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Differential subordination of harmonic means of analytic functions

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ABSTRACT. In this paper,we examine the differential subordination involving the harmonic means of analytic functions $p(z)$ and $p(z) + \frac{zp'(z)}{\delta p(z)+\gamma}$ with $\delta \geq 0$ and $\gamma \geq 0$, defined in the open unit disc Δ such that $p(z)$ is subordinate to certain well known analytic functions.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions f normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ defined in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in Δ, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there is an analytic self-map ω in Δ satisfying $\omega(0) = 0$ and $f(z) = g \circ \omega$. Assume that $\psi : \mathbb{C}^2 \to \mathbb{C}$ and $h \in \mathcal{A}$ is univalent,we say that a function $p \in \mathcal{A}$ satisfies the first-order differential subordination if the function $\Delta \ni z \to \psi(p(z), zp'(z))$ is well defined and analytic,

$$\psi(p(z), zp'(z)) \prec h(z), \tag{1.1}$$

then we say that p is the solution of the above differential subordination. An univalent function $q \in \mathcal{A}$ is called a dominant of solutions of differential subordination (1.1) if $p \prec q$ for all solutions $p \in \mathcal{A}$ of (1.1) . A dominant $\tilde{q}$ of (1.1) is called the best dominant of (1.1) if $\tilde{q} \prec q$ for all dominants q of (1.1). Note that the differential subordination (1.1) can be written as the differential equation

$$\psi(p(z), zp'(z)) = h(\omega(z)),$$

where ω is a Schwarz function.

Among the various measures considered in the study of averages, Harmonic Mean(HM) is the reciprocal of the average of the reciprocals of the given data values. The harmonic mean is one of the Pythagorean means, along with the arithmetic and geometric mean, which is no greater than either of them $(HM < GM < AM)$. Harmonic Mean of any two quantities q_1 and q_2 is given by $2q_1q_2/(q_1 + q_2)$, whereas the Geometric Mean(GM) and Arithmetic Mean(AM) are $\sqrt{q_1q_2}$ and $(q_1 + q_2)/2$ respectively. When these quantities happen to be functions $f_i \in \mathcal{A}$, then the weighted harmonic mean of such functions, is given by $\Sigma w_i/\Sigma(w_i/f_i)$, where w_i are integers. Let $\beta \in (0, 1]$ and $a, b \in \mathbb{C}$, for $a + \beta(a - b) \neq 0$, the harmonic mean of a and b is defined as $\dfrac{ab}{a + \beta(a - b)}$. For analytic functions f and g defined in the unit disc Δ, the harmonic mean is defined as follows:

$$\frac{f(z)g(z)}{g(z) + \beta(f(z) - g(z))}, \beta \in (0, 1) \text{ provided } g(z) + \beta(f(z) - g(z)) \neq 0.$$

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The concept of differential Subordinations related to the harmonic mean was introduced and studied by Chojnacka et al.[1] and Cho et al.[3]. For given $\beta \in (0,1)$, and $\psi \in \mathcal{A}$, Cho. N. E and Lecko. A in [3] studied the following differential subordination involving harmonic mean of the type

$$\frac{p(z)(p(z)+zp'(z)\psi(z))}{p(z)+(1-\beta)zp'(z)\psi(z)} \prec h(z), \qquad z \in \Delta \tag{1.2}$$

where $h(z)$ is univalent and analytic in Δ. The above differential subordination with $\beta = 1/2$ and suitable ψ and h was considered by Kanas and Tudor in [11].

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Here, we introduce the subclass Q [for details on corners of curves [13] (pp.51-65)].

Definition 1.1. *By Q, we denote the class of convex functions h with the following properties:*
(a) $h(\Delta)$ is bounded by finitely many smooth arcs which form corners at their end points (including corners at infinity):
(b) $E(h)$ is the set of all points $\zeta \in \mathbb{T}$ which corresponds to corners $h(\zeta)$ of $\partial h(\Delta)$;
(c) $h'(\zeta) \neq 0$ exists at every $\zeta \in \mathbb{T}\backslash E(h)$.

Following the works of Chojnacka. O, Lecko. A [4], in this article, we investigate differential subordination of the form (1.2), where ψ is linear function and h is an univalent function.

For the choice of $\psi(z) = 1/(\delta p(z) + \gamma)$, we obtain sufficient conditions such that the function $p(z)$ is subordinate to an analytic function $q(z)$. The following are some choices we have made for $q(z)$ throughout this article:

(i) For $q(z) = \sqrt{1+z}$, where the image domain is bounded by the right-half of the Bernoulli lemniscate given by $|w^2 - 1| < 1$ [14].

(ii) For $q(z) = 1 + \frac{4z}{3} + \frac{2z^2}{3}$, which maps Δ into the image set bounded by the Cardioid given by $(9x^2+9y^2-18x+5)^2 - 16(9x^2+9y^2-6x+1) = 0$, [15] and furter studied in [16].

(iii) The function $q(z) = ae^{Mz}$ for $a \geq 0$ and $M > 0$ is a generalized version of the function e^z discussed in [18].

(iii) The function $q(z) = (1+Az)/(1+Bz)$, with $A, B \in [-1,1]$ and $B < A$ discussed in W. Janowski[5].

(iv) The function $q(z) = a + Mz, a \geq 0, M > 0$ has been elaborately studied in [1].

We require the following Lemma 1 [6], to prove our main result.

Lemma 1.1. *Let $h \in Q$ and p be a non-constant analytic function with $p(0) = h(0)$. If p is not subordinate to h, then there exist $z_0 \in \Delta\backslash\{0\}$ and $\zeta_0 \in \mathbb{T}\backslash E(h)$ such that $p(z_0) = h(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 h'(\zeta_0)$ for some $m \geq 1$.*

2. Main result

Without loss of generality, we assume that the values of β to be in the interval (0,1] and $\alpha, \delta \geq 0$ with a ≥ 0, M > 0 throughout this article.

Theorem 2.1. *Let p be an analytic function with $p(0) = 1$, If*

$$\left| \frac{p(z)\left[p(z) + \dfrac{zp'(z)}{\delta p(z)+\gamma}\right]}{p(z) + (1-\beta)\dfrac{zp'(z)}{\delta p(z)+\gamma}} - 1 \right| < \frac{4\sqrt{2}(\delta\sqrt{2}+\gamma)+\sqrt{2}}{4[\delta\sqrt{2}+\gamma]+(1-\beta)} - 1 \tag{2.3}$$

then $p(z) \prec \sqrt{1+z}$.

Proof. Let $h(z) = \sqrt{1+z}$. Since h is univalent, $p(0) = h(0) = 1$. Suppose on the contrary, that p is not subordinate to h. Since $h \in Q$ with $E(h) = \phi$, there exists $z_0 \in \Delta\backslash\{0\}$ and $\zeta_0 \in \mathbb{T}$ such that $p(z_0) = h(\zeta_0)$ and for some $m \geq 1$, $z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) = \dfrac{m\zeta_0}{2\sqrt{1+\zeta_0}}$.
Now ,

$$\frac{p(z_0)\left[p(z_0)+\dfrac{z_0p'(z_0)}{\delta p(z_0)+\gamma}\right]}{p(z_0)+(1-\beta)\dfrac{z_0p'(z_0)}{\delta p(z_0)+\gamma}} - 1 = \frac{p(z_0)[p(z_0)-1][\delta p(z_0)+\gamma] + z_0p'(z_0)[p(z_0)-(1-\beta)]}{p(z_0)[\delta p(z_0)+\gamma]+(1-\beta)z_0p'(z_0)}$$

$$= \frac{[\sqrt{1+\zeta_0}[\delta\sqrt{1+\zeta_0}+\gamma][\sqrt{1+\zeta_0}-1] + \frac{m\zeta_0}{2\sqrt{1+\zeta_0}}[\sqrt{1+\zeta_0}-1+\beta]}{\sqrt{1+\zeta_0}[\delta\sqrt{1+\zeta_0}+\gamma]+(1-\beta)\frac{m\zeta_0}{2\sqrt{1+\zeta_0}}}$$

$$= \frac{2(1+\zeta_0)^{3/2}[\delta\sqrt{1+\zeta_0}+\gamma] + m\zeta_0\sqrt{1+\zeta_0}}{2(1+\zeta_0)[\delta\sqrt{1+\zeta_0}+\gamma]+m\zeta_0(1-\beta)} - 1.$$

Therefore,

$$\left|\frac{p(z_0)[p(z_0)+\dfrac{z_0p'(z_0)}{\delta p(z_0)+\gamma}]}{p(z_0)+(1-\beta)\dfrac{z_0p'(z_0)}{\delta p(z_0)+\gamma}} - 1\right| = \left|\frac{2(1+\zeta_0)^{3/2}[\delta\sqrt{1+\zeta_0}+\gamma] + m\zeta_0\sqrt{1+\zeta_0}}{2(1+\zeta_0)[\delta\sqrt{1+\zeta_0}+\gamma]+m\zeta_0(1-\beta)} - 1\right|. \tag{2.4}$$

For each $m \geq 1$, $\beta \in (0, 1]$, we define

$$q_m(z) = \frac{2(1+z)^{3/2}[\delta\sqrt{1+z}+\gamma] + mz\sqrt{1+z}}{2(1+z)[\delta\sqrt{1+z}+\gamma]+mz(1-\beta)} - 1.$$

Clearly $q_m(1) < q_m(-1)$, since

$$q_m(1) = \frac{2(2^{3/2}[\delta\sqrt{2}+\gamma] + m\sqrt{2}}{4[\delta\sqrt{2}+\gamma]+m(1-\beta)} - 1$$

and

$$q_m(-1) = -1.$$

Thus,

$$|q_m(z)| \geq q_m(1) = \frac{2(2^{3/2}[\delta\sqrt{2}+\gamma] + m\sqrt{2}}{4[\delta\sqrt{2}+\gamma]+m(1-\beta)} - 1. \tag{2.5}$$

Observe that the function $r(m) = q_m(1), m \geq 1$ is increasing as

$$r'(m) = \frac{8\delta\beta + 4\sqrt{2}\gamma\beta}{(4\delta\sqrt{2}+4\gamma+m(1-\beta))'(2)} > 0, \qquad m \geq 1.$$

Hence

$$|q_m(z)| \geq q_m(1) = r(m) \geq r(1).$$

From (2.4),we have

$$\left|\frac{p(z)\left[p(z)+\dfrac{zp'(z)}{\delta p(z)+\gamma}\right]}{p(z)+(1-\beta)\dfrac{zp'(z)}{\delta p(z)+\gamma}} - 1\right| \geq \frac{4\sqrt{2}(\delta\sqrt{2}+\gamma)+\sqrt{2}}{4[\delta\sqrt{2}+\gamma]+(1-\beta)} - 1,$$

which is a contradiction to (2.3). Therefore $p(z) \prec \sqrt{1+z}$. □

Remark 2.1. *For the function,*

$$p_1(z) = 1 - 0.02075z^5$$

and $\delta = \gamma = 1$ *with* $\beta = 0$ *we observe that the condition* (2.3) *is satisfied and hence using Theorem 2.1, we obtain* $p_1(z)$ *is subordinate to* $\sqrt{1+z}$*, which is given in the following figure,*

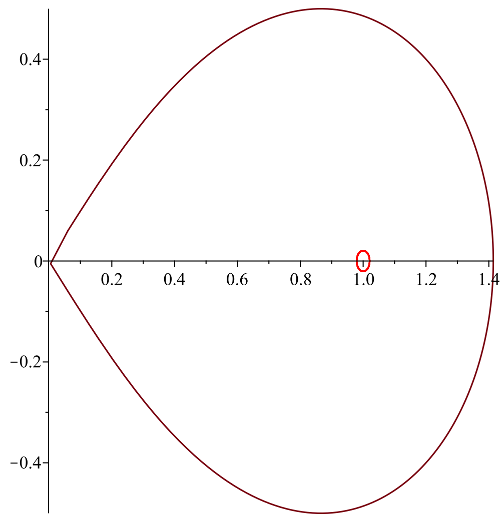

FIGURE 1. $p_1(z) \prec \sqrt{1+z}$

For the choice of $p(z) = \frac{zf'(z)}{f(z)}$, for $f \in \mathcal{A}$,

(i) $\beta = \gamma = 1, \delta = 0$ and
(ii) $\beta = \delta = 1, \gamma = 0$

we have the following results:

Corollary 2.1. *(i) If*

$$\left| \frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 + 2\frac{zf'(z)}{f(z)} - 1 \right| < \frac{5\sqrt{2}}{4} - 1 \approx 0.78$$

then $\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}$.

(ii) If

$$\left| \frac{f''(z)}{f(z)} \right| < \frac{8 - 3\sqrt{2}}{4\sqrt{2}} \approx 1.328$$

then $\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}$

By taking $f(z) = sinz$ in (ii) of Corollary 2.1 we observe that f satisfies the sufficient condition and hence we obtain the result that $zcotz \prec \sqrt{1+z}$, which is depicted in the Figure 2.

Theorem 2.2. *Let* p *be an analytic function with* $p(0) = 1$ *and if*

$$\left| \frac{p(z)\left[p(z) + \dfrac{zp'(z)}{\delta p(z) + \gamma}\right]}{p(z) + (1-\beta)\dfrac{zp'(z)}{\delta p(z) + \gamma}} - 1 \right| < \frac{9\delta + 6\gamma + \frac{8}{3}(2+\beta)}{9\delta + 3\gamma + \frac{8}{3}(1-\beta)} \tag{2.6}$$

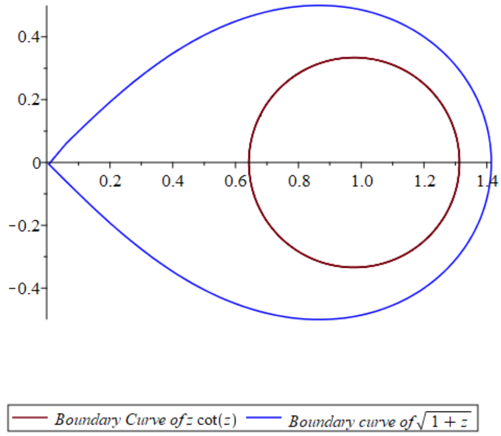

FIGURE 2. $zcotz \prec \sqrt{1+z}$

then $p(z) \prec 1 + \dfrac{4z}{3} + \dfrac{2z^2}{3}$.

Proof. Let $h(z) = 1 + \dfrac{4z}{3} + \dfrac{2z^2}{3}$. Since h is univalent, $p(0) = h(0) = 1$. Suppose on the contrary, that p is not subordinate to h. Since $h \in Q$ with $E(h) = \phi$, there exists $z_0 \in \Delta\backslash\{0\}$ and $\zeta_0 \in \mathbb{T}$ such that $p(z_0) = h(\zeta_0)$ and for some $m \geq 1$,

$$z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) = \frac{4}{3} m\zeta_0(1+\zeta_0).$$

Now ,

$$\frac{p(z_0)\left[p(z_0) + \dfrac{z_0 p'(z_0)}{\delta p(z_0)+\gamma}\right]}{p(z_0) + (1-\beta)\dfrac{z_0 p'(z_0)}{\delta p(z_0)+\gamma}} - 1 = \frac{p(z_0)[p(z_0)-1][\delta p(z_0)+\gamma] + z_0 p'(z_0)[p(z_0)-(1-\beta)]}{p(z_0)[\delta p(z_0)+\gamma] + (1-\beta) z_0 p'(z_0)}$$

$$= \frac{\left(1+\dfrac{4\zeta_0}{3}+\dfrac{2\zeta_0^2}{3}\right)\left[\delta\left(1+\dfrac{4\zeta_0}{3}+\dfrac{2\zeta_0^2}{3}\right)+\gamma\right]\left[\frac{4\zeta_0}{3}+\frac{2\zeta_0^2}{3}\right] + \frac{4}{3}m\zeta_0(1+\zeta_0)\left[\frac{4\zeta_0}{3}+\frac{2\zeta_0^2}{3}+\beta\right]}{\left(1+\dfrac{4\zeta_0}{3}+\dfrac{2\zeta_0^2}{3}\right)\left[\delta\left(1+\dfrac{4\zeta_0}{3}+\dfrac{2\zeta_0^2}{3}\right)+\gamma\right] + (1-\beta)\dfrac{4}{3}m\zeta_0(1+\zeta_0)}$$

$$= \frac{\zeta_0\left[\left(1+\dfrac{4\zeta_0}{3}+\dfrac{2\zeta_0^2}{3}\right)^2\delta + \gamma\left(1+\dfrac{4\zeta_0}{3}+\dfrac{2\zeta_0^2}{3}\right)\left[\frac{4}{3}+\frac{2\zeta_0}{3}\right] + \frac{4}{3}m(1+\zeta_0)\left[\frac{4\zeta_0}{3}+\frac{2\zeta_0^2}{3}+\beta\right]\right]}{\delta\left(1+\dfrac{4\zeta_0}{3}+\dfrac{2\zeta_0^2}{3}\right)^2 + \gamma\left(1+\dfrac{4\zeta_0}{3}+\dfrac{2\zeta_0^2}{3}\right) + (1-\beta)\dfrac{4}{3}m\zeta_0(1+\zeta_0)}.$$

Therefore,

$$\left|\frac{p(z_0)[p(z_0)+\dfrac{z_0p'(z_0)}{\delta p(z_0)+\gamma}]}{p(z_0)+(1-\beta)\dfrac{z_0p'(z_0)}{\delta p(z_0)+\gamma}}-1\right|$$

$$=\left|\frac{\zeta_0\left[\left(1+\frac{4\zeta_0}{3}+\frac{2\zeta_0^2}{3}\right)^2\delta+\gamma\left(1+\frac{4\zeta_0}{3}+\frac{2\zeta_0^2}{3}\right)\left[\frac{4}{3}+\frac{2\zeta_0}{3}\right]+\frac{4}{3}m(1+\zeta_0)\left[\frac{4\zeta_0}{3}+\frac{2\zeta_0^2}{3}+\beta\right]\right]}{\delta\left(1+\frac{4\zeta_0}{3}+\frac{2\zeta_0^2}{3}\right)^2+\gamma\left(1+\frac{4\zeta_0}{3}+\frac{2\zeta_0^2}{3}\right)+(1-\beta)\frac{4}{3}m\zeta_0(1+\zeta_0)}\right|$$

For each $m\geq 1$, $\beta\in(0,1]$, we can define

$$q_m(z)=\frac{\left[\left(1+\frac{4z}{3}+\frac{2z^2}{3}\right)^2\delta+\gamma\left(1+\frac{4z}{3}+\frac{2z^2}{3}\right)\left[\frac{4}{3}+\frac{2z}{3}\right]+\frac{4}{3}m(1+z)\left[\frac{4z}{3}+\frac{2z^2}{3}+\beta\right]\right]}{\delta\left(1+\frac{4z}{3}+\frac{2z^2}{3}\right)^2+\gamma\left(1+\frac{4z}{3}+\frac{2z^2}{3}\right)+(1-\beta)\frac{4}{3}mz(1+z)}$$

Clearly $q_m(1)<q_m(-1)$, since

$$q_m(1)=\frac{9\delta+6\gamma+\frac{8}{3}m(2+\beta)}{9\delta+3\gamma+\frac{8}{3}m(1-\beta)}$$

and

$$q_m(-1)=\frac{\delta+6\gamma}{\delta+3\gamma},$$

we have,

$$|q_m(z)|\geq q_m(1)=\frac{9\delta+6\gamma+\frac{8}{3}m(2+\beta)}{9\delta+3\gamma+\frac{8}{3}m(1-\beta)}. \tag{2.7}$$

Observe that ,the function $r(m)=q_m(1), m\geq 1$ is increasing as

$$r'(m)=\frac{24[(1+2\beta)\delta+9\gamma\beta]}{[9\delta+3\gamma+\frac{8}{3}m(1-\beta)]^2}>0.$$

Hence

$$|q_m(z)|\geq q_m(1)=r(m)\geq r(1).$$

From (2),we have

$$\left|\frac{p(z)\left[p(z)+\dfrac{zp'(z)}{\delta p(z)+\gamma}\right]}{p(z)+(1-\beta)\dfrac{zp'(z)}{\delta p(z)+\gamma}}-1\right|\geq\frac{9\delta+6\gamma+\frac{8}{3}(2+\beta)}{9\delta+3\gamma+\frac{8}{3}(1-\beta)}$$

which is a contradiction to (2.6). Therefore $p(z)\prec 1+\dfrac{4z}{3}+\dfrac{2z^2}{3}$. □

Remark 2.2. *The function $p_2(z)=1-0.0195z^5$ and $\delta=\gamma=1$ with $\beta=0$ we observe that the condition* (2.6) *is satisfied and hence using Theorem 2.2, we obtain $p_2(z)$ is subordinate to $1+\dfrac{4z}{3}+\dfrac{2z^2}{3}$, which is given in the figure 3.*

When $p=\frac{zf'(z)}{f(z)}$, for $f\in\mathcal{A}$ in the above theorem,we have the following result.

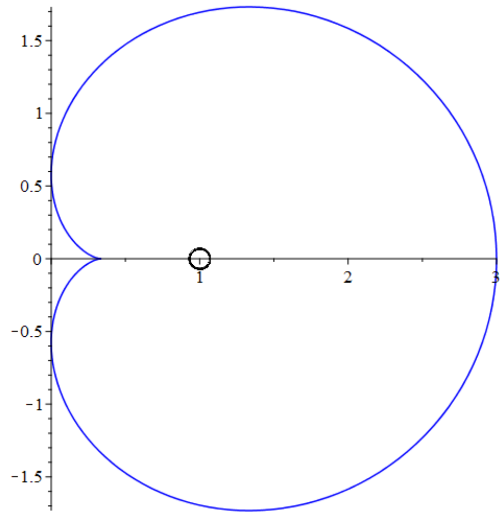

FIGURE 3. $p_2(z) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}$,

Corollary 2.2. *For $f \in \mathcal{A}$, and let $\beta = 1$,$\delta = 0$,$\gamma = 1$ and if*

$$\left| \frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 + 2\frac{zf'(z)}{f(z)} - 1 \right| < \frac{17}{3}$$

then $\frac{zf'(z)}{f(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}$.

Theorem 2.3. *If p is an analytic function with $p(0) = a$ and*

$$\left| \frac{p(z)[p(z) + \frac{zp'(z)}{\delta p(z)+\gamma}]}{p(z) + (1-\beta)\frac{zp'(z)}{\delta p(z)+\gamma}} - a \right| < a\frac{(\delta a\mathrm{e}^{\mathrm{M}} + \gamma)(\mathrm{e}^{\mathrm{M}} - 1) + \mathrm{M}(\mathrm{e}^{\mathrm{M}} - 1 + \beta)}{\delta a\mathrm{e}^{\mathrm{M}} + \gamma + (1-\beta)\mathrm{M}} \tag{2.8}$$

then $p(z) \prec a\mathrm{e}^{\mathrm{Mz}}$.

Proof. As the proof is similar to that of previous theorems , we omit the proof .

□

Remark 2.3. *The function $p_3(z) = 3 - 0.4816z^5, \delta = \gamma = M = 1$ with $\beta = 0$ and $a = 3$ we observe that the condition* (2.8) *is satisfied and hence using Theorem 2.2, we obtain $p_3(z)$ is subordinate to $3e^z$, which is given in the figure 4.*

When $p(z) = \frac{zf'(z)}{f(z)}$,for $f \in \mathcal{A}$ in the above theorem,we have the following result.

Corollary 2.3. *For $f \in \mathcal{A}$, and let $\beta = 1, \delta = 0, \gamma = 1$ and if*

$$\left| \frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 + 2\frac{zf'(z)}{f(z)} - a \right| < a[e^M(1 + M) - 1]$$

then $\frac{zf'(z)}{f(z)} \prec ae^{Mz}$.

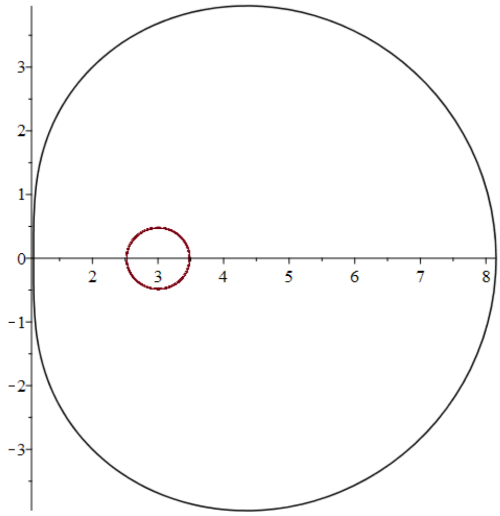

FIGURE 4. $p_3(z) \prec 3e^z$

Corollary 2.4. *For $f \in \mathcal{A}$ and $\beta = 1, \delta = 1, \gamma = 0$ and if*

$$\left|\frac{f''(z)}{f(z)}\right| < a(e^M - 1) + M$$

then $\frac{zf'(z)}{f(z)} \prec ae^{Mz}$.

Theorem 2.4. *Let p be an analytic function with $p(0) = 1$. For $-1 < B < A \leq 1$ if*

$$\left|\frac{p(z)\left[p(z) + \dfrac{zp'(z)}{\delta p(z) + \gamma}\right]}{p(z) + (1-\beta)\dfrac{zp'(z)}{\delta p(z) + \gamma}} - 1\right| < \tag{2.9}$$

$$(A - B)\frac{(1+A)^2\delta + (1+A)(1+B)\gamma + A - B + \beta(1+B)}{(1+B)[(1+A)^2\delta + (1+A)(1+B)\gamma + (A-B)(1-\beta)]},$$

then

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

Proof. Let $h(z) = \dfrac{1 + Az}{1 + Bz}$, $z \in \Delta$. Since $h(z)$ is univalent and $p(0) = h(0) = 1$, $p(z) \prec \dfrac{1 + Az}{1 + Bz}$ can be replaced by the inclusion $p(\Delta) \subset h(\Delta)$. Suppose on the contrary, that p is not subordinate to h, since $h \in Q$ with $E(h) = \phi$, there exists $z_0 \in \Delta\backslash\{0\}$ and $\zeta_0 \in \mathbb{T}$ such that $p(z_0) = h(\zeta_0)$ and for some $m \geq 1$, $z_0 p'(z_0) = m\zeta_0 \dfrac{A - B}{(1 + B\zeta_0)^2}$.

Now,

$$\frac{p(z_0)\left[p(z_0)+\dfrac{z_0p'(z_0)}{\delta p(z_0)+\gamma}\right]}{p(z_0)+(1-\beta)\dfrac{z_0p'(z_0)}{\delta p(z_0)+\gamma}}-1=\frac{p(z_0)[p(z_0)-1][\delta p(z_0)+\gamma]+z_0p'(z_0)[p(z_0)-1+\beta]}{p(z_0)[\delta p(z_0)+\gamma]+(1-\beta)z_0p'(z_0)}$$

$$=\frac{\left(\dfrac{1+A\zeta_0}{1+B\zeta_0}\right)\left(\delta\dfrac{1+A\zeta_0}{1+B\zeta_0}+\gamma\right)\left(\dfrac{1+A\zeta_0}{1+B\zeta_0}-1\right)+m\zeta_0\dfrac{A-B}{(1+B\zeta_0))^2}\left(\dfrac{1+A\zeta_0}{1+B\zeta_0}-1+\beta\right)}{\left(\dfrac{1+A\zeta_0}{1+B\zeta_0}\right)\left(\delta\dfrac{1+A\zeta_0}{1+B\zeta_0}+\gamma\right)+(1-\beta)m\zeta_0\dfrac{A-B}{(1+B\zeta_0)^2}}$$

$$=\frac{(A-B)\zeta_0[\delta(1+A\zeta_0)^2+\gamma(1+A\zeta_0)(1+B\zeta_0)+m((A-B)\zeta_0+\beta(1+B\zeta_0))]}{(1+B\zeta_0)[\delta(1+A\zeta_0)^2+\gamma(1+A\zeta_0)(1+B\zeta_0)+m\zeta_0(A-B)(1-\beta)]}.$$

Therefore,

$$\left|\frac{p(z)\left[p(z)+\dfrac{zp'(z)}{\delta p(z)+\gamma}\right]}{p(z)+(1-\beta)\dfrac{zp'(z)}{\delta p(z)+\gamma}}-1\right| \tag{2.10}$$

$$=(A-B)\left|\frac{(1+A\zeta_0)^2\delta+(1+A\zeta_0)(1+B\zeta_0)\gamma+m[(A-B)\zeta_0+\beta(1+B\zeta_0)]}{(1+B\zeta_0))[\delta(1+A\zeta_0)^2+\gamma(1+A\zeta_0)(1+B\zeta_0)+m\zeta_0(A-B)(1-\beta)]}\right|.$$

Let

$$q_m(z)=\frac{(1+Az)^2\delta+(1+Az)(1+Bz)\gamma+m[(A-B)z+\beta Bz+\beta]}{[(1+Az)^2\delta+(1+Az)(1+Bz)\gamma+(A-B)(1-\beta)mz](1+Bz)}.$$

Clearly $q_m(1)<q_m(-1)$ as we have

$$q_m(1)=\frac{(1+A)^2\delta+(1+A)(1+B)\gamma+m[(A-B)+\beta B+\beta]}{[(1+A)^2\delta+(1+A)(1+B)\gamma+(A-B)(1-\beta)m](1+B)}$$

and

$$q_m(-1)=\frac{(1-A)^2\delta+(1-A)(1-B)\gamma+m[-(A-B)-\beta B+\beta]}{[(1-A)^2\delta+(1-A)(1-B)\gamma-(A-B)(1-\beta)m](1-B)}.$$

Thus

$$|q_m(z)|\geq\frac{(1+A)^2\delta+(1+A)(1+B)\gamma+m[(A-B)+\beta B+\beta]}{[(1+A)^2\delta+(1+A)(1+B)\gamma+(A-B)(1-\beta)m](1+B)}.$$

The function $r(m)=q_m(1), m\geq 1$ is increasing, since

$$r'(m)=\frac{\beta(1+A)[(1+A)^2\delta+(1+A)(1+B)\gamma]}{(1+B)\left[(1+A)^2\delta+(1+A)(1+B)\gamma+m(A-B)(1-\beta)\right]^2},$$

which implies $r'(m)\geq 0,$ for $m\geq 1$. Hence $|q_m(z)|\geq q_m(1)=r(m)\geq r(1)$. Using (2.10) , we have

$$\left|\frac{p(z)\left[p(z)+\dfrac{zp'(z)}{\delta p(z)+\gamma}\right]}{p(z)+(1-\beta)\dfrac{zp'(z)}{\delta p(z)+\gamma}}-1\right| \tag{2.11}$$

$$\geq(A-B)\frac{(1+A)^2\delta+(1+A)(1+B)\gamma+A-B+\beta(1+B)}{(1+B)[(1+A)^2\delta+(1+A)(1+B)\gamma+(A-B)(1-\beta)]},$$

which is a contradiction to (2.9). Thus we have

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

□

Remark 2.4. *The function* $p_4(z) = 1 - 0.005z^6, \delta = \gamma = 1$ *with* $\beta = 0$ *and* $A = 0.35, B = 0.2$ *we observe that the condition* (2.9) *is satisfied and hence using Theorem 2.4, we obtain* $p_4(z)$ *is subordinate to* $\dfrac{1+0.35z}{1+0.2z}$, *which is given in the following figure,*

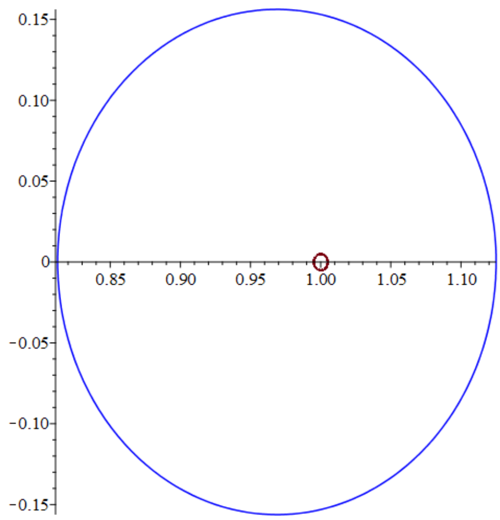

FIGURE 5. $p_4(z) \prec \dfrac{1+0.35z}{1+0.2z}$

For the choice $A = 1, B = 0$ and $\delta = 0$ in Theorem 2.4 , we have the following corollary.

Corollary 2.5. *Let* p *be an anlytic function with* $p(0) = 1$. *If*

$$\left| \frac{p(z)\left[p(z) + \dfrac{zp'(z)}{\gamma}\right]}{p(z) + (1-\beta)\dfrac{zp'(z)}{\gamma}} - 1 \right| < \frac{2\gamma+\beta+1}{2\gamma-\beta+1}$$

then

$$\left|p(z) - 1\right| < 1.$$

Hence $p(z)$ *is in the disc with centre and radius 1.*

Remark 2.5. *Under the assumptions,* $0 \in \overline{h(\Delta)}$ *and* $\dfrac{2\gamma+\beta+1}{2\gamma-\beta+1} \geq 1$, *we shall discuss the best dominant. Rewritting Theorem 2.4, interms of the differential subordination. we have the following:*

Corollary 2.6. *Let* $\alpha, \beta \in (0,1]$ *and* p *be an analytic function with* $p(0) = 1$. *If*

$$\frac{p(z)[p(z) + \alpha zp'(z)]}{p(z) + (1-\beta)\alpha zp'(z)} \prec 1 + \left(1 - \frac{2\beta}{2\gamma+\beta+1}\right) z, \tag{2.12}$$

then

$$p(z) \prec q(z) := 1 + \left(\frac{1 - \dfrac{2\beta}{2\gamma+\beta+1}}{1+\alpha\beta} \right) z. \tag{2.13}$$

Moreover, the function q is the best dominant of (2.12).

Proof. We shall prove that q is the best dominant of (2.12). In order to obtain the required result, we will find an univalent analytic solution q for the following differential equation

$$\frac{q(z)[q(z)+\alpha z q'(z)]}{q(z)+(1-\beta)\alpha z q'(z)} = 1 + \left(1 - \frac{2\beta}{2\gamma+\beta+1}\right) z, \qquad z \in \Delta. \tag{2.14}$$

such that $q(0) = 1$. We apply the technique of power series to find the analytic solution of (2.14) of the form

$$q(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + ..., \qquad z \in \Delta.$$

Since q is required to be univalent, we have $a_1 = q'(0) \neq 0$.
From (2.14), we have

$$q(z)[q(z)-1] + \alpha z q'(z)[q(z)-(1-\beta)] = \left(1 - \frac{2\beta}{2\gamma+\beta+1}\right)[zq(z) + \alpha(1-\beta)z^2 q'(z)].$$

Using $q(z)$ and $q'(z)$ in the above equation and simplifing, we have

$$(1+\alpha\beta)a_1 z + (a_2 + a_1^2(1+\alpha) + 2\beta a_2)z^2 + (a_3 + a_1 a_2(2+3\alpha) + 3a_3\beta)z^3 + \dots$$
$$= \frac{1 - \dfrac{2\beta}{2\gamma+\beta+1}}{1+\alpha\beta} z + \frac{1 - \dfrac{2\beta}{2\gamma+\beta+1}}{1+\alpha(1-\beta)}(1+\alpha(1-\beta))a_1 z^2 + \frac{1 - \dfrac{2\beta}{2\gamma+\beta+1}}{1+\alpha(1-\beta)}(1+2\alpha(1-\beta))a_2 z^3 + \dots$$

Comparing the coefficients on both sides ,we get

$$(1+\alpha\beta)a_1 = 1 - \frac{2\beta}{2\gamma+\beta+1}, \tag{2.15}$$

$$a_2(1+2\beta) + a_1^2(1+\alpha) = \left(1 - \frac{2\beta}{2\gamma+\beta+1}\right)[1+\alpha(1-\beta)]a_1, \tag{2.16}$$

$$a_3(1+3\beta) + a_1 a_2(2+3\alpha) = \left(1 - \frac{2\beta}{2\gamma+\beta+1}\right)[1+2\alpha(1-\beta)]a_2, \tag{2.17}$$

$$a_4(1+4\beta) + a_1 a_3(2+4\alpha) + a_2^2(1+2\alpha) = \left(1 - \frac{2\beta}{2\gamma+\beta+1}\right)[1+3\alpha(1-\beta)]a_3, \tag{2.18}$$

and so on. In general, for $n = 2k-1$, $k \geq 2$

$$\begin{aligned} a_{2k-1}(1+(2k-1)\beta) + (2+(2k-1)\alpha)(a_1 a_{2k-2} + a_2 a_{2k-3} + \cdots + a_{k-1}a_k) \\ = \left(1 - \frac{2\beta}{2\gamma+\beta+1}\right)(1+(2k-2)\alpha(1-\beta)a_{2k-2}, \end{aligned} \tag{2.19}$$

and for $n = 2k$, $k \geq 2$,

$$\begin{aligned} a_{2k}(1+2k\beta) + (2+2k\alpha)(a_1 a_{2k-1} + a_2 a_{2k-2} + \cdots + a_{k-1}a_{k+1} + \frac{1}{2}a_{k^2}) \\ = \left(1 - \frac{2\beta}{2\gamma+\beta+1}\right)(1+(2k-1)\alpha(1-\beta)a_{2k-1}. \end{aligned} \tag{2.20}$$

From (2.15), we have

$$a_1 = \frac{1 - \dfrac{2\beta}{2\gamma+\beta+1}}{1+\alpha\beta}.$$

Substituting a_1 in (2.16) we obtain $a_2 = 0$. And then using $a_2 = 0$ in (2.16), we obtain $a_3 = 0$.

By Mathematical Induction, we prove that

$$a_2 = a_3 = a_4 = \cdots = a_{2k-2} = 0.$$

and hence the equation (2.19) reduces to

$$a_{2k-1}(1+(2k-1)\beta) = 0 \implies a_{2k-1} = 0.$$

Also, the equation (2.20) becomes

$$a_{2k}(1+2k\beta) = 0 \implies a_{2k} = 0.$$

Thus we have proved that $a_n = 0$, for all $n \geq 2$. In this way, we obtain

$$q(z) = 1 + \left(\frac{1 - \dfrac{2\beta}{2\gamma+\beta+1}}{1+\alpha\beta} \right) z,\; z \in \Delta,$$

is the unique analytic univalent solution of (2.13). □

The proof of the following theorem is similar to that of Theorem 2.4 and hence the proof can be omitted.

Theorem 2.5. *Let p be an analytic function with $p(0) = a$ and if*

$$\left| \frac{p(z)\left[p(z) + \dfrac{zp'(z)}{\delta p(z)+\gamma}\right]}{p(z) + (1-\beta)\dfrac{zp'(z)}{\delta p(z)+\gamma}} - a \right| < M \frac{(a+M)[(a+M)\delta+\gamma] + M + a\beta}{(a+M)[(a+M)\delta+\gamma] + M(1-\beta)}, \tag{2.21}$$

then

$$p(z) \prec a + Mz$$

.

Remark 2.6. *The function $p_5(z) = 1 - 0.08532z^6, \delta = \gamma = 1$ with $\beta = 0.3$ and $a = M = 1$ we observe that the condition (2.21) is satisfied and hence using Theorem 2.5, we obtain $p_5(z)$ is subordinate to $1+z$, which is given in Figure 6.*

Remark 2.7. *(i) By taking $a = 0$ in Theorem 2.5, we have the result obtained in* [1].
(ii) For $\delta = 0, \gamma = 1$, we have the result obtained in ([1]*,Corollary 2.3).*

3. Conclusions

Throughout the Remarks 2.1, 2.2, 2.3, 2.4 and 2.6 we have shown that the sufficient conditions involving harmonic means of analytic functions have been satisfied by each of the functios, $p_i(z), i = 1, 2, \ldots, 5$ such that they are subordinate to well known Ma-Minda type functions like $\sqrt{1+z}, 1 + \frac{4z}{3} + \frac{2z^2}{3}, ae^{Mz}, (1+Az)/(1+Bz)$ and $a + Mz$ respectively for particular values of a, M, A and B, hence we claim that sets of functions satisfying the mentioned conditions [Condition (2.3), (2.6), (2.8), (2.9) and (2.21) of Theorem 2.1, 2.2, 2.3, 2.4 and 2.5 respectively] are non -empty.

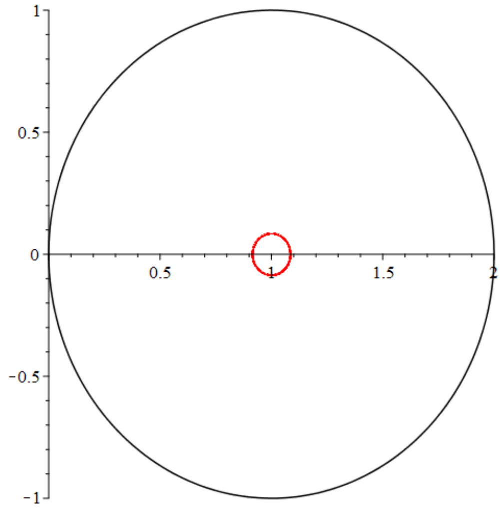

FIGURE 6. $p_5(z) \prec 1 + z$

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