \frac{t}{2} \rfloor$, $c^*(v_i) = c_i$,

$$c^*(v_{\lfloor \frac{t}{2} \rfloor + i}) = \begin{cases} c_i, & i \equiv 0, 1 \pmod 3; \\ c_{\lfloor \frac{t}{2} \rfloor + \lceil \frac{i}{3} \rceil}, & i \equiv 2 \pmod 3. \end{cases}$$

for $1 \le i \le \lfloor \frac{t}{2} \rfloor - 2$,

$$c^*(u_i) = \begin{cases} c(v_{i+1}), & i \equiv 0 \pmod 3; \\ c(v_{i+2}), & i \equiv 1 \pmod 3; \\ c_{\lceil \frac{t}{2} \rceil + \lceil \frac{t}{6} \rceil + \lceil \frac{i}{3} \rceil}, & i \equiv 2 \pmod 3. \end{cases}$$

for $j = \lfloor \frac{t}{2} \rfloor + i$, when $1 \le i \le \lfloor \frac{t}{2} \rfloor - 2$,

$$c^*(u_j) = \begin{cases} c(v_{j+1}), & i \equiv 0 \pmod 3; \\ c(v_{j+2}), & i \equiv 1 \pmod 3; \\ c_{\lceil \frac{t}{2} \rceil + \lceil \frac{t}{6} \rceil + \lceil \frac{j}{3} \rceil}, & j \equiv 2 \pmod 3. \end{cases}$$

and $c^*(u_t) = c^*(u_{\lfloor \frac{t}{2} \rfloor}) = c_4$, and $c^*(u_{t-1}) = c^*(u_{\lfloor \frac{t}{2} \rfloor - 1}) = c_1$.

This is a dominator coloring of Y_t, as the vertices u_{i-1}, u_i, u_{i+1} and v_{i-1}, v_i, v_{i+1}, for $1 \le i \le \lfloor \frac{t}{2} \rfloor - 2$, when $i \equiv 2 \pmod 3$, dominate the color class $\{u_i\}$ and $\{v_i\}$, respectively, with respect to c. Also, the vertices u_{j-1}, u_j, u_{j+1} and v_{j-1}, v_j, v_{j+1}, where $j = \lfloor \frac{t}{2} \rfloor + i$, for $1 \le i \le \lfloor \frac{t}{2} \rfloor - 2$, and when $i \equiv 2 \pmod 3$, dominate the color class $\{u_j\}$ and $\{v_j\}$, respectively, in c. The vertices v_t, u_t dominate $\{v_t\}$ and $v_{\lfloor \frac{t}{2} \rfloor}, u_{\lfloor \frac{t}{2} \rfloor}$ dominate $\{v_{\lfloor \frac{t}{2} \rfloor}\}$, in this coloring.

Case 4: For $t \equiv 3 \pmod 6$, let $c'' : V(Y_t) \to \{c_1, c_2, \dots, c_t\}$ be a coloring such that for all $1 \le i \le t-1$, $c''(v_i) = c^*(v_i)$, and $c''(u_i) = c^*(u_i)$, $c''(u_t) = c^*(v_{t-1})$ and $c''(v_t) = c_t$, where c^* is a dominator coloring of Y_t; $t \equiv 2 \pmod 6$, as defined in Case 3.

This is a dominator coloring of Y_t; $t \equiv 3 \pmod 6$, as all vertices except v_t and u_t, dominate the color classes, as explained in Case 3, and the vertices v_t and u_t dominate the color class $\{v_t\}$.

Case 5: When $t \equiv 4 \pmod 6$, $\bar{c} : V(Y_t) \to \{c_1, c_2, \ldots, c_t\}$ such that for $1 \le i \le \lfloor \frac{t}{2} \rfloor$, $\bar{c}(v_i) = c_i$,

$$\bar{c}(v_{\lfloor \frac{t}{2} \rfloor + i}) = \begin{cases} c_i, & i \equiv 0, 1 \pmod 3; \\ c_{\lfloor \frac{t}{2} \rfloor + \lceil \frac{i}{3} \rceil}, & i \equiv 2 \pmod 3. \end{cases}$$

for $1 \le i \le \lfloor \frac{t}{2} \rfloor - 1$,

$$\bar{c}(u_i) = \begin{cases} c(v_{i+1}), & i \equiv 0 \pmod 3; \\ c(v_{i+2}), & i \equiv 1 \pmod 3; \\ c_{\lceil \frac{t}{2} \rceil + \lceil \frac{t}{6} \rceil + \lceil \frac{i}{3} \rceil}, & i \equiv 2 \pmod 3, \end{cases}$$

for $j = \lfloor \frac{t}{2} \rfloor + i$, when $1 \le i \le \lfloor \frac{t}{2} \rfloor - 1$,

$$\bar{c}(u_j) = \begin{cases} c(v_{j+1}), & i \equiv 0 \pmod 3; \\ c(v_{j+2}), & i \equiv 1 \pmod 3; \\ c_{\lceil \frac{t}{2} \rceil + \lceil \frac{t}{6} \rceil + \lceil \frac{j}{3} \rceil}, & j \equiv 2 \pmod 3, \end{cases}$$

where $t + 1 = 1$, and $\bar{c}(u_{\lfloor \frac{t}{2} \rfloor}) = c_4$, $\bar{c}(u_t) = c_t$.

Based on the domination properties of the vertices mentioned in Case 3, it follows that $\bar{c}$ is a dominator coloring of Y_t, when $t \equiv 4 \pmod 6$.

Case 6: For $t \equiv 5 \pmod 6$, let $c''' : V(Y_t) \to \{c_1, c_2, \ldots, c_t\}$ be a coloring such that for all $1 \le i \le t - 1$, $c'''(v_i) = \bar{c}(v_i)$, and $c'''(u_i) = \bar{c}(u_i)$, $c'''(u_t) = c_4$ and $c'''(v_t) = c_t$, where $\bar{c}$ is a dominator coloring of Y_t; $t \equiv 4 \pmod 6$, as defined in Case 5.

This is a dominator coloring of Y_t; $t \equiv 5 \pmod 6$, as all vertices except v_t and u_t, dominate the color classes, as explained in Case 5, and the vertices v_t and u_t dominate the color class $\{v_t\}$.

In all the cases, the coloring c of Y_t, there exists a rainbow path between the vertices v_i, v_j, as the vertices of the cycle $v_i - v_2 - \ldots - v_t - v_1$ are given the rainbow dominator coloring of C_t. Also, this guarantees the existence of a rainbow path between the vertices u_i, u_j, and u_i, v_j, for any $1 \le i \ne j \le t$, as the vertices of the $v_i - v_j$ path are internal vertices of the u_i, u_j, and u_i, v_j, path in Y_t. This completes the proof. □

Proposition 2.4. *For $t \ge 3$, $\chi_{rd}(\overline{Y}_t) = t$.*

Proof. In the complement $\overline{Y}_t$ of a prism graph Y_t described in Theorem 2.4, each v_i is adjacent to all the vertices of $\overline{Y}_t$, except u_i, v_{i+1}, v_{i-1}, for all $1 \le i \le t$, where $t + j = j$, for any $1 \le j \le t - 1$. Hence, any color can be assigned to at most two vertices, either v_i, v_{i+1}, or u_i, u_{i+1}, or v_i, u_i, for $1 \le i \le t$, in any proper coloring of $\overline{Y}_t$, where the suffixes are taken modulo t. Therefore, $\chi_{rd}(\overline{Y}_t) \ge t$.

The coloring c of $\overline{Y}_t$ such that $c(v_i) = c(u_i) = c_i$, for $1 \le i \le t$, is its rainbow dominator coloring, as there exists a path of length 2 between any pair of non-adjacent vertices, v_i, u_i or v_i, v_{i+1} or u_i, u_{i+1} of $\overline{Y}_t$ through the vertices u_{i+2}, u_{i+3} and v_{i+3}, respectively; proving the result. □

Let $C_t = v_1 - v_2 - \ldots - v_t - v_1$ and $C'_t = u_1 - u_2 - \ldots - u_t - u_1$ be two cycles. A *web graph* of order $3t$, denoted by Wb_t, is a graph obtained by making $u_i \in V(C'_t)$ and $v_i \in V(C_t)$ adjacent, and adjoining a vertex w_i to each v_i.

Proposition 2.5. *For $t \ge 3$, $\chi_{rd}(Wb_t) \begin{cases} t + 2, & \text{when } t \text{ is even}; \\ t + 3, & \text{when } t \text{ is odd}. \end{cases}$*

Proof. In a web graph Wb_t; $t \geq 3$, with $V(Wb_t) = \{u_i : 1 \leq i \leq t\} \cup \{v_i : 1 \leq i \leq t\} \cup \{w_i : 1 \leq i \leq t\}$, let w_i be the pendant vertex adjacent to v_i's, for all $1 \leq i \leq t$, that form a cycle of order t, and let u_i's be the vertices of a cycle of order t such that each u_i is adjacent to the corresponding v_i's. Here, as there are t support vertices v_i; $1 \leq i \leq t$, we require at least t colors in any dominator coloring of Wb_t. As these t colors cannot be assigned to any other vertex other than the support vertices of Wb_t, to color the u_i's and w_i's we need at least $\chi(C_t)$ colors; yielding $\chi_{rd}(Wb_t) \geq t + \chi(C_t)$.

The coloring $c : V(Wb_t) \to \{c_1, c_2, \ldots, c_{t+\chi(C_t)}\}$ such that $c(v_i) = c_i$; $1 \leq i \leq t$, $c(w_i) = c(u_i) = c_1$; $1 \leq i \leq t-1, i \equiv 1 \pmod 2$, $c(w_t) = c(u_t) = c_3$; $t \equiv 1 \pmod 2$, and $c(w_i) = c(u_i) = c_2$; $1 \leq i \leq t, i \equiv 0 \pmod 2$, is a rainbow dominator coloring of Wb_t with $t+\chi(C_t)$ colors, as the vertices u_i, v_i, w_i dominate the color class $\{v_i\}$, for all $1 \leq i \leq t$, in c, and between any w_i and w_j, or u_i and u_j, or w_i and u_j, or v_i and v_j, there exists a rainbow path $v_i - v_{i+1} - \ldots - v_{j-1} - v_j$, which are all given unique colors in c. Hence the result. □

Proposition 2.6. *For $t \geq 3$, $\chi_{rd}(\overline{Wb_t}) = t + \lceil \frac{t}{2} \rceil$.*

Proof. For $t \geq 3$, let Wb_t be a web graph with $V(Wb_t) = \{u_i : 1 \leq i \leq t\} \cup \{v_i : 1 \leq i \leq t\} \cup \{w_i : 1 \leq i \leq t\}$ as described in Proposition 2.5. Define a coloring $c : V(\overline{Wb_t}) \to \{c_1, c_2, \ldots, c_{t+\lceil \frac{i}{2} \rceil}\}$ such that $c(w_i) = c(v_i) = c_i$, for $1 \leq i \leq t$, and $c(u_i) = c(u_{i+1}) = c_{t+\lceil \frac{i}{2} \rceil}$, for $1 \leq i \leq t$, when $i \equiv 1 \pmod 2$.

This is a rainbow coloring of $\overline{Wb_t}$, as any two non-adjacent u_i's have a path of length 2 through some w_j, and there exists a $u_i - w_{i+1} - v_i$ path between any u_i and v_i. Also, between two non-adjacent v_i, v_j, there exists a path $v_i - w_{i^*} - v_j$, where $1 \leq i \neq j \neq i^* \leq t$. In addition to it, each u_i; $1 \leq i \neq j \leq t$, dominates the color class $\{v_j, w_j\}$, every w_i dominates the color class $\{u_j, u_{j+1}\}$, and every v_i dominates the color class $\{w_i, w_{j+2}\}$, with respect to c. Hence, c is a rainbow coloring of $\overline{Wb_t}$ with $t+\lceil \frac{t}{2} \rceil$ colors. As the vertices w_i; $1 \leq i \leq t$, and u_j; $j \equiv 0 \pmod 2$, for $1 \leq j \leq t$, form a clique of order $t + \lfloor \frac{t}{2} \rfloor$ in $\overline{Wb_t}$, it follows that $\chi_{rd}(\overline{Wb_t}) \geq t + \lfloor \frac{t}{2} \rfloor$.

If t is even, we are done. If t is odd, the vertex u_t in any dominator coloring of $\overline{Wb_t}$ must be assigned a unique color, as it can either be assigned the color assigned to the vertices u_1, u_{t-1}, or v_t. However, as the colors assigned to these vertices are assigned to one more vertex, to which u_t is adjacent to, such an assignment of colors is not possible. Hence, $\chi_{rd}(\overline{Wb_t}) = t + \lceil \frac{t}{2} \rceil$, when t is odd. □

Based on Proposition 2.5, and Proposition 2.6, the rainbow dominator chromatic number of a sunlet graph, and its complement is given in the following result, where a *sunlet graph* of order $2t$, denoted by Sl_t, is obtained by adjoining a vertex to every vertex of a cycle C_t.

Proposition 2.7. *For $t \geq 3$, $\chi_{rd}(Sl_t) = t+1$, and $\chi_{rd}(\overline{Sl_t}) = t$.*

Proof. As every vertex of a cycle C_t in a sunlet graph Sl_t is a support vertex, it has to have a unique color in any of its dominator coloring. Therefore, any path between two vertices of this cycle C_t in Sl_t is a rainbow path in this coloring. Also, as any path between two pendant vertices of Sl_t is also through the vertices of C_t in Sl_t, it can be observed that $\chi_{rd}(Sl_t) = t + 1$.

All pendant vertices of Sl_t induce a clique in $\overline{Sl_t}$, and every pendant vertex of Sl_t is not adjacent to only its support vertex of Sl_t in the graph $\overline{Sl_t}$. Hence, any color can be assigned only to at most two vertices of $\overline{Sl_t}$, in any of its proper coloring.

A proper coloring of $\overline{Sl_t}$ that assigns a color c_i for every pendant vertex and its corresponding support vertex in Sl_t, as given in the coloring c of $\overline{Wb_t}$ in Proposition 2.6 is an optimal rainbow dominator coloring of $\overline{Sl}_t$, owing to Proposition 2.6. Hence, $\chi_{rd}(Sl_t) = t+1$, and $\chi_{rd}(\overline{Sl}_t) = t$, for all $t \geq 3$. □

3. Conclusion

In this article, we investigated the rainbow dominator coloring of some cycle related graphs, and their complements, by obtaining their rainbow dominator coloring pattern and the corresponding chromatic number. As this is just a beginning of the study on this topic, it offers wide avenues for future explorations that includes obtaining bounds for the rainbow dominator chromatic number of graphs, determining the rainbow dominator coloring of several classes of graphs and its derived graphs, and addressing several realisation problems.

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