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On Strictly Convex Linear Metric Spaces

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ABSTRACT. Ahuja, Narang and Trehan extended the concept of strict convexity from normed linear spaces to linear metric spaces [G. C. Ahuja, T. D Narang, and S. Trehan. Best approximation on convex sets in metric linear spaces. *Math. Nachr.* **78** (1977), no.1, 125-130] and since then, various other forms of strict convexity in linear metric spaces have emerged in the literature. In this article, we provide a brief overview of the existing literature on various forms of strict convexity in linear metric spaces, introduce some new forms of strict convexity in linear metric spaces, and examine their connections. The article also contains some results on the uniqueness of best approximation, the convexity of Chebyshev sets in strictly convex linear metric spaces, characterizations of strictly convex linear metric spaces in terms of best approximation and some other properties. We also mention some related open problems in the article.

1. INTRODUCTION

A normed linear space $(X, ||.||)$ is called *strictly convex* [9] if its unit sphere does not contain nontrivial line segments. Strict convexity in normed linear spaces is known to be equivalent to the following [15]:
(i) A normed linear space $(X, ||.||)$ is strictly convex if and only if $x \neq y$, $||x|| = ||y|| = 1$ together imply $||\frac{x+y}{2}|| < 1$.
(ii) A normed linear space $(X, ||.||)$ is strictly convex if and only if $x \neq y$, $||x|| = ||y|| = 1$ together imply $||\lambda x + (1-\lambda)y|| < 1$ for every $0 < \lambda < 1$.

Clarkson [8] and Krein [3] were the first to introduce and investigate strictly convex normed linear spaces independently. The notion was extended to linear metric spaces by Ahuja et al. [2] in order to analyse the uniqueness of best approximations in such spaces. Subsequently, various other forms of strict convexity in linear metric spaces were proposed and studied by Sastry et al. ([26], [27], [28]), Vasil'ev [33] and others. In this paper, we explore the extant literature on various forms of strict convexity in linear metric spaces, propose some new forms of strict convexity and analyse their inter connections. We discuss some results on the uniqueness of best approximation and on the convexity of Chebyshev sets in strictly convex linear metric spaces. We also provide characterizations of strictly convex linear metric spaces in terms of uniqueness of best approximation and some other properties. Some unresolved problems in the related areas have also been mentioned in the paper.

2. PRELIMINARIES

In this section, we establish our terminology and recall some definitions that are needed in the sequel.

Following Wilansky [35], by a *linear metric space* (X, d), we mean a real linear metric space of dimension at least one such that the metric d is translation invariant i.e. $d(x +$

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$z, y+z) = d(x,y)$, $x,y,z \in X$ and the linear operations are continuous with respect to d ; $B(x,r) = \{y \in X : d(x,y) < r\}$ is an open ball with center x and radius $r > 0$; $B[x,r] = \{y \in X : d(x,y) \leq 1\}$ is a closed ball with center x and radius r; $S[x,r] = \{y \in X : d(x,y) = r\}$ is a sphere with center x and radius r; $\{(1-\lambda)x + \lambda y : 0 \leq \lambda \leq 1\} \equiv [x,y]$ is a closed line segment in X; $\{(1-\lambda)x + \lambda y : 0 < \lambda < 1\} \equiv (x,y)$ is an open line segment in X. For $A \subset X,\ x \in X$, we define $d(x,A) = \inf\{d(x,y) : y \in A\}$ and for two subsets A and B of X, dist $(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$. For $x \in X$, $P_A(x) = \{y \in X : d(x,y) = d(x,A)\}$ is the set of all best approximations to x in A. The set A is called (i) an *existence set* or a *proximinal set* if $P_A(x) \neq \phi$ for each $x \in X$, (ii) a *uniqueness set* or a *semi-Chebyshev set* if $P_A(x)$ is atmost singleton for each $x \in X$, (iii) a *Chebyshev set* if $P_A(x)$ is a singleton for each $x \in X$, (iv) a *convex set* if $[x,y] \subset A$ for all $x,y \in A$. The mapping which takes each element $x \in X$ to the set $P_A(x)$ is called *metric projection*.

For any two sets A and B of a metric space (X,d), $a \in A$ and $b \in B$ are said to be *proximinal points* if $d(a,b) =$ dist (A,B). The pair (A,B) is said to be a *proximinal pair* or *distance pair* [19], if there exists a pair $(a,b) \in A \times B$ of proximinal points. If there exists at most one pair of proximinal points for (A,B), then the pair (A,B) is called a *semi-Chebyshev pair* and if the pair (A,B) is proximinal as well as semi-Chebyshev, then it is called a *Chebyshev pair*. It is clear that when one of the two sets A and B is a singleton, then the problem of finding proximinal points reduces to the problem of best approximation.

A set M in a metric space (X,d) is said to be
(i) *approximatively compact* [1] if for every $x \in X$ and every sequence $\langle y_n \rangle$ in M such that $\{d(x,y_n)\}$ converges to $d(x,M)$, there exists a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ converging to an element of M.
(ii) *spherically compact* [2] if for each $x \in X \backslash M$, there is a real number $r > d(x,M)$ such that the set $\{y \in M : d(x,y) \leq r\}$ is compact.
(iii) *boundedly compact* [22] if every bounded sequence in M has a convergent subsequence.

The metric d of a linear metric space (X,d) is said to be *strictly monotone* [33] if $x \neq 0, 0 \leq t < 1$ imply $d(tx,0) < d(x,0)$, equivalently, for $0 < \alpha < \beta$, we have $d(\alpha x, 0) < d(\beta x, 0)$, and d is said to be *absolutely monotone* if $d(tx,0) \leq d(x,0)$ whenever $|t| \leq 1$.

If M is a closed linear subspace of a linear metric space (X,d), then the quotient space X/M is also a linear metric space under the metric $\bar{d}$ defined by
$\bar{d}(x+M, y+M) := d(x-y, M) :=$ Inf $\{d(x-y,m) : m \in M\}$ [11].

An element z in a linear metric space (X,d) is said to be *algebraically between two distinct elements x and y* of X if $z = \alpha x + (1-\alpha)y$ for some $\alpha \in (0,1)$, and $z \in X$ is called *metrically between x and y* if $d(x,y) = d(x,z) + d(z,y)$. z is called *algebraic mid-point of x and y* if $z = \frac{x+y}{2}$, and z is called a *metric mid-point of x and y* if $d(x,z) = d(z,y) = \frac{1}{2}d(x,y)$ [12].

A pair (A,B) of non-empty subsets A, B of a metric space (X,d) is said to have $(d)-$*property* [24] if $d(x_1,y_1) = d(A,B), d(x_2,y_2) = d(A,B)$ imply $d(x_1,x_2) = d(y_1,y_2)$.

A metric space (X,d) is said to be *round* [29] if closure of every open ball in X is the corresponding closed ball, and it is said to be *sleek* [30] if interior of every closed ball is the corresponding open ball.

A mapping f defined on a metric space (X,d) is called *nonexpansive* [21] if $d(f(x),f(y)) \leq d(x,y)$ for all $x,y \in X$.

3. Different forms of Strict Convexity in linear metric spaces

We first recall the notion of strict convexity in linear metric spaces introduced by Ahuja et al. [1].

A linear metric space (X,d) is said to be *strictly convex* if for any $r > 0$ and $x,y \in X, d(x,0) \leq r, d(y,0) \leq r$ imply $d(\frac{x+y}{2}, 0) < r$ unless $x = y$.

This notion of strict convexity was called S.C.I in [27], in which S.C.II, S.C.III and S.C.IV were introduced and studied. It is to be noted that all these forms of strict convexity are equivalent in normed linear spaces, whereas, it is not so in linear metric spaces. Before we define these notions and discuss their relationships in linear metric spaces, we reformulate the notion S.C.I as below.

A linear metric space (X, d) is said to be *strictly convex* if $d(\lambda x+(1-\lambda)y, 0) < \lambda d(x, 0)+(1-\lambda)d(y, 0)$ for all $x, y \in S[0, r], x \neq y$ and $0 < \lambda < 1$.

Let us call this notion as S.C.V for now. We shall show that, both the notions S.C.I and S.C.V, are equivalent. To start with, we recall all the forms of strict convexity known in linear metric spaces and discuss their inter-relationships.

A linear metric space (X, d) is said to satisfy
S.C.II [27] if for any $r > 0$ and $x, y \in X, x \neq y, d(x, 0) = r, d(y, 0) = r$ imply $B(0, r) \cap (x, y) \neq \phi$.
S.C.III [27] if for any $r > 0, x \neq y; x, y \in B[0, r]$ imply $(x, y) \subset B(0, r)$.
S.C.IV [27] if $x \neq 0, y \neq 0, d(x + y, 0) = d(x, 0) + d(y, 0)$ imply $y = tx$ for some $t > 0$.
The notion S.C.IV was called pseudo strict convexity in [28]. It was shown in [27] that S.C.I, S.C.II and S.C.III are all equivalent and so we shall call (X, d) strictly convex if it has any (and hence all) of these S.C.I or S.C.II or S.C.III. Whereas, S.C.I and S.C.IV are equivalent in normed linear spaces [7], it was shown in [27] that for linear metric spaces S.C.IV need not imply S.C.I. It is not known whether S.C.I imply S.C.IV in general but it is true on the real line.

It is known [27] that strictly convex linear metric spaces inherit many of the geometric properties enjoyed by normed linear spaces. It was shown in [27] that in a non-zero strictly convex linear metric space, every half-ray emanating from the center of a ball passes through its surface when the surface is non-empty ([27, Corollary 1.6]), and closed balls with non-empty surface are compact if the space is finite dimensional ([27, Theorem 1.6]).

For strictly convex linear metric spaces, we have

Theorem 3.1. [26] *Let (X, d) be a strictly convex linear metric space and $x \neq 0 \in X$. Define $f_x : \mathbb{R}^+ \to \mathbb{R}^+$ by $f_x(t) = d(tx, 0)$. Then f_x is continuous and strictly increasing.*

A linear metric space (X, d) has property (B.C.) [26] if $r \geq 0, d(x, 0) = d(y, 0) = r$ imply $d(\frac{x+y}{2}, 0) \leq r, x, y \in X$.

A linear metric space has (B.C.) if and only if all balls in it are convex [26]. Moreover, if (X, d) has (B.C.), then the function f_x is an increasing function. Obviously, strictly convex linear metric spaces have (B.C.) and therefore in a strictly space, all balls are convex ([26, Corollary 1.5]). However, the converse is not true [26, Example 1.3].

Given two linear metrics on a linear space X, their Euclidean combination on $X \times X$ is a linear metric. While the Euclidean combination of two strictly convex norms on X is strictly convex on $X \times X$, it need not be so in the case of linear metrics as shown in the following example:

Example 3.1. [27] *Consider the strictly convex linear metric space $(\mathbb{R}, d)$, where $d(s, t) = |s - t|^{\frac{1}{2}}, s, t \in \mathbb{R}$. Then $d^*((s_1, t_1), (s_2, t_2)) = [|s_1 - s_2| + |t_1 - t_2|]^{\frac{1}{2}}$ is the Euclidean combination of d with itself and is a linear metric on $\mathbb{R} \times \mathbb{R}$ but is not strictly convex, as $d^*((1, 0), (0, 0)) = d^*((0, 1), (0, 0)) = 1$ and $d^*((\frac{1}{2}, \frac{1}{2}), (0, 0)) = 1$*

The following example shows that S.C.IV need not imply S.C.I

Example 3.2. [27] *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}$$

and d be the linear metric on $\mathbb{R}$ defined by $d(0,t) = |f(t)|$ for all $t \in \mathbb{R}$. Then $(\mathbb{R}, d)$ has (B.C.) and S.C.IV but not S.C.I

The following example shows that S.C.IV need not imply (B.C.)

Example 3.3. [27] *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ \frac{1}{2}(1 + \frac{1}{t}), & t > 1 \end{cases}$$

and d be the linear metric on $\mathbb{R}$ defined by $d(0,t) = |f(t)|$ for all $t \in \mathbb{R}$. Then $(\mathbb{R}, d)$ has S.C.IV but not (B.C.).

While on the real line, S.C.I $\Longrightarrow$ S.C.IV, the following problem is still open:
Problem 1: Does S.C.I imply S.C.IV in general?
We show that S.C.I $\Longrightarrow$ S.C.V.

Theorem 3.2. *If a linear metric space has S.C.I, then it has S.C.V.*

Proof. Let (X, d) be a linear metric space satisfying S.C.I and $x, y \in X$ be such that $x, y \in S[0, r], x \neq y, r > 0$. Let $0 < t < 1$. Since $0 < t < 1$, there exists a $\delta > 0$ such that $0 < t - \delta < t < t + \delta < 1$. Let $z = (t - \delta)x + (1 - t + \delta)y$ and $w = (t + \delta)x + (1 - t - \delta)y$. Since $x, y \in B[0, r]$ and $B[0, r]$ is convex as X has S.C.I, we have $z, w \in B[0, r]$. Consider

$$\begin{aligned} d(tx + (1 - t)y, 0) &= d\left(\frac{z + w}{2}, 0\right) \\ &< r \text{ by S. C. I} \\ &= tr + (1 - t)r \\ &= td(x, 0) + (1 - t)d(y, 0). \end{aligned}$$

Therefore, $d(tx + (1 - t)y, 0) < td(x, 0) + (1 - t)d(y, 0)$ and so (X, d) has S.C.V. □

Problem 2: Is there any hierarchical relationship between the properties S.C.IV and S.C.V?
Remark 1: Since S.C.I, S.C.II and S.C.III are equivalent, it follows that S.C.II $\Longrightarrow$ S.C.V and S.C.III $\Longrightarrow$ S.C.V.
Remark 2: The natural question whether S.C.V implies S.C.I has the positive answer, as, S.C.V clearly implies S.C.II and S.C.II implies S.C.I by Theorem 1.8 [26].
Remark 3: A linear metric space has S.C.V, if for each $x \neq 0 \in X, \lambda \in (0, 1)$, we have $d(\lambda x, 0) < \lambda d(x, 0)$. This can be seen as under.
Let $x, y \in X$ be such that $d(x, 0) = r, d(y, 0) = r, x \neq y, r > 0$.
Consider

$$\begin{aligned} d(\lambda x + (1 - \lambda)y, 0) &\le d(\lambda x, 0) + d((1 - \lambda)y, 0) \\ &< \lambda d(x, 0) + (1 - \lambda)d(y, 0) \\ i.e.\ d(\lambda x + (1 - \lambda)y, 0) &< \lambda d(x, 0) + (1 - \lambda)d(y, 0). \end{aligned}$$

Remark 4: Proceeding as in Remark 3, we can show that a linear metric space (X, d) has S.C.I if for each $x \neq 0 \in X, \lambda \in (0, 1)$ implies $d(\lambda x, 0) < \lambda d(x, 0)$.
Remark 5: Consider the following stronger form of S.C.V:
A linear metric space (X, d) has S.C.VI if $d(\lambda x + (1 - \lambda)y, 0) < \lambda d(x, 0) + (1 - \lambda)d(y, 0)$ for all $x, y \in X, x \neq y$ and $0 < \lambda < 1$.

Problem 3: Clearly S.C.VI implies S.C.I, S.C.II, S.C.III and S.C.V. It will be interesting to know whether the converses hold, and, also its relationship with S.C.IV.

A closed ball $B[0,r]$ in a linear metric space (X,d) is said to be strictly convex (Vasilev [33] called such balls as strongly convex) if for any distinct $x,y \in B[0,r]$ and any $\lambda \in (0,1)$, the point $(1-\lambda)x+\lambda y$ belongs to the topological interior of $B[0,r]$.

The strict convexity of closed balls in linear metric spaces was introduced by Albinus [4] and it was linked to the strict convexity of the space by Vasilev [33]. Albinus [4] called a linear metric space 'strictly convex' if all its closed balls are strictly convex.

Concerning the compactness of closed balls in strictly convex linear metric spaces, we have

Theorem 3.3. [27] *Let (X,d) be a finite-dimensional strictly convex linear metric space such that $S[0,r] \neq \phi$, then $B[0,r]$ is a compact set.*

Concerning the connectedness of the surface of closed balls in strictly convex linear metric spaces, we have

Theorem 3.4. [27] *In a strictly convex linear metric space of dimension two or more, the surface of every closed ball is arcwise connected.*

This result need not be true if strict convexity is replaced by ball convexity.

Example 3.4. [27] *Let d be the metric on $\mathbb{R}^2$ defined by $d((x,y),(0,0)) = max\{\frac{|x|}{1+|x|},|y|\}$. Then $(\mathbb{R}^2,d)$ is a linear metric space which has (B.C.) but for $r \geq 1, S[0,r]$ is neither compact nor arcwise connected.*

The following example shows that the distance between two points can be the sum of their distances from an intermediate point but it need not be so for every intermediate point even in a strictly convex linear metric space.

Example 3.5. [27] *Define $f:\mathbb{R}\to\mathbb{R}$ be by*

$$f(t) = \begin{cases} \frac{2t}{1+t}, & 0 \leq t \leq 1 \\ t, & t \geq 1 \end{cases}$$

and d be the linear metric on $\mathbb{R}$ defined by $d(s,t) = f|s-t|$ for all $s,t \in \mathbb{R}$. Then $(\mathbb{R},d)$ is a strictly convex linear metric space. We have

$$d(3,4)+d(4,5) = d(3,5) \neq d(3,\frac{7}{2})+d(\frac{7}{2},5)$$

It is well known that a normed linear space is strictly convex if and only if the surface of any closed ball does not contain any line segment on its surface. The following characterization of linear metric spaces show that this is true in case of linear metric spaces too.

Theorem 3.5. [26] *Let (X,d) be a linear metric space. Then the following are equivalent:*
(i) *$r>0, d(x,0)=r=d(y,0)$ and $x\neq y$ imply $B(0,r)\cap(x,y)\neq\phi$*
(ii) *(X,d) is strictly convex.*
(iii) *$r>0, x\neq y, x,y\in B[0,r]$ imply $(x,y)\subset B(0,r)$*

From the implication $(ii) \Rightarrow (iii)$ of Theorem 3.10, we have the following:

Corollary 3.1 ([26], Corollary 2.3). *Let (X,d) be a linear metric space. Then the following are equivalent:*
(i) X is strictly convex.
(ii) $d\left(\frac{x+y}{2},0\right) < r$ for any $x \neq y \in X$ with $d(x,0) = r = d(y,0)$, where $r > 0$ is any real number.

Consider the following form of strict convexity:

A linear metric space (X, d) has S.C.VII if for all $x, y \in X, x \neq y, d(x, 0) = d(y, 0) = r$, we have $d(\lambda x + (1-\lambda)y, 0) < r$ for all $\lambda \in (0,1)$. This form of strict convexity in normed linear spaces is known in the literature (A normed linear space X is strictly convex if for all $x, y \in X, x \neq y, ||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1$ for all $\lambda \in (0,1)$- ([8], [15, p. 426]). Concerning S.C.VII, we have the following two results:

Theorem 3.6. *The following assertions are equivalent:*
(i)*(X, d) has S.C.VII.*
(ii) *If $x, y \in X, x \neq y, d(x, 0) = d(y, 0) = r$, then $d(\frac{x+y}{2}, 0) < r$.*
(iii) *If $x, y \in B[0, r], x \neq y$ then $(x, y) \subset B(0, r)$.*

Proof. $(i) \implies (ii)$ is obvious.
$(ii) \implies (iii)$ follows from Theorem 3.10.
$(iii) \implies (i)$ Let $x, y \in X, x \neq y, d(x, 0) = d(y, 0) = r$. Then $x, y \in B[0, r]$ and so by the hypothesis, $(x, y) \subset B(0, r)$ and hence (X, d) has S.C.VII □

Problem 4: It will be interesting to know the relationship of S.C.VII with other forms of strict convexity. It appears that in the light of Theorems 3.10 and 3.12, S.C.I and S.C.VII are equivalent. In fact, this is true and it follows from Corollary 3.11 and Theorem 3.12. This assertion is further strengthened by the following result:

Theorem 3.7. *A linear metric space (X, d) satisfies S.C.VII if and only if $S_r = \{z \in X : d(z, 0) = r\}$ contains no nontrivial line segments.*

To prove this theorem, we shall be using the following lemma:

Lemma 3.1 ([15], p.175). *Suppose that C is a convex subset of a linear topological space X. If $x \in C, y \in C^o$ and $0 < t < 1$, then $tx + (1-t)y \in C^o$, where C^o denotes interior of the set C.*

Proof. For $x \in C, y \in C^o, 0 < t < 1$ implies $tx + (1-t)y \in tC + (1-t)C^o \subset C$ as C is convex. Since $tC + (1-t)C^o$ is an open subset of C and C^o is the largest open subset of C, we have $tC + (1-t)C^o \subset C^o$ and hence $tx + (1-t)y \in C^o$. □

Proof of Theorem 3.13. Let $x, y \in X, x \neq y, d(x, 0) = d(y, 0) = r$. Since (X, d) satisfies S.C.VII, we obtain that (x, y) lies entirely in the interior of the ball $B_r = \{z \in X : d(z, 0) \leq r\}$ and so S_r contains no nontrivial line segments.

Conversely, suppose that no nontrivial line segments lie in S_r and x, y are two different points of S_r. Then some of the points of (x, y) lie in B_r^o, so it follows easily from the above lemma that all the points of (x, y) lie in B_r^o and hence $d(tx + (1-t)y, 0) < r$ i.e. (X, d) satisfies S.C.VII.

Remark 6: ([15], p. 175) The last portion of the argument in the proof of the above theorem also establishes the following characterization of S.C.VII that involves only the midpoints of line segments rather than the entire line segments:

A linear metric space satisfies S.C.VII if and only if $d(\frac{x+y}{2}, 0) < r$ whenever $x, y \in X, x \neq y, d(x, 0) = d(y, 0) = r$.

The following result shows that the metric of a strictly convex linear metric space is strictly monotone.

Theorem 3.8. *If (X, d) is a strictly convex linear metric space and $0 \neq x \in X$, then $f_x : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f_x(t) = d(tx, 0)$ is strictly monotone and so d is strictly monotone.*

Proof. Since (X, d) is strictly convex, f_x is strictly increasing (Theorem 3.1) and so $f_x(t) < f_x(s), 0 \leq t < s$. In particular, $0 \leq t < 1,$ implies that $f_x(t) < f_x(1)$ i.e. $d(tx, 0) < d(x, 0)$ and so d is strictly monotone. □

For strictly monotone metrics (and so for strictly convex linear metric spaces), we have

Theorem 3.9. *If the metric d of a linear metric space (X, d) is strictly monotone, C is a convex subset of X, and $T : C \to C$ is a non-expansive map, then $T_\lambda : C \to C$ defined by $T_\lambda(y) = (1-\lambda)x + \lambda T(y);\ x, y \in C,\ 0 \leq \lambda < 1$, is contractive.*

Proof. For $y_1, y_2 \in C$, consider

$$\begin{aligned} d(T_\lambda(y_1), T_\lambda(y_2)) &= d((1-\lambda)x + \lambda T(y_1), (1-\lambda)x + \lambda T(y_2)) \\ &= d(\lambda T(y_1), \lambda T(y_2)) \\ &< d(T(y_1), T(y_2)) \text{ as } d \text{ is strictly monotone} \\ &\leq d(y_1, y_2) \text{ as } T \text{ is non-expansive.} \end{aligned}$$

Therefore, $d(T_\lambda(y_1), T_\lambda(y_2)) < d(y_1, y_2)$ i.e. T_λ is contractive.

□

For absolutely monotone metrics, we have

Theorem 3.10. *If the metric d of a linear metric space (X, d) is absolutely monotone, C is a convex subset of X, and T is a non-expansive map, then $T_\lambda : C \to C$ defined by $T_\lambda(y) = (1-\lambda)x + \lambda Ty;\ x, y \in C,\ 0 \leq \lambda \leq 1$ is non-expansive.*

Proof. For $y_1, y_2 \in C$, consider

$$\begin{aligned} d(T_\lambda(y_1), T_\lambda(y_2)) &= d((1-\lambda)x + \lambda T(y_1), (1-\lambda)x + \lambda T(y_2)) \\ &= d(\lambda T(y_1), \lambda T(y_2)) \\ &\leq d(T(y_1), T(y_2)) \text{ as } \lambda \leq 1 \text{ and } d \text{ is absolutely monotone} \\ &\leq d(y_1, y_2) \text{ as } T \text{ is non-expansive.} \end{aligned}$$

Therefore, $d(T_\lambda(y_1), T_\lambda(y_2)) \leq d(y_1, y_2)$ i.e. T_λ is non-expansive.

□

It is known ([27], [36]) that in a linear metric space satisfying (B.C.), closure of an open ball need not be the corresponding closed ball. Infact, in a linear metric space (X, d) with (B.C.), the closure of every open ball is the corresponding closed ball if and only if the function $f_x : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f_x(t) = d(tx, 0)$ is a strictly increasing function for every non-zero x in the space X. Since the function f_x is strictly increasing if the space X is strictly convex (Theorem 3.1) and a strictly convex space has (B.C.), we obtain that in a strictly convex linear metric space, the closure of an open ball is the corresponding closed ball. In fact, we have:

Theorem 3.11. [33] *A linear metric space (X, d) is strictly convex*
(i) if and only if for any $r > 0$, the ball $B[x, r]$ is strictly convex and $\overline{B}(x, r) = B[x, r]$.
(ii) if and only if every $B(x, r)$ is convex and no sphere $S[x, r]$ contains segments.

It is well known [15] that a normed linear space is strictly convex if and only if any two closed balls in X having disjoint interior, do not intersect at more than one point. This result was extended to linear metric spaces in [13] as under:

Theorem 3.12 ([13], Theorem 2.7). *(i) In a strictly convex linear metric space, no two closed balls having disjoint interiors intersect at more than one point.*
(ii) A linear metric space (X, d) with ball convexity is strictly convex if no two closed balls in X having disjoint interiors intersect at more than one point.

Theorem 3.19 results in the following characterization of strictly convex linear metric spaces amongst the linear metric spaces that have ball convexity:

Theorem 3.13 ([13], Theorem 2.8). *A linear metric space (X, d) with ball convexity is strictly convex if and only if no two closed balls in X having disjoint interiors intersect at more than one point.*

The inheritance of strict convexity of linear metric spaces by the quotient spaces has been recently discussed in [11] in which the following result has been proved:

Theorem 3.14 ([11], Proposition 2.2). *Let (X, d) be a strictly convex linear metric space and M be a proximinal subspace of X, then the quotient space $(X/M, \bar{d})$ is also strictly convex.*

Remark 7. The proximinality of M cannot be replaced by closedness of M even in case of normed linear spaces [11].

4. Applications to Best Approximations

The concept of strict convexity has been applied extensively in problems dealing with the uniqueness of best approximations in normed linear spaces. Ahuja et al. [1] introduced strictly convex linear metric spaces in order to analyse the problem of uniqueness of best approximation in linear metric spaces. We now briefly survey some of the known results in this direction.

Albinus [4] proved the following:

Theorem 4.15. [4] [5] *Suppose X is a linear metric space with monotone quasi norm and dim$X \geq 2$. Then the following conditions are equivalent:*
(i) Every subspace of X is a uniqueness set.
(ii) Every one dimensional subspace of X is a uniqueness set.
(iii) Every finite dimensional subspace of X is a Chebyshev set.
(iv) Every one dimensional subspace of X is a Chebyshev set.
(v) The space X is strictly convex.

Remark 8: Wriedt [37] proved that any non-normable strictly convex linear metric space contains a closed subspace of codimension 2 which is not an existence set.

Ahuja et al. [2] proved the following:

Theorem 4.16. *Every convex proximinal set in a strictly convex linear metric space is a Chebyshev set.*

The following result proved by Narang [18] shows that the converse of the above mentioned result is also true.

Theorem 4.17. *A linear metric space (X, d) is strictly convex if and only if each proximinal convex subset of X is Chebyshev, or equivalently, if and only for each convex subset S of X and distinct points x and y of S, $S_x \cap S_y = \emptyset$, where $S_z = \{x \in S : d(x, z) = d(x, S)\}$, or equivalently, if and only if balls in (X, d) are convex, and $S[0, r] = \{x \in X : d(x, 0) = r\}$ does not contain any line segment for $r > 0$, or equivalently, if and only if $x, y \in B[0, r] \Rightarrow (x, y) \subset B(0, r)$.*

Since all balls in a strictly convex linear metric space are convex, such a space is locally convex. Using this fact, Narang [17] proved the following:

Theorem 4.18. *A convex boundedly weakly compact set (a set in which every bounded sequence has a weakly convergent subsequence) in a strictly convex linear metric space is Chebyshev.*

The following example shows that a finite dimensional subspace or a convex subset of a non strictly convex linear metric space need not be semi-Chebyshev.

Example 4.6. *Consider $X = \mathbb{R}^2$ with metric d defined by $d((x_1, y_1), (x_2, y_2)) = max\{d(x_1, x_2), d(y_1, y_2)\}$ and $K = \{(x, 0) : x \in \mathbb{R}\}$. For $x = (0, 1)$, $P_K(x) = \{(x, 0) : |x| \leq 2\}$, K is a finite dimensional subspace of non strictly convex linear metric space $(\mathbb{R}^2, d)$ and K is not semi-Chebyshev.*

However we have the following characterization [20] of strictly convex linear metric spaces:

Theorem 4.19. (i) *A linear metric space (X, d) is strictly convex if and only if each linear subspace of X is semi Chebyshev.* (ii) *A linear metric space (X, d) is strictly convex if and only if each convex subspace of X is semi Chebyshev.*

Proof. As each linear subspace is a convex set, it is sufficient to prove necessary part of (ii) and the sufficient part of (i).
Necessary part of (ii): Let (X, d) be a strictly convex linear metric space and K be convex subset of X. Then K is semi Chebyshev [2].
Sufficient part of (i): Suppose (X, d) is a linear metric space such that each linear subspace of X is semi Chebyshev. Then X is strictly convex [20, Theorem 2]. □

Remark 9: For Banach spaces, Theorem 3.6 was proved by Sthechkin ([32], [9]).
The following characterization of strictly convex linear metric spaces was given by Vasilev [33]:

Theorem 4.20. *A linear metric space (X, d) is strictly convex if and only if all locally compact closed convex sets in X are Chebyshev.*

The following results on uniqueness of best approximation were proved by Sthechkin [32] in strictly convex normed linear spaces:

Theorem 4.21. *Let X be a strictly convex normed linear space, $A \subseteq X$, $x_0 \in X \backslash A$, $y_0 \in P_A(x_0)$. Then $P_A(x) = \{y_0\}$ for every x in the semi-interval $(x_0, y_0]$.*

Theorem 4.22. *In a strictly convex normed linear space X, the set $Q(A) = \{x \in X : P_A(x)$ contains at most is dense in X, for every subset A of X.*

Theorem 4.23. *Let X be a strictly convex Banach space and A a relatively boundedly compact subset of X(i.e. intersection of A with any ball is compact in X). Then $Q(A)$ is a set of second category, i.e. $X \backslash Q(A)$ is of first Baire Category.*

Theorem 4.24. *A Banach space X is strictly convex if and only if every subset A of X is a near uniqueness set, i.e. $Q(A)$ is dense in X for every subset A of X.*

Sthechkin [32] posed the following problem, that is still open:
Problem 5: If a Banach space X has the property that $Q(A)$ is dense in X or $Q(A)$ is of second category for every compact subset A of X, then must X be strictly convex?
One of the major unsolved problem in approximation theory is: Whether every Chebyshev subset of a Hilbert space is convex? Various partial answers to this problem are known in the literature ([6], [10], [14], [16], [21], [34]). In this connection, Phelps [23] proved that a Chebyshev subset of a strictly convex normed linear space is convex if the associated metric projection is non-expansive. This result was extended to linear metric spaces by Narang [21] as under:

Theorem 4.25. [21] *Let X be a linear metric space with a metric d that is convex (i.e. for all $x, y \in X$ and for all $\delta_1 \geq 0, \delta_2 \geq 0$ such that $\delta_1 + \delta_2 = d(x, y)$, there is a point $z \in X$ with $d(x, z) = \delta_1$ and $d(z, y) = \delta_2$) and translation invariant and suppose that (X, d) has S.C.IV. If S is a Chebyshev subset of X, then S is convex if the associated metric projection is non-expansive.*

The strict convexity of linear metric spaces has also been used to discuss the problem of proximinal points by Narang in [19] and [22] and the following results have been proved:

Theorem 4.26. [19] *If A and B are mid-point convex sets in a strictly convex linear metric space (X, d) such that $(A - A) \cap (B - B) = \{0\}$, then the pair (A, B) is semi-Chebyshev.*

Theorem 4.27. [22] *Let (X, d) be a strictly convex linear metric space. If A is locally compact or boundedly compact or spherically compact or approximatively compact closed convex subset of X and B is a compact convex subset of X, then (A, B) is a distance pair.*

5. Connections of Strict Convexity with some other properties

It is well known [12] that in a normed linear space, algebraic betweenness implies metric betweenness but not conversely. In fact, Smiley [31] proved that a real normed linear space X is strictly convex if and only if algebraic and metric betweenness coincide in X. For strictly convex real linear metric spaces, Grover et al. [12] proved that implication of algebraic betweenness from metric betweenness characterizes pseudo strictly convexity.

Recall that a linear metric space is called pseudo strict convex if $x \neq 0, y \neq 0$, $d(x + y, 0) = d(x, 0) + d(y, 0)$ imply that $y = tx$ for some $t > 0$ [28]. In normed linear spaces, pseudo strict convexity is actually equivalent to strict convexity.

Theorem 5.28. *A linear metric space is pseudo strictly convex if and only if metric betweenness implies algebraic betweenness.*

In the following result, Grover et al. [12] proved that in linear metric spaces, the two notions of algebraic betweenness and metric betweenness coincide if and only if the linear metric space is a strictly convex normed linear space.

Theorem 5.29. *For a linear metric space (X, d), the following statements are equivalent:*
(i) The notions of algebraic betweenness and metric betweenness coincide.
(ii) The notions of algebraic mid-point and metric mid-point coincide.
(iii) The function $||.|| : X \to \mathbb{R}^+$, defined by $||x|| = d(x, 0)$ is a strictly convex norm function.

Raj and Eldred [24] characterized strictly convex normed linear spaces in terms of $(d)-$ property and proved the following result:

Theorem 5.30. *A normed linear space X is strictly convex if and only if every pair (A, B) of non-empty closed and convex subsets of X has the $(d)-$ property.*

Sangeeta and Narang [25] partially extended Raj and Eldred's result to linear metric spaces as under:

Theorem 5.31. *In a strictly convex linear metric space (X, d), every pair (A, B) of non-empty closed and convex subsets of X has the $(d)-$ property.*

As seen above, the converse of the above result holds in normed linear spaces but it is still not known whether it holds in linear metric spaces.
Problem 6: [25] If in a linear metric space, every pair (A, B) of non-empty closed and convex subsets has the $(d)-$ property, then must (X, d) be strictly convex ?

It is well known that all normed linear spaces are round as well as sleek, but this is not the case [29] in metric spaces or even linear metric spaces. However, there are some necessary and sufficient conditions known in the literature for metric spaces and linear metric spaces to be round ([29], [33]) or sleek [30]. Vasilev [33] gave the following characterizations of strictly convex linear metric spaces:

Theorem 5.32. *A linear metric space (X, d) is strictly convex if and only if for any $r > 0$, the ball $B[0, r]$ is strictly convex and the space X is round.*

The following analogue of this result was proved by Singh and Narang [29]:

Theorem 5.33. *A linear metric space (X, d) is strictly convex if and only if for any $r > 0$, the ball $B[0, r]$ is strictly convex and the space is sleek.*

Thus strictly convex linear metric spaces are both round and sleek, and if a linear metric space has the strict ball convexity, then the notions of being round and sleek are equivalent.

6. Conclusion

One of the very natural trends in mathematical research is to refine the framework of the known results and to see which of the results survive in more general settings. For instance, it is natural to analyse the questions, that have already been dealt with in normed linear spaces, in linear metric spaces. Although, the available literature in the theory of best approximation in strictly convex normed linear spaces is very rich but only a few results have been extended to strictly convex linear metric spaces. The concept of strictly convex normed linear spaces has also proved out to be particularly fertile and beneficial in the study of geometry of Banach spaces, orthogonality, semi inner product spaces, theory of non-linear operators, fixed point theory etc. It is anticipated that this article may spark more research on these topics, as well as in other branches of mathematics where the underlying spaces are linear metric spaces.

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