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Sharp estimates of second and third-order Hankel determinants for a certain class of analytic functions related to exponential function using Hohlov operator

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ABSTRACT. In this paper a new subclass $\mathcal{SI}(m,r,d)$ of analytic functions, where $m,r \in \mathbb{C},\ d \in \mathbb{C}\backslash\mathbb{Z}_0^- = \{0,-1,-2,...\}$ is introduced and sharp bounds are obtained for the second and third-order Hankel determinants for functions belonging to $\mathcal{SI}(m,r,d)$ related to exponential functions using Hohlov operator.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions

$$f(z) = z + a_2z^2 + a_3z^3 + ... \tag{1.1}$$

defined in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ represent the subclass of $\mathcal{A}$ consisting of univalent functions. Two widely investigated subclasses of $\mathcal{S}$ are convex univalent functions $\mathcal{C}$ and starlike univalent functions $\mathcal{S}^*$. For $z \in \mathbb{U}$, a necessary and sufficient condition for $f(z) \in \mathcal{A}$ to be the $\mathcal{S}^*$and $\mathcal{C}$ is that $\mathcal{R}\left(\frac{zf^{'}(z)}{f(z)}\right) > 0$ and $\mathcal{R}\left(1 + \frac{zf^{''}(z)}{f'(z)}\right) > 0$ respectively [16] .
Suppose $\mathcal{M}$ denotes the class of analytic function $\Lambda(z) = 1 + c_1z + c_2z^2 + ...$ and satisfying the conditions $\mathcal{R}\Lambda(z) > 0, z \in \mathbb{U}$. Then, we have for $\Lambda \in \mathcal{M}$, there exists a Schwarz function $w(z)$ analytic in $\mathbb{U}$ that satisfies the condition $w(0) = 0$ and $|w(z)| < 1$ such that $\Lambda(z) = \frac{1+w(z)}{1-w(z)}$ [17].

Suppose $h_1, h_2 \in \mathcal{A}$, then h_1 is subordinate to h_2 if $h_2(z) = h_1(w(z))$, z $\in \mathbb{U}$ and $w(z)$ is the Schwarz function [12] .

The q-th Hankel determinant [13] $H_{q,s}(f), (s, q \in \mathbb{N} = \{1, 2, ...\})$ for the function $f \in \mathcal{S}$ of the form (1.1) is defined as

$$H_{q,s}(f) = \begin{vmatrix} a_s & a_{s+1} & \cdots & a_{s+q-1} \\ a_{s+1} & a_{s+2} & \cdots & a_{s+q} \\ \vdots & \vdots & \vdots & \ddots \\ a_{s+q-1} & a_{s+q} & \cdots & a_{s+2q-2} \end{vmatrix}.$$

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The Hankel determinants of the first order, second order and third order of f can be given by

$$H_{2,1}(f) = a_3 - a_2^2,$$
$$H_{2,2}(f) = a_2a_4 - a_3^2,$$
$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

The Hankel determinants of second and third order in geometric function theory have been extensively studied by many researchers [1, 2, 4, 13, 19, 20].

The function $\varphi_0(z) = e^z$ has positive real part in $\mathbb{U}$, $\varphi_0(\mathbb{U}) = \{w \in \mathbb{C} : |logw| < 1\}$ is symmetric with respect to real axis and starlike with respect to 1 and $\varphi_0^{'}(0) > 0$. Mendiratta et al. [11] introduced the family $\mathcal{S}_e^*$ of analytic functions as

$$\mathcal{S}_e^* = \{f \in \mathcal{A} : \frac{zf^{'}(z)}{f(z)} \prec e^z, z \in \mathbb{U}\}$$

or equivalently

$$\mathcal{S}_e^* = \{f \in \mathcal{A} : |log\frac{zf^{'}(z)}{f(z)}| < 1, z \in \mathbb{U}\}.$$

Hohlov [6, 7] introduced the familiar convolution operator $\mathcal{I}_d^{m,r}f(z)$, where $m, r \in \mathbb{C}$, $d \in \mathbb{C}\backslash\mathbb{Z}_0^- = \{0, -1, -2, ...\}$ as follows:

$$\mathcal{I}_d^{m,r}f(z) = z + \sum_{n=2}^{\infty} \phi_n a_n z^n, z \in \mathbb{U}$$

where,

$$\phi_n = \frac{(m)_{n-1}(r)_{n-1}}{(d)_{n-1}(1)_{n-1}}$$

and $(\Upsilon)_n$ is the Pochhammer symbol or the shifted factorial defined as

$$(\Upsilon)_n = \begin{cases} 1 & \text{, for } (n = 0) \\ \Upsilon(\Upsilon+1)(\Upsilon+2)...(\Upsilon+n-1) & \text{, for } (n = 1, 2, 3, ...). \end{cases}$$

2. Preliminary results

The next Lemmas (2.1) to (2.3) are required for proving our results.

Lemma 2.1. [3] *If $\Lambda \in \mathcal{M}$ and has the form $\Lambda(z) = 1 + c_1z + c_2z^2 + ...$, then $|c_n| \leq 2, \quad n = 1, 2, 3...$ and the inequality is sharp.*

Lemma 2.2. [8, 14] *If $\Lambda \in \mathcal{M}$ and has the form $\Lambda(z) = 1 + c_1z + c_2z^2 + ...$, then*

$$|c_{i+j} - c_ic_j| < 2 \text{ for } 0 \leq \eta \leq 1,$$
$$|c_lc_i - c_jc_p| \leq 4 \text{ for } l + i = j + p; \quad l, i \in \mathbb{N},$$
$$|c_{i+2j} - \eta c_ic_j^2| \leq 2(1 + 2\eta), (\eta \in \mathbb{R})$$
$$|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2},$$

and for the complex number μ, we have

$$|c_2 - \mu c_1^2| \leq 2max\{1, |2\mu - 1|\}.$$

Lemma 2.3. [9, 10, 14] *If the function $\Lambda \in \mathcal{M}$ is given by $\Lambda(z) = 1 + c_1 z + c_2 z^2 + ...$, then there exists some κ, x, ν with $|\kappa| \leq 1$, $|x| \leq 1$, $|\nu| \leq 1$ and such that*

$$2c_2 =(4 - c_1^2)x + c_1^2,$$
$$4c_3 =2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + \nu(1 - |x|^2)2(4 - c_1^2) + c_1^3,$$
$$8c_4 =x[4x + (x^2 - 3x + 3)c_1^2](4 - c_1^2) - 4(1 - |x|^2)[c(x - 1)\nu + \bar{x}\nu^2 - (1 - |\nu|^2)\kappa](4 - c_1^2) + c_1^4.$$

3. Main results

Motivated by the results of Breaz et al. [2] , Juma et al. [5] , Mendiratta et al. [11] and Sudharsan et al. [18], we define a new subclass $\mathcal{SI}(m, r, d)$ of $\mathcal{S}$.

Definition 3.1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{SI}(m, r, d)$, if it satisfies the following condition*

$$\frac{z(\mathcal{I}_d^{m,r} f(z))'}{\mathcal{I}_d^{m,r} f(z)} \prec e^z,$$

where

$$\mathcal{I}_d^{m,r} f(z) = z + \sum_{n=2}^{\infty} \frac{(m)_{n-1}(r)_{n-1}}{(d)_{n-1}(1)_{n-1}} a_n z^n,\ z \in \mathbb{U},\ m, r \in \mathbb{C},\ d \in \mathbb{C}\backslash\mathbb{Z}_0^- = \{0, -1, -2, ...\}.$$

Remark 2.1: *If $m = r = d = 1$, the family $\mathcal{SI}(1, 1, 1) = \mathcal{S}^*(e^z)$ was introduced by Mendiratta et al. in* [11]. *The class defined on the other hand in the atricles* [2, 5, 20] *consider domains that are different from the one considered in definition 3.1. Further the new subclass $\mathcal{SI}(m, r, d)$ defined in this paper is associated with Hohlov operator.*

Remark 2.2: *(i) If we define the function $\hat{k} : \mathbb{U} \to \mathbb{C}$ by $\hat{k}(z) = z + \beta z^2$, $\beta \in \mathbb{C}$, a simple computation yields*

$$\mathcal{I}_d^{m,r} \hat{k}(z) = z + \beta \frac{(m)_1(r)_1}{(d)_1(1)_1} z^2,$$

for $m = r = d = 1$ and

FIGURE 1. The image of $\hat{\psi}(z)$ (Blue color) is contained in e^z (Red color)

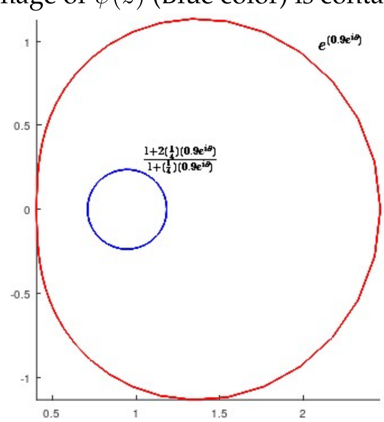


FIGURE 1: Figure for Remark 2.2 (i)

$$\hat{\psi}(z) = \frac{z(\mathcal{I}_d^{m,r}\hat{k}(z))'}{\mathcal{I}_d^{m,r}\hat{k}(z)} = \frac{1+2Bz}{1+Bz},$$

where $\mathcal{I}_d^{m,r}\hat{k}(z) = 1 + Bz$ *and* $B = \beta(\frac{mr}{d})$. *Considering the transformation* $\hat{\psi}(z) = \frac{1+2Bz}{1+Bz}$, $z \in \mathbb{U}$ *and assuming that* $0 \leq B \leq \frac{1}{2}$, *we obtain that* ψ *maps the unit disc* $\mathbb{U}$ *on to the open disc that is symmetric with respect to the real axis.*

If $\beta = \frac{d}{4mr}$, *then* $B = \frac{1}{4}$. *These show that for* $z = 0.9e^{i\theta}$, $\hat{\psi}(\mathbb{U}) \subset e^z$, *which is* $\hat{\psi}(z) \prec e^z$ *that is* $\hat{k} \in \mathcal{SI}(m,r,d)$ *for* $z = 0.9e^{i\theta}$ *and for* $m = 1, r = 1$ *and* $d = 1$, *whenever* $\beta = \frac{d}{4mr}$. *It follows from Figure 1 that there exist values of the parameters* $m, r \in \mathbb{C},\ d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, *such that* $\mathcal{SI}(m,r,d) \neq \emptyset$.

(ii) If we define the function $\hat{k_1} : \mathbb{U} \to \mathbb{C}$ *by* $\hat{k_1}(z) = z + \beta_1 z^2$, $\beta_1 \in \mathbb{C}$, *a simple computation yields*

$$\mathcal{I}_d^{m,r}\hat{k_1}(z) = z + \beta_1 \frac{(m)_1(r)_1}{(d)_1(1)_1} z^2,$$

for $m = -i, r = i, d = 1$ *and*

$$\hat{\psi_1}(z) = \frac{z(\mathcal{I}_d^{m,r}\hat{k_1}(z))'}{\mathcal{I}_d^{m,r}\hat{k_1}(z)} = \frac{1+2B_1z}{1+B_1z},$$

where $\mathcal{I}_d^{m,r}\hat{k_1}(z) = 1 + B_1z$ *and* $B_1 = \beta_1(\frac{mr}{d})$. *Considering the transformation* $\hat{\psi_1}(z) = \frac{1+2B_1z}{1+B_1z}$, $z \in \mathbb{U}$ *and assuming that* $0 \leq B_1 \leq \frac{1}{2}$, *we obtain that* $\hat{\psi_1}$ *maps the unit disc* $\mathbb{U}$ *on to the open disc that is symmetric with respect to the real axis.*

FIGURE 2. The image of $\hat{\psi_1}(z)$ (Blue color) is contained in e^z (Red color)

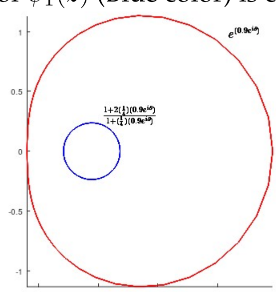


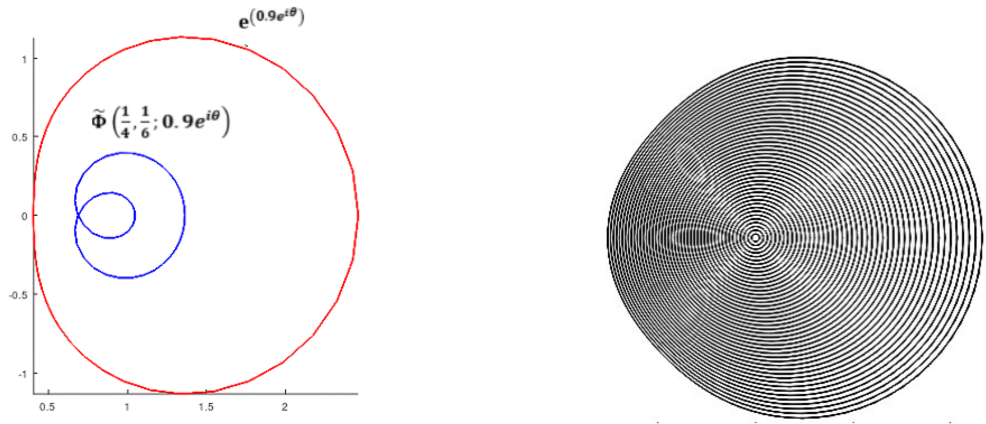
FIGURE 2: Figure for Remark 2.2 (ii)

If $\beta_1 = \frac{d}{9mr}$, *then* $B_1 = \frac{1}{9}$. *These show that for* $z = 0.9e^{i\theta}$, $\hat{\psi_1}(\mathbb{U}) \subset e^z$, *which is* $\hat{\psi_1}(z) \prec e^z$ *that is* $\hat{k_1} \in \mathcal{SI}(m,r,d)$ *for* $z = 0.9e^{i\theta}$ *and for* $m = -i, r = i$ *and* $d = 1$, *whenever* $\beta_1 = \frac{d}{9mr}$. *It follows from Figure 2, that for* $m = -i, r = i$ *and* $d = 1$, *other than* $m, r, d = 1$ *there exist values of the parameters* $m, r \in \mathbb{C},\ d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, *such that* $\mathcal{SI}(m,r,d) \neq \emptyset$.
The following two examples apart from examples (i) and (ii) above illustrate how the class of mappings defined in this paper is different from the other well known classes.

(iii) If we define the function $\tilde{f} : \mathbb{U} \to \mathbb{C}$ *by* $\tilde{f}(z) = z + \beta z^2 + \alpha z^3$, $\beta, \alpha \in \mathbb{C}$, *a simple computation yields*

$$\tilde{\Phi}(z) = \frac{z(\mathcal{I}_d^{m,r}\tilde{f}(z))'}{\mathcal{I}_d^{m,r}\tilde{f}(z)} = \frac{1 + 2Bz + 3Cz^2}{1 + Bz + Cz^2}, \; z \in \mathbb{U}$$

If $B = \frac{1}{4}, C = \frac{1}{6}, \beta = \frac{1}{4}, \alpha = \frac{1}{12}$, *we see in the Figure 3(A) that* $\tilde{\Phi}(z)(\mathbb{U}) \subset \; e^z$ *and from the univalence of* e^z *we have* $\tilde{\Phi}(z) \prec \; e^z$, *that is* $\tilde{\Phi} \in \mathcal{SI}(m, r, d)$ *for the choice of the parameters. Moreover, from this figure we see that* $\tilde{\Phi}$ *is not univalent in* $\mathbb{U}$ *while the Figure 3(B) shows that* $\tilde{f}$ *is univalent in* $\mathbb{U}$.



(A) The image of $\tilde{\Phi}(z)$ (Blue color) is contained in $e^{(0.9e^{i\theta})}$ (Red color)

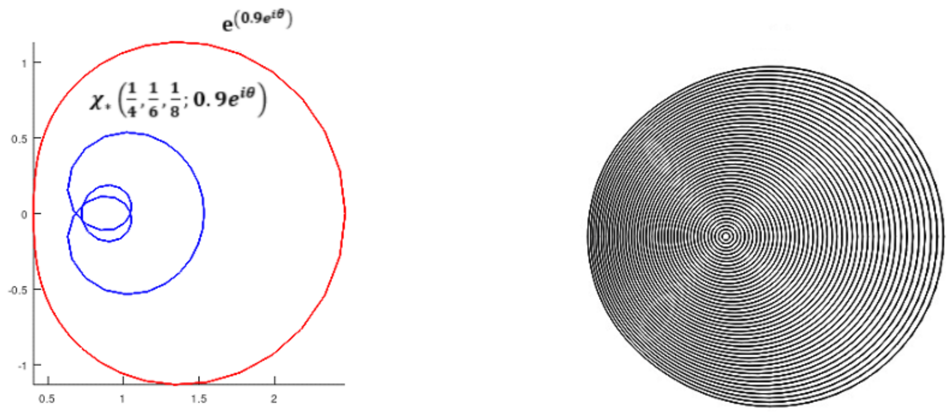
(B) The Domain $\tilde{f}(\mathbb{U})$

FIGURE 3: Figures for Remark 2.2 (iii)

(iv) Consider $g_*(z) : \mathbb{U} \to \mathbb{C}$ *with* $g_*(z) = z + \beta z^2 + \alpha z^3 + \gamma z^4$, $\beta, \alpha, \gamma \in \mathbb{C}$. *A simple computation gives*

$$\chi_*(z) = \frac{z(\mathcal{I}_d^{m,r}g_*(z))'}{\mathcal{I}_d^{m,r}g_*(z)} = \frac{1 + 2Bz + 3Cz^2 + 4D^3}{1 + Bz + Cz^2 + Dz^3}, \; z \in \mathbb{U}$$

If $B = \frac{1}{4}, C = \frac{1}{6}, D = \frac{1}{8}, \beta = \frac{1}{4}, \alpha = \frac{1}{12}$ *and* $\gamma = \frac{1}{48}$, *we see in the Figure 4(A) that* $\chi_*(\mathbb{U}) \subset \; e^z$ *and from the univalence of* e^z *we have* $\chi_*(z) \prec \; e^z$, *that is* $\chi_* \in \mathcal{SI}(m, r, d)$ *for the choice of the parameters. Moreover, from this figure we see that* χ_* *is not univalent in* $\mathbb{U}$ *while the Figure 4(B) shows that* g_* *is univalent in* $\mathbb{U}$. *It can be noted that Figure 1, Figure 2, Figure 3 and Figure 4 are made with GNU Octave computer software.*



(A) The image of $\chi_*(z)$ (Blue color) is contained in $e^{(0.9e^{i\theta})}$ (Red color)

(B) The Domain $g_*(\mathbb{U})$

FIGURE 4: Figures for Remark 2.2 (iv)

(v) To conclude, the examples given in the Remark 2.2(i)-(iv) show that $\mathcal{SI}(m,r,d) \neq \emptyset$. From the example of the Remark 2.2(i) and (ii) it follows that $\mathcal{SI}(m,r,d) \not\subset \mathcal{S}$. In addition, the examples of Remark 2.2 (iii) and (iv) show that the corresponding functions of the form $\tilde{f}$and $g*$ belong to $\mathcal{SI}(m,r,d) \cap \mathcal{S}$, i.e. $\mathcal{SI}(m,r,d) \cap \mathcal{S} \neq \emptyset$. Also the above given examples and comments provide the much needed motivation to show that they are significant for the proposed class $\mathcal{SI}(m,r,d)$ as well as their properties presented in section 3.

Remark 2.3: *We observe that a function $f \in \mathcal{A}$ is in the function class $\mathcal{SI}(m,r,d)$ if it satisfies the condition*

$$\frac{z(\mathcal{I}_d^{m,r}f(z))'}{\mathcal{I}_d^{m,r}f(z)} \prec e^z, \ z \in \mathbb{U}$$

and this implies that $f \in \mathcal{SI}(m,r,d)$ if it satisfies the inequality

$$\left| log \frac{z(\mathcal{I}_d^{m,r}f(z))'}{\mathcal{I}_d^{m,r}f(z)} \right| < 1, \ z \in \mathbb{U}.$$

we now provide an analytic example of a function in $\mathcal{A}$ for the function class $\mathcal{SI}(m,r,d)$. Consider the function

$$P(z) = z + \frac{z^2}{4}, \ z \in \mathbb{U}$$

Then it is easy to see that

$$W = \frac{z(\mathcal{I}_d^{m,r}f(z))'}{\mathcal{I}_d^{m,r}f(z)} = \frac{1+\frac{\phi_2}{2}z}{1+\frac{\phi_2}{4}z}$$

where,

$$\phi_2 = \frac{(m)_1(r)_1}{(d)_1}.$$

Now choosing $m=-i, r=i, d=1$, we have

$$\phi_2 = \frac{(-i)_1(i)_1}{(1)_1} = 1.$$

This gives

$$W = \frac{1+\frac{z}{2}}{1+\frac{z}{4}},\ \phi_2 = 1.$$

For $|z| = r < 1$, *a simple calculation shows that*

$$\begin{aligned}|logW| &= \left| log\frac{1+\frac{z}{2}}{1+\frac{z}{4}} \right| \\ &< 1\end{aligned}$$

Indeed for $|z| = r = 0.9 < 1$, *a computation yields*

$$\begin{aligned}logW &= log\Big(\frac{1+\frac{z}{2}}{1+\frac{z}{4}}\Big) \\ &= log\Big(1+\frac{z}{2}\Big) - log\Big(1+\frac{z}{4}\Big) \\ &= \Big(\frac{z}{2} - \frac{1}{2}\Big(\frac{z}{2}\Big)^2 + \frac{1}{3}\Big(\frac{z}{2}\Big)^3 - \frac{1}{4}\Big(\frac{z}{2}\Big)^4 + ...\Big) \\ &\quad - \Big(\frac{z}{4} - \frac{1}{2}\Big(\frac{z}{4}\Big)^2 + \frac{1}{3}\Big(\frac{z}{4}\Big)^3 - \frac{1}{4}\Big(\frac{z}{4}\Big)^4 + ...\Big) \\ &= \Big(\frac{1}{2} - \frac{1}{4}\Big)z + \frac{1}{2}\Big(\frac{-1}{4} + \frac{1}{16}\Big)z^2 + \frac{1}{3}\Big(\frac{1}{8} - \frac{1}{64}\Big)z^3 + ...\end{aligned}$$

Therefore,

$$\begin{aligned}|logW| &\leq \frac{1}{4}|z| + \frac{3}{32}|z|^2 + \frac{7}{192}|z|^3 + ... \\ &\leq \frac{1}{4}(0.9) + \frac{3}{32}(0.9)^2 + \frac{7}{192}(0.9)^3 + ... \\ &\leq 0.225 + 0.0759375 + 0.026578125 + ... \\ &= 0.327515625 \\ &< 1\end{aligned}$$

which shows that

$$|logW| < 1.$$

Hence

$$P(z) = z + \frac{z^2}{4},\ z \in \mathbb{U}$$

belongs to the function class $\mathcal{SI}(m, r, d)$.

Theorem 3.1. *Suppose* $f \in \mathcal{SI}(m, r, d)$ *has the series expansion (1.1), we have*

$$|a_3 - a_2^2| \leq \frac{1}{2\phi_3}. \tag{3.2}$$

The inequality is sharp.

Proof. Suppose $f \in \mathcal{SI}(m,r,d)$, then from the definition of subordination we have,

$$\frac{z(\mathcal{I}_d^{m,r} f(z))'}{\mathcal{I}_d^{m,r} f(z)} = e^{w(z)},$$

where $w(z)$ is the Schwarz function satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$. Now

$$\frac{z(\mathcal{I}_d^{m,r} f(z))'}{\mathcal{I}_d^{m,r} f(z)} = 1 + z(\phi_2 a_2) + z^2(2\phi_3 a_3 - \phi_2^2 a_2^2) + z^3(3\phi_4 a_4 + \phi_2^3 a_2^3 - 3\phi_2\phi_3 a_2 a_3)$$
$$+z^4(4\phi_5 a_5 - 2\phi_3^2 a_3^2 - \phi_2^4 a_2^4 + 4\phi_2^2\phi_3 a_2^2 a_3 - 4\phi_2\phi_4 a_2 a_4) + ... \quad (3.3)$$

Consider

$$\Lambda(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$$

Since $\Lambda(z) \in \mathcal{M}$ and $w(z) = \frac{\Lambda(z) - 1}{\Lambda(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + ...}{2 + c_1 z + c_2 z^2 + c_3 z^3 + ...}$, a simple computation yields

$$e^{w(z)} = 1 + \frac{1}{2}c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{8}\right) z^2 + \left(\frac{c_1^3}{48} - \frac{c_1 c_2}{4} + \frac{c_3}{2}\right) z^3$$
$$+ \left(\frac{c_1^4}{384} + \frac{c_1^2 c_2}{16} - \frac{c_3 c_1}{4} - \frac{c_2^2}{8} + \frac{c_4}{2}\right) z^4 + ... \quad (3.4)$$

From (3.3) and (3.4), we have

$$a_2 = \frac{c_1}{2\phi_2}, \quad (3.5)$$

$$a_3 = \frac{c_2}{4\phi_3} + \frac{c_1^2}{16\phi_3}, \quad (3.6)$$

$$a_4 = \frac{c_3}{6\phi_4} + \frac{c_1 c_2}{24\phi_4} - \frac{c_1^3}{288\phi_4}, \quad (3.7)$$

$$a_5 = \frac{c_1^4}{1152\phi_5} - \frac{c_1^2 c_2}{96\phi_5} + \frac{c_1 c_3}{48\phi_5} + \frac{c_4}{8\phi_5}. \quad (3.8)$$

But

$$|a_3 - a_2^2| = \left|\frac{c_2}{4\phi_3} + \frac{c_1^2}{16\phi_3} - \left(\frac{c_1}{2\phi_2}\right)^2\right| = \left|\frac{2\phi_2^2(2c_2) + c_1^2(\phi_2^2 - 4\phi_3)}{16\phi_2^2\phi_3}\right|.$$

From lemma (2.3), we have

$$|a_3 - a_2^2| = \left|\frac{8x\phi_2^2 + c_1^2(\phi_2^2 - 4\phi_3)}{16\phi_2^2\phi_3}\right|$$

For $|x| = t \in [0,1], c_1 = c \in [0,2]$ we have

$$|a_3 - a_2^2| \leq \frac{8t\phi_2^2 + c^2(\phi_2^2 - 4\phi_3)}{16\phi_2^2\phi_3}.$$

If

$$G(c,t) = \frac{8t\phi_2^2 + c^2(\phi_2^2 - 4\phi_3)}{16\phi_2^2\phi_3}$$

we obtain,

$$\frac{\partial G}{\partial t} = \frac{8\phi_2^2}{16\phi_2^2\phi_3} \geq 0.$$

We observe from the above that the function $G(c,t)$ increases in $[0,1]$ about t and

$$G(c,1) = \frac{8\phi_2^2 + c^2(\phi_2^2 - 4\phi_3)}{16\phi_2^2\phi_3}$$

is the maximum value. Let

$$F(c) = \frac{8\phi_2^2 + c^2(\phi_2^2 - 4\phi_3)}{16\phi_2^2\phi_3} = \frac{1}{2\phi_3} + \frac{c^2(\phi_2^2 - 4\phi_3)}{2(8\phi_2^2\phi_3)}$$

Since $F'(c) = 0$, we see that $c = 0$ is a root and $F(0)$ is the maximum value given by,

$$|a_3 - a_2^2| \leq F(0) = \frac{1}{2\phi_3}.$$

For $f(z) = z + \frac{1}{2\phi_2}z^2 + (\frac{1}{2\phi_3} + \frac{1}{4\phi_2^2})z^3 + ...$, equality is attained. □

Theorem 3.2. *Suppose $f \in \mathcal{SI}(m,r,d)$ has the series expansion (1.1), we have*

$$|a_2a_3 - a_4| \leq \frac{1}{3\phi_4}. \tag{3.9}$$

The inequality is sharp.

Proof. Using (3.5),(3.6) and (3.7), we have

$$|a_2a_3 - a_4| = \left| \frac{c_1}{2\phi_2}\left(\frac{c_2}{4\phi_3} + \frac{c_1^2}{16\phi_3}\right) - \left(\frac{c_3}{6\phi_4} + \frac{c_1c_2}{24\phi_4} - \frac{c_1^3}{288\phi_4}\right)\right|$$

by applying the lemma (2.3),

$$|a_2a_3 - a_4| = \left| c_1^3\left(\frac{3}{32\phi_2\phi_3} - \frac{15}{288\phi_4}\right) + \frac{c_1x^2(4-c_1^2)}{24\phi_4} - \frac{(4-c_1^2)(1-|x|^2)\nu}{12\phi_4} + c_1x(4-c_1^2)\left(\frac{1}{16\phi_2\phi_3} - \frac{5}{48\phi_4}\right)\right|$$

For $|x| = t \in [0,1], c_1 = c \in [0,2]$ we have,

$$|a_2a_3 - a_4| \leq \frac{3c^3 + 2ct(4-c^2)}{32\phi_2\phi_3} + \frac{12ct^2(4-c^2) + 24(4-c^2) + 24t^2(4-c^2) + 30ct(4-c^2) + 15c^3}{288\phi_4}$$

If

$$G(c,t) := \frac{3c^3 + 2ct(4-c^2)}{32\phi_2\phi_3} + \frac{12ct^2(4-c^2) + 24(4-c^2) + 24t^2(4-c^2) + 30ct(4-c^2) + 15c^3}{288\phi_4}$$

Hence, we have

$$\frac{\partial G}{\partial t} = \frac{2c(4-c^2)}{32\phi_2\phi_3} + \frac{24ct(4-c^2) + 48t(4-c^2) + 30c(4-c^2)}{288\phi_4} \geq 0.$$

We observe from the above that the function $G(c,t)$ increases in $[0,1]$ about t and

$$G(c,1) = \frac{3c^3}{32\phi_2\phi_3} + \frac{12c^4+96-24c^3-15c^3}{288\phi_4}$$

is the maximum value. Let

$$F(c) = \frac{3c^3}{32\phi_2\phi_3} + \frac{12c^4+96-24c^3-15c^3}{288\phi_4}.$$

Since $F'(c)=0$, we see that $c=0$ is a root and $F(0)$ is the maximum value given by,

$$|a_2a_3 - a_4| \le F(0) = \frac{1}{3\phi_4}.$$

For $f(z) = z - \frac{1}{32\phi_2}z^2 + \frac{3}{\phi_3}z^3 + \left(\frac{1}{3\phi_4} - \frac{3}{32\phi_2\phi_3}\right)z^4 + ...$, equality is attained. □

Theorem 3.3. *Suppose $f \in \mathcal{SI}(m,r,d)$ has the series expansion (1.1), we have,*

$$|a_2a_4 - a_3^2| \le \frac{1}{2\phi_3^2}. \tag{3.10}$$

The inequality is sharp.

Proof. From equations (3.5),(3.6) and (3.7), we have

$$\begin{aligned}|a_2a_4 - a_3^2| =& \left|\frac{c_1}{2\phi_2}\left(\frac{c_3}{6\phi_4} + \frac{c_1c_2}{24\phi_4} - \frac{c_1^3}{288\phi_4}\right) - \left(\frac{c_2}{4\phi_3} + \frac{c_1^2}{16\phi_3}\right)^2\right| \\ =& \left|\frac{c_1c_3}{12\phi_2\phi_4} + \frac{c_1^2c_2}{48\phi_2\phi_4} - \frac{c_1^4}{2(288)\phi_2\phi_4} - \left(\frac{c_2^2}{16\phi_3^2} + \frac{2c_2c^2}{4(16)\phi_3^2}\right)\right|\end{aligned}$$

By lemma (2.3), we obtain

$$\begin{aligned}|a_2a_4 - a_3^2| =& \left|c_1^4\left(\frac{17}{576\phi_2\phi_4} - \frac{5}{256\phi_3^2}\right) + c_1^2x(4-c_1^2)\left(\frac{5}{96\phi_2\phi_4} - \frac{3}{64\phi_3^2}\right)\right. \\ & \left. - \frac{c_1^2x^2(4-c_1^2)}{48\phi_2\phi_4} - \frac{x^2(4-c_1^2)^2}{32\phi_3^2} + \frac{(1-|x|^2)\nu c_1(4-c_1^2)}{24\phi_2\phi_4}\right|\end{aligned}$$

For $|x| = t \in [0,1], c_1 = c \in [0,2]$ we have,

$$\begin{aligned}|a_2a_4 - a_3^2| \le& c^4\left(\frac{17}{576\phi_2\phi_4} - \frac{5}{256\phi_3^2}\right) + c^2t(4-c^2)\left(\frac{5}{96\phi_2\phi_4} - \frac{3}{64\phi_3^2}\right) + \\ & \frac{c^2t^2(4-c^2)}{48\phi_2\phi_4} + \frac{t^2(4-c^2)^2}{32\phi_3^2} + \frac{c(4-c^2)(1-t^2)}{24\phi_2\phi_4}\end{aligned}$$

Let

$$\begin{aligned}G(c,t) :=& c^4\left(\frac{17}{576\phi_2\phi_4} - \frac{5}{256\phi_3^2}\right) + c^2t(4-c^2)\left(\frac{5}{96\phi_2\phi_4} - \frac{3}{64\phi_3^2}\right) \\ & + \frac{t^2(4-c^2)c(c-2)}{48\phi_2\phi_4} + \frac{t^2(4-c^2)^2}{32\phi_3^2} + \frac{c(4-c^2)}{24\phi_2\phi_4}\end{aligned}$$

$$\frac{\partial G}{\partial t} = c^2(4-c^2)\left(\frac{5}{96\phi_2\phi_4} - \frac{3}{64\phi_3^2}\right) + \frac{2t(4-c^2)c(c-2)}{48\phi_2\phi_4} + \frac{2t(4-c^2)^2}{32\phi_3^2} \ge 0.$$

We observe from the above that the function $G(c,t)$ increases in $[0,1]$ about t and

$$G(c,1) = c^4\left(\frac{17}{576\phi_2\phi_4}-\frac{5}{256\phi_3^2}\right)+c^2(4-c^2)\left(\frac{5}{96\phi_2\phi_4}-\frac{3}{8(8\phi_3^2)}\right)+\frac{(4-c^2)(c^2-2c)}{6(8\phi_2\phi_4)}$$
$$+\frac{(4-c^2)(4-c^2)}{8(4\phi_3^2)}+\frac{c(4-c^2)}{24\phi_2\phi_4} := F(c)$$

is the maximum value. Since $F'(c) = 0$, we see that $c = 0$ is a root and $F(0)$ is the maximum value given by,

$$|a_2a_4 - a_3^2| \leq F(0) = \frac{1}{2\phi_3^2}.$$

For $f(z) = z + \frac{4}{\phi_3}z^2 + (\frac{2}{\phi_3\phi_4})z^3 + (\frac{1}{8\phi_3} + \frac{1}{\phi_3\phi_4^2})z^4 + ...$, equality is attained. □

Theorem 3.4. *Suppose $f \in \mathcal{SI}(m,r,d)$ has the series expansion (1.1), we have*

$$|H_3(1)| \leq \frac{648\phi_4^2\phi_5 + 272\phi_3^3\phi_5 + 315\phi_3^2\phi_4^2}{1728\phi_5\phi_4^2\phi_3^3} \tag{3.11}$$

The inequality is sharp.

Proof. Using lemma (2.1) and (2.2) in equations (3.5) and (3.6), we obtain

$$|a_2| \leq \frac{1}{\phi_2} \tag{3.12}$$

and

$$|a_3| \leq \frac{3}{4\phi_3} \tag{3.13}$$

Rearranging (3.7)

$$|a_4| = \frac{1}{6}\left|\frac{5c_1c_2}{24\phi_4} + \frac{c_1}{24\phi_4}(c_2 - \frac{c_1^2}{2}) + \frac{c_3}{\phi_4}\right|$$

By triangle inequality and lemma (2.1) and (2.2)

$$|a_4| \leq \frac{1}{6}\left(\frac{5|c_1|}{12\phi_4} + \frac{|c_1|}{24\phi_4}(2 - \frac{|c_1|^2}{2}) + \frac{2}{\phi_4}\right)$$

For $c_1 = c$ belongs to $[0,2]$, we have,

$$|a_4| \leq \frac{1}{6}\left(\frac{5c}{12\phi_4} + \frac{c}{24\phi_4}(2 - \frac{c^2}{2}) + \frac{2}{\phi_4}\right)$$

The function has max value at $c = 2$, thus

$$|a_4| \leq \frac{17}{36\phi_4}. \tag{3.14}$$

Rearranging equation (3.8) we have

$$|a_5| = \frac{1}{4}\left|\frac{c_1^4}{288\phi_5} + \frac{c_1c_3}{12\phi_5} + \frac{c_4}{2\phi_5} - \frac{c_1^2c_2}{24\phi_5}\right|$$
$$|a_5| = \frac{1}{4}\left|\frac{1}{2\phi_5}(c_4 - \frac{c_1^2c_2}{48}) - \frac{c_1^2}{96\phi_5}(c_2 - \frac{c_1^2}{c_3}) + \frac{c_1}{12\phi_5}(c_3 - \frac{c_1c_2}{4})\right|$$

Using triangle inequality and lemma (2.1) and (2.2), we have

$$|a_5| \leq \frac{35}{96\phi_5}. \tag{3.15}$$

Using equations (3.13),(3.14),(3.15),we have

$$|H_3(1)| \leq \frac{648\phi_4^2\phi_5 + 272\phi_3^3\phi_5 + 315\phi_3^2\phi_4^2}{1728\phi_5\phi_4^2\phi_3^3}.$$

The equality is attained for the function

$$f(z) = z + z^2 + \left(\frac{7}{64\phi_3}\left(\frac{5}{3\phi_5} + \frac{24}{7\phi_3^2}\right) - \left(\frac{17}{6\phi_4}\right)^2\left(\frac{-1}{51}\right)\right)z^5 + ...$$

□

Remark 2.4: It is interesting to observe that for functions in the new class $\mathcal{SI}(m, r, d)$ the proposed results in theorems $3.1 - 3.4$, we have obtained sharp bounds for the second and third order Hankel determinants related to exponential function using Hohlov operator. In comprison for the particular case $m = r = d = 1$ even though the class $\mathcal{SI}(1, 1, 1) = \mathcal{S}^*(e^z)$, the class introduced by Mendiratta et al. in [11], the results obtained are not associated with Hankel determinants.
Indeed Mendiratta et al. [11] have established properties such as inclusion relations, coefficient estimates, subordination results, convolution properties, radius problems besides other results.

Further more, based on the class $\mathcal{SI}(m, r, d)$, we make the following key inference for specific values $\phi_n(n = 2, 3...\infty)$.
For $\phi_3 \geq 1$ in theorem 3.1 the upper bound obtained for $|a_3 - a_2^2|$ is best possible when compared to Zhang et al. [22].
For $\phi_4 \geq 1, \phi_3 \geq \frac{3}{2}$ in theorem 3.2 an improvement of the best possible upper bound for $|a_2a_3 - a_4|$ and $|a_2a_4 - a_3^2|$ are obtained when compared with the results obtained by Zaprawa [21]. In comparison with the results obtained by Raza et al. [15] in theorem 3.4 when $\phi_3, \phi_4, \phi_5 \geq 3$, the upper bound obtained for $|H_3(1)|$ are still best possible. Also in the proof of theorem 3.4, we have $|a_2| \leq \frac{1}{\phi_2}, |a_3| \leq \frac{3}{4\phi_3}, |a_4| \leq \frac{17}{36\phi_4}, |a_5| \leq \frac{35}{96\phi_5}$ and noted that the first three of these are best improvement of the result in theorem 2.3 studied by Mendiratta et al. [11] for particular values of ϕ_2, ϕ_3, and ϕ_4. Further the upper bound for $|a_5|$ for our new class is the best possible when compared with the corollary 1 of Zaprawa. [21]. The above observations are significant implications of the class $\mathcal{SI}(m, r, d)$.

4. Conclusions

Hankel determinants play a very significant role for finding the coefficient estimates in the theory of univalent functions. Hankel determinants are one of the vital tools in various fields of mathematics and in the areas of engineering, biology, cryptography, signal processing, and neural networks. In this paper we have made a connection to a subclass of analytic functions using Hohlov operator. In our current investigation we have obtained sharp estimates of second and third-order Hankel determinants for a certain class of analytic functions related to exponential functions using Hohlov operator. We believe these results would be very useful in various fields related to mathematics, science and engineering. As a further development this study can be extended by researchers working in this area related to other trigonometric functions using familiar existing operators.

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