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Impulsive Integro-Differential Equations via Densifiability Techniques

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ABSTRACT. This article employs a new fixed-point theorem that relies on the degree of nondensifiability to explore the existence of solutions to integro-differential equations within Banach spaces. Additionally, an example is presented to validate the core findings. Furthermore, the results obtained in this research enhance and broaden certain prior results in this field.

1. INTRODUCTION

Integro-differential equations are widely used in various fields, including physics, population dynamics, electrical engineering, finance, biology, ecology, and sociology. Researchers have extensively examined the qualitative features of integro-differential equations, including existence, uniqueness, controllability, and stability. (See for instance, [18, 19, 21, 7, 12, 30, 3, 5, 4, 14]).

Impulsive differential equations and inclusions have many applications in physics and engineering, making them a popular topic (Ballinger and Liu. In [6]). Impulsive issues are useful for explaining processes that change quickly and cannot be represented using classical differential equations. Many authors have studied mild solutions to instantaneous impulsive differential equations and inclusions, including Benchohra *et al.* [9, 1, 8, 10, 11, 31], Ravichandran and Trujillo [37], Shu *et al.* [39], and Wang *et al.* [40]. The nonlocal condition improves solution resolution and precision for physical measurements compared to the ordinary condition $y(0) = y_0$. The nonlocal condition can describe things that ordinary initial value problems cannot, such as population dynamics under rapid changes (e.g. harvesting, illnesses). For further remarks and citations, see to [13, 29, 38, 33] and the references therein.

This manuscript draws significant inspiration from the studies conducted in [32, 33], where the authors extensively explored nonlocal problems related to integro-differential equations and nonlocal impulsive problems for nonlinear differential equations within Banach spaces. The arguments presented in this paper, in fact, extend the findings of [32] to a broader range of impulsive systems. Leveraging the compactness of the resolvent operator $\{R(t)\}_{t\geq 0}$ (refer to the details below), we have derived an intermediate result crucial for establishing the existence of mild solutions, subject to various conditions on the provided data, for a class of Volterra integro-differential equations.

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In particular, Ezzinbi *et al*. [20] studied the existence of mild solutions for a class of nonlinear impulsive integro-differential equations with a nonlocal initial condition:

$$\begin{cases} y'(t) = Zy(t) + N(t, y(t) + \int_0^t H(t-\varrho)y(\varrho)d\varrho, \text{ if } 0 \le t \le T, t \neq t_j, \\ y(0) + g(y) = y_0, \\ y(t_j^+) - y(t_j^-) = I_j(y(t_j)), j = 1, 2, \ldots, p, 0 < t_1 < t_2 < \cdots < t_p < T, \end{cases}$$

where Z generates a C_0-semigroup on a Banach space E, $H(t)$ is a closed linear operator on E with time independent domain $D(Z) \subset D(H)$. $N : [0,T] \times E \to E$ and $g : PC([0,T], E) \to E$ are continuous functions where the set $PC([0,T], E)$ is a Banach space.

In the 1980s [15], the authors introduced the concept of ϑ-dense curves. Cherruault [16] and Mora [34] were primarily responsible for its creation. Mora and Mira [35] established the notion of degree of nondensifiability (DND), which is based on ϑ-dense curves. García [26, 24] demonstrated a novel fixed-point result using the DND that is more applicable than the Darbo fixed point theorem and its generalizations.

In [17], Benchohra *et al* studied the following problem:

$$\begin{cases} {}^C D_{t_1^+}^{\zeta,\Psi} y(t) = N(t, y(t)); \quad t \in D := [t_1, t_2], \\ y(t_1) = y(t_2) = \theta, \end{cases}$$

where $\zeta \in (1,2], {}^C D_{t_1^+}^{\zeta,\Psi}$ is the Ψ-Caputo fractional derivative, $N : D \times E \to E$ is a given function, θ is the null vector in the space E.

Our findings fundamentally rely on the notable contributions made by García [26, 25] in investigating the existence of mild solutions for impulsive integro-differential equations within Banach spaces:

$$(1.1) \quad \begin{cases} y'(t) = Z(t)y(t) + N(t, y(t)) + \int_0^t \gamma(t,\varrho)y(\varrho)d\varrho, \text{ if } t \in \Theta, t \neq t_j \\ y(0) = y_0 \in E, \\ y(t_j^+) - y(t_j^-) = I_j(y(t_j)), \quad j = 1, 2, \cdots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < T, \end{cases}$$

where $\Theta = [0,T], N : \Theta \times E \to E$ is a continuous function, $(E, \|\cdot\|_E)$ is a Banach space, and $Z(t) : G(Z(t)) \subset E \to E, \gamma(t,\varrho)$ are closed linear operators on E, with dense domain $G(Z(t))$, which is independent of t, and $G(Z) \subset G(\gamma(t,\varrho))$.

Next, we investigate the existence of mild solutions for impulsive integro-differential equations with a nonlocal initial condition:

$$(1.2) \quad \begin{cases} y'(t) = Z(t)y(t) + N(t, y_t) + \int_0^t \gamma(t,\varrho)y(\varrho)d\varrho, \text{ if } t \in \Theta, t \neq t_j, \\ y(0) + g(y) = y_0 \in E, \\ y(t_j^+) - y(t_j^-) = I_j(y(t_j)), \quad j = 1, 2, \cdots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < T, \end{cases}$$

where $g : PC(\Theta, E) \to E$ is continuous function and the set $PC(\Theta, E)$ is given later.
This paper is organized as follows. In Section 2, some necessary concepts and important definitions and lemmas are given. In Section 3, we show the existence of mild solutions for impulsive integro-differential equations with local and nonlocal initial conditions for the problems (1.1) and (1.2) by applying a novel fixed point theorem based on DND. An example is also given in Section 4 to illustrate the theory of the abstract main result.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts that used in the remainder of this paper.
Let E be a real Banach space with the norm $\|\cdot\|_E$ and M_E is the class of non-empty and bounded subsets of E, let $B(E)$ be the space of all bounded linear operators from E into E, with the norm

$$\|T\|_{B(E)} = \sup_{y\in E} \|T(y)\|_E.$$

We denote by $(L^1(\Theta, E), \|\cdot\|_1)$ the Banach space of measurable functions that are Bochner integrable from $\Theta := [0, T]$ into E, with the norm

$$\|y\|_1 = \int_0^T \|y(t)\|_E dt.$$

$L^\infty(\Theta, E)$, is the Banach space of measurable functions which are essentially bounded, with the norm

$$\|\cdot\|_\infty = \inf\{C > 0 : \|y(t)\|_E \leq C, \text{ a.e. } t \in \Theta\}.$$

By $C(\Theta, E)$ we denote the Banach space of all continuous functions from Θ into E with

$$\|y\| = \sup_{t\in\Theta} \|y(t)\|_E.$$

We consider the following linear Cauchy problem

$$\begin{cases} y'(t) = Z(t)y(t) + \int_0^t \gamma(t,\varrho)y(\varrho)d\varrho, & \text{for } t \geq 0, \\ y(0) = y_0 \in E. \end{cases} \tag{2.3}$$

Definition 2.1. ([27]) *A resolvent operator for a Cauchy problem (2.3) is a bounded linear operator-valued function $R \in B(E)$ for $t \geq 0$, verifying the following conditions:*

(1) *$R(0) = I$ (the identity map of E) and $\|R(t)\|_{B(E)} \leq Me^{\eta t}$ for some constants $M > 0$ and $\eta \in \mathbb{R}$.*
(2) *For each $y \in E$, $t \to R(t)y$ is strongly continuous for $t \geq 0$.*
(3) *$R \in B(E)$ for $t \geq 0$. For $y \in E$, $R(\cdot)y \in C^1(\mathbb{R}_+, E) \cap C(\mathbb{R}_+, E)$ and*

$$\begin{aligned} R'(t)y &= Z(t)R(t)y + \int_0^t \gamma(t,\varrho)R(\varrho)yd\varrho \\ &= R(t)Z(t)y + \int_0^t R(t,\varrho)\gamma(\varrho)yd\varrho, \end{aligned}$$

for $t \geq 0$.

From now on, we assume that:

$(P1)$ The operator Z is the infinitesimal generator of a uniformly continuous semigroup $\{T(t)\}_{t>0}$.
$(P2)$ For $t \geq 0$, $\gamma(t,\varrho)$ is closed linear operator from $G(Z)$ to E and $\gamma(t,\varrho) \in B(E)$. For any $y \in E$, the map $t \to \gamma(t,\varrho)y$ is bounded, differentiable and the derivative $t \to \gamma'(t,\varrho)y$ is bounded uniformly continuous on $\mathbb{R}_+$.

The following theorem gives a satisfactory answer to the problem of existence of resolvent to (2.3).

Theorem 2.1. ([28]) *Assume that $(P1) - (P2)$ hold, then there exists a unique resolvent operator for the Cauchy problem (2.3).*

Definition 2.2. ([34, 36]) *Suppose that $\vartheta \geq 0$ and $\Bbbk \in M_E$, a continuous mapping $\zeta : \phi := [0, 1] \to E$ is an $\vartheta-$dense curve in $\Bbbk$ if:*

- $\zeta(\phi) \subset \Bbbk$.
- *For any* $y_1 \in \Bbbk$, *there is* $y_2 \in \zeta(\phi)$ *such that* $\|y_1 - y_2\|_E \leq \vartheta$.

Ff for $\vartheta > 0$, *there is an* $\vartheta-$*dense curve in* $\Bbbk$, *then* $\Bbbk$ *is densifiable.*

Definition 2.3. *(*[35, 22]*) Let* $\vartheta > 0$, *and denote by* $\Gamma_{\vartheta,\Bbbk}$ *the class of all* $\vartheta-$*dense curves in* $\Bbbk \in M_E$. *The DND is a mapping* $\varkappa : M_E \to \mathbb{R}_+$ *defined as:*

$$\varkappa(\Bbbk) = \inf\{\vartheta \geq 0 : \Gamma_{\vartheta,\Bbbk} \neq \varnothing\},$$

for each $\Bbbk \in M_E$.

Remark 2.1. *It is important to highlight that a thorough examination of the degree of nondensifiability (DND) was conducted in* [22]. *Specifically, the study established that the DND does not function as a measure of noncompactness* [22]. *Nonetheless, it exhibits characteristics remarkably akin to those of MNC (see Proposition 2.6 in* [22]*).*

Lemma 2.1 ([23, 22])**.** *Let* $\Bbbk_1, \Bbbk_2 \in M_E$. *Then, we have:*

(a) $\varkappa(\Bbbk_1) = 0 \iff \Bbbk_1$ *is a precompact set, for each nonempty, bounded and arc-connected subset* $\Bbbk_1$ *of* E.
(b) $\varkappa(\bar{\Bbbk_1}) = \varkappa(\Bbbk_1)$, *where* $\bar{\Bbbk_1}$ *denotes the closure of* $\Bbbk_1$.
(c) $\varkappa(\lambda\Bbbk_1) = |\lambda|\varkappa(\Bbbk_1)$, *for* $\lambda \in \mathbb{R}$.
(d) $\varkappa(x + \Bbbk_1) = \varkappa(\Bbbk_1)$, *for all* $x \in E$.
(e) $\varkappa(Conv\Bbbk_1) \leq \varkappa(\Bbbk_1)$ *and* $\varkappa(Conv\Bbbk_1 \cup \Bbbk_2) \leq \max\{\varkappa(Conv\Bbbk_1), \varkappa(Conv\Bbbk_2)\}$, *where* $\varkappa(Conv\Bbbk_1)$ *represent the convex hull of* $\Bbbk_1$.
(f) $\varkappa(\Bbbk_1 + \Bbbk_2) \leq \varkappa(\Bbbk_1) + \varkappa(\Bbbk_2)$.

Let

$$X = \left\{ \begin{array}{c} \varpi : \mathbb{R}_+ \to \mathbb{R}_+ : \varpi \text{ is monotone increasing} \\ \text{and } \lim\limits_{n\to\infty} \varpi^n = 0 \text{ for any } t \in \mathbb{R}_+ \end{array} \right\},$$

where $n \in \mathbb{N}$ and $\varpi^n(t)$ denotes the $n-$th composition of ϖ with itself.

Remark 2.2. *It is important to observe that the fixed point theorem based on DND in* [26] *takes a form closely resembling the renowned Darbo fixed-point theorem* [2]. *Nevertheless, as demonstrated in* [26, 23] *through various examples, both outcomes are fundamentally distinct. The presented theorem in* [26] *operates under more inclusive conditions than the Darbo fixed-point theorem or its well-known generalizations.*

Lemma 2.2. *(*[26]*) Let* $\Bbbk \subset C(\Theta, E)$ *be non-empty and bounded. Then:*

$$\sup_{t\in\Theta} \varkappa(\Bbbk(t)) \leq \varkappa(\Bbbk).$$

3. Existence of Mild Solutions for Integro-Differential Equations

Let $PC \equiv PC(\Theta, E)$ be the set of all function y from Θ into E such that y is continuous at $t \neq t_j$ and left continuous at $t = t_j$ and the right limit $y(t_j^+)$ exists for $j = 1, 2, \cdots, p$. We recall from [32] that $PC(\Theta, E)$ is a Banach space with the following norm

$$\|y\|_{PC} = \sup_{t\in\Theta} \|y(t)\|_E.$$

Definition 3.4. *We say that a function* $y(\cdot) \in PC(\Theta, E)$ *is a mild solution of problem (1.1), if* y *satisfies the following integral equation*

$$y(t) \quad = \quad R(t,0)y_0 + \int_0^t R(t,\varrho)N(\varrho, y(\varrho))d\varrho$$

$$+ \sum_{0<t_j<t} R(t,t_j)I_j(y(t_j)), \quad \textit{for each } t \in \Theta.$$

Now, we assume the following hypotheses:

$(H1)$ The function $N : \Theta \times E \to E$ satisfies the Carathéodory conditions, and there exist $p_f \in L^1(\Theta, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a nondecreasing continuous function such that:

$$\|N(t,y)\|_E \leq p_f(t)\psi(\|y\|_E), \text{ for } y \in E, \text{ and for a.e. } t \in \Theta.$$

$(H2)$ The functions $I_j : PC(\Theta, E) \to E$ are continuous and there exist constant $L_j > 0$, $j = 1, 2, \cdots, p$, such that for any $y \in E$

$$\|I_j(y)\|_E \leq L_j\|y\|_{PC}.$$

$(H3)$ The resolvent operator is uniformly continuous and there exist $M \geq 1$ such that

$$\|R(t,\varrho)\|_{B(E)} \leq M, \text{ for every } t \in \Theta.$$

$(H4)$ There exist $K \in L^\infty(\Theta, \mathbb{R}_+)$ and $h \in X$ where for any non-empty, bounded and convex subset $\Bbbk \subset E$,

$$\varkappa(N(t,\Bbbk)) \leq K(t)h(\varkappa(\Bbbk)),$$

holds for a.e. $t \in \Theta$.

$(H5)$ There exists $r > 0$ such that

$$r \geq M\left[r + \psi(r)\|p_f\|_{L^1} + r\sum_{j=1}^{p} L_j\right].$$

Theorem 3.2. *Assume that the conditions $(H1) - (H5)$ are satisfied and that*

$$TM\|K\|_\infty \leq 1.$$

Thus, (1.1) has at least one solution defined on Θ.

Proof. Firstly, define the operator

$$\begin{aligned} Uy(t) &= R(t,0)y_0 + \int_0^t R(t,\varrho)N(\varrho, y(\varrho))d\varrho \\ &+ \sum_{0<t_j<t} R(t,t_j)I_j(y(t_j)), \quad \text{for each } t \in \Theta. \end{aligned}$$

We consider the set

$$\widehat{\Bbbk} = \Big\{y \in PC : \|y\|_{PC} \leq r\Big\}.$$

We note that $\widehat{\Bbbk}$ is bounded, closed and convex subset.

Step 1 : We prove that $U\widehat{\Bbbk} \subset \widehat{\Bbbk}$.

Indeed for any $y \in \widehat{\Bbbk}$ and under $(H_1) - (H_5)$ we obtain

$$\begin{aligned} \|Uy(t)\|_E &= \|R(t,0)y_0 + \int_0^t R(t,\varrho)N(\varrho,y(\varrho))d\varrho + \sum_{0<t_j<t} R(t,t_j)I_j(y(t_j))\|_E \\ &\leq \|R(t,0)\|_{B(E)}\|y_0\|_E + \int_0^t \|R(t,\varrho)\|_{B(E)}\|N(\varrho,y(\varrho))\|_E d\varrho \\ &+ \sum_{0<t_j<t} \|R(t,t_j)\|_{B(E)}\|I_j(y(t_j))\|_E \end{aligned}$$

$$\leq M\|y_0\|_E + M\int_0^t p_f(\varrho)\psi(\|y(t)\|_E)d\varrho + M\sum_{0<t_j<t} L_j\|y(t_j)\|_E$$
$$\leq Mr + M\psi(r)\|p_f\|_{L^1} + Mr\sum_{j=1}^{p} L_j$$
$$\leq r.$$

Thus $U(\widehat{\mathbb{k}}) \subset \widehat{\mathbb{k}}$. By (H_1) and the Lebesgue dominated convergence theorem, U is continuous on $\widehat{\mathbb{k}}$.

Step 2 : We prove that U satisfies the contractive condition.
Let $\digamma$ be any non-empty and convex subset of $\widehat{\mathbb{k}}$, and for each $t \in \Theta$, let $\vartheta_t = \varkappa(\digamma(t))$. Then $K \in L^\infty(\Theta, \mathbb{R}_+)$ and $h \in X$ where for a.e $t \in \Theta$,

$$\varkappa(N(t, \digamma(t))) \leq K(t)h(\varkappa(\vartheta_t)).$$

Therefore, for $\varepsilon \leq 0$, there is a continuous mapping $\zeta_t : \phi \to E$, with $\zeta_t(\phi) \subset N(t, \digamma(t))$, such that for all $y \in \digamma$, there is $\eta \in \phi$ with

$$\|N(t, y(t)) - \zeta_t(\eta)\|_E \leq K(t)h(\vartheta_t) + \varepsilon, \text{ for a.e } t \in \Theta. \tag{3.4}$$

Construct now the mapping $\tilde{\zeta} : \phi \to ((C(\Theta, E)), \|\cdot\|_\infty)$ as follows:

$$\eta \in \phi \to \tilde{\zeta}(\eta, t) = R(t,0)y_0 + \int_0^t R(t,\varrho)\zeta_\varrho(\eta)d\varrho$$
$$+ \sum_{0<t_j<t} R(t,t_j)I_j(y(t_j)), \text{ for a.e } t \in \Theta.$$

So, $\tilde{\zeta}$ is continuous and $\tilde{\zeta}(\phi) \subset U(\digamma)$. By (3.4), given $y \in \digamma$ we have $\eta \in \phi$ where

$$\|Uy(t) - \tilde{\zeta}_t(\eta)\|_E \leq \int_0^t \|R(t,\varrho)\|_{B(E)}\|N(\varrho, y(\varrho)) - \zeta_\varrho(\eta)\|d\varrho$$
$$\leq M\int_0^t K(\varrho)h(\vartheta_\varrho) + \varepsilon d\varrho.$$

Setting $\vartheta := \varkappa(\digamma)$, we can deduce that $h(\vartheta_t) \leq h(\vartheta)$ for a.e $t \in \Theta$, and

$$\|Uy(t) - \tilde{\zeta}_t(\eta)\|_E \leq TM\|K\|_\infty h(\vartheta)$$
$$\leq h(\vartheta).$$

Thus, from the arbitrariness of $t \in \Theta$, that $\varkappa(U\digamma) \leq h(\vartheta)$. □

4. Integro-Differential Equations with Nonlocal Condition

Definition 4.5. *We say that a function $y(\cdot) \in PC(\Theta, E)$ is a mild solution of problem (1.2), if y satisfies the following integral equation*

$$y(t) = R(t,0)[y_0 - g(y)] + \int_0^t R(t,\varrho)N(\varrho, y(\varrho))d\varrho$$
$$+ \sum_{0<t_j<t} R(t,t_j)I_j(y(t_j)), \quad \textit{for each } t \in \Theta.$$

Now, we assume the following hypotheses:

$(C1)$ The function $g : PC \to E$ is continuous, and there exists a constant $L_2 > 0$ such that

$$\|g(y)\|_E \leq L_2 \|y\|_{PC}, \text{ for } y \in PC.$$

$(C2)$ There exists $r > 0$ such that

$$r \geq M\left[r + L_2 r + \psi(r)\|p_f\|_{L^1} + r\sum_{j=1}^{p} L_j\right]$$

Theorem 4.3. *Assume that the conditions* $(H1) - (H4)$ *and* $(C1) - (C2)$ *are satisfied and that*

$$TM\|K\|_\infty \leq 1.$$

So, (1.2) has at least one solution defined on Θ.

Proof. Let

$$\begin{aligned} \mathcal{M}y(t) &= R(t,0)[y_0 - g(y)] + \int_0^t R(t,\varrho)N(\varrho, y(\varrho))d\varrho \\ &+ \sum_{0<t_j<t} R(t,t_j)I_j(y(t_j)), \quad \text{for each } t \in \Theta. \end{aligned}$$

Step 1 : We prove $\mathcal{M}\widehat{\mathbb{k}} \subset \widehat{\mathbb{k}}$.
For any $y \in \widehat{\mathbb{k}}$ we obtain

$$\begin{aligned} \|\mathcal{M}y(t)\|_E &= \|R(t,0)[y_0 - g(y)] + \int_0^t R(t,\varrho)N(\varrho,y(\varrho))d\varrho + \sum_{0<t_j<t} R(t,t_j)I_j(y(t_j))\|_E \\ &\leq \|R(t,0)\|_{B(E)}\|y_0 - g(y)\|_E + \int_0^t \|R(t,\varrho)\|_{B(E)}\|N(\varrho,y_\varrho)\|_E d\varrho \\ &+ \sum_{0<t_j<t} \|R(t,t_j)\|_{B(E)}\|I_j(y(t_j))\|_E \\ &\leq M[\|y_0\|_E + L_2\|y\|_{PC}] + M\int_0^t p_f(\varrho)\psi(\|y\|_E)d\varrho + M\sum_{0<t_j<t} L_1\|y(t_j)\|_E \\ &\leq Mr + ML_2 r + M\psi(r)\|p_f\|_{L^1} + Mr\sum_{j=1}^{p} L_j \\ &\leq r. \end{aligned}$$

Thus $\mathcal{M}(\widehat{\mathbb{k}}) \subset \widehat{\mathbb{k}}$. Furthermore, combining assumption (H_1) and the Lebesgue dominated convergence theorem, $\mathcal{M}$ is continuous on $\widehat{\mathbb{k}}$.

Step 2 : Let $\digamma \subset \widehat{\mathbb{k}}$, and for each $t \in \Theta$, let $\vartheta_t = \varkappa(\digamma(t))$. Thus, $K \in L^\infty(\Theta, \mathbb{R}_+)$ and $h \in X$ where for a.e $t \in \Theta$

$$\varkappa(N(t,\digamma(t))) \leq K(t)h(\varkappa(\vartheta_t)).$$

By the same technique of the step 2 in the Theorem 3.2, we get:
$\tilde{\zeta}$ is continuous and $\tilde{\zeta}(\tilde{t}) \subset \mathcal{M}(\digamma)$. By (3.4), given $y \in \digamma$ we can find $\eta \in \tilde{t}$ where

$$\begin{aligned} \|\mathcal{M}y(t) - \tilde{\zeta}_t(\eta)\| &\leq \int_0^t \|R(t,\varrho)\|_{B(E)}\|N(\varrho,y(\varrho)) - \zeta_\varrho(\eta)\|d\varrho \\ &\leq M\int_0^t K(\varrho)h(\vartheta_\varrho) + \varepsilon d\varrho. \end{aligned}$$

Setting $\vartheta := \varkappa(F)$, we can deduce that $h(\vartheta_t) \leq h(\vartheta)$ for a.e $t \in \Theta$, and

$$\begin{aligned}\|\mathcal{M}y(t) - \tilde{\zeta}_t(\eta)\| &\leq TM\|K\|_\infty h(\vartheta)\\ &\leq h(\vartheta).\end{aligned}$$

So, from the arbitrariness of $t \in \Theta$, that $\varkappa(\mathcal{M}F) \leq h(\vartheta)$.
Then y is a fixed point of $\mathcal{M}$, which is a mild solution of (1.2). □

5. An Example

Consider the following class of partial integro-differential system:

$$(5.5)\quad \begin{cases} \frac{\partial}{\partial t}z(t,\widetilde{y}) = \frac{\partial}{\partial \widetilde{y}}z(t,\widetilde{y}) - \int_0^t \Gamma(t-\varrho)\frac{\partial}{\partial \widetilde{y}}z(\varrho,\widetilde{y})d\varrho \\ \qquad\qquad -\frac{1}{1+e^t}\left(\frac{1}{1+t^2} + ln(1+|z(t,\widetilde{y})|)\right) \text{ if } t \in \Theta = [0,1] \text{ and } \widetilde{y} \in (0,1), \\ z(t,0) = z(t,1) = 0, \text{ for } t \in \Theta, \\ z(0,\widetilde{y}) = e^{\widetilde{y}}, \text{ for } \widetilde{y} \in (0,1)\ , \\ z(t_j^+,\widetilde{y}) - z(t_j^-,\widetilde{y}) = I_j(z(t_j,\widetilde{y})), \text{ for } \widetilde{y} \in (0,1), j = 1,2,\cdots,p. \end{cases}$$

Let Z be defined by

$$(Zz)(\widetilde{y}) = \frac{\partial}{\partial \widetilde{y}}z(t,\widetilde{y}).$$

And

$$G(Z) = \{z \in L^2(0,1) \ / \ z, \frac{\partial}{\partial \widetilde{y}}z \in L^2(0,1)\ ;\ z(0) = z(1) = 0\}.$$

The operator Z is the infinitesimal generator of a C_0-semigroup on $L^2(0,1)$ with domain $G(Z)$, and with more appropriate conditions on operator $\gamma(\cdot) = \Gamma(\cdot)Z$, the problem (5.5) has a resolvent operator $(R(t))_{t\geq 0}$ on $L^2(0,1)$ which is norm continuous.

Now, define

$$y(t)(\widetilde{y}) = z(t,\widetilde{y}),$$

$$N(t,y(t))(\widetilde{y}) = N(t,z(t))(\widetilde{y})$$

and $N : \Theta \times L^2(0,1) \longrightarrow L^2(0,1)$ given by

$$N(t,z(t))(\widetilde{y}) = \frac{1}{1+e^t}\left(\frac{1}{1+t^2} + ln(1+|z(t,\widetilde{y})|)\right), \quad \text{for } t \in \Theta,$$

Now, for $t \in \Theta$, we have

$$\begin{aligned}\|N(t,z(t))\|_{L^2} &= \|\frac{1}{1+e^t}\left(\frac{1}{1+t^2} + ln(1+|z(t,\widetilde{y})|)\right)\|_{L^2}\\ &\leq \frac{1}{1+e^t}(1+\|z(t,\widetilde{y})\|_{L^2})\\ &\leq p_f(t)\psi(\|z(t)\|_{L^2}).\end{aligned}$$

Therefore, assumption $(H1)$ is satisfied with

$$p_f(t) = \frac{1}{1+e^t},\ t \in \Theta \text{ and } \psi(\widetilde{y}) = 1+\widetilde{y},\ \widetilde{y} \in (0,1).$$

Now we shall check that condition of $(H5)$ is satisfied. Indeed, we have

$$r \geq Mr + M(1+r) + TML_1 r.$$

Thus

$$r \geq \frac{M}{1-3M}.$$

For $\digamma \subset C(\Theta, L^2(0,1))$ and $t \in \Theta$ fixed, let ζ be an ϑ_t-dense curve in $\digamma(t)$ for some $\vartheta_t \geq 0$. Then, for $z \in \digamma$, there is $\eta \in \phi$ satisfying:

$$\|y(t) - \zeta(\eta, t)\|_{L^2} \leq \vartheta_t.$$

Therefore, we have:

$$\begin{aligned}
\|N(t, z(t)) - N(t, \zeta(\eta, t))\|_{L^2} &\leq \frac{1}{1+e^t}\|ln(1+|z(t,\widetilde{y})|) - ln(1+|\zeta(\eta,t)|)\|_{L^2} \\
&\leq \frac{1}{1+e^t}\left\|ln\left(1+\frac{|z(t,\widetilde{y}) - \zeta(\eta,t)|}{1+|\zeta(\eta,t)|}\right)\right\|_{L^2} \\
&\leq \frac{1}{1+e^t} ln(1+\|z(t,\widetilde{y}) - \zeta(\eta,t)\|_{L^2}) \\
&\leq \frac{1}{1+e^t} ln(1+\vartheta_t),
\end{aligned}$$

and $h(t) = ln(1+t)$. This function is continuous, and $h \in X$, so (H_4) is verified by $K(t) = \frac{1}{1+e^t}$. Consequently, all the hypotheses of Theorem 3.2 are satisfied and we conclude that the problem (5.5) has at least one solution $y \in C(\Theta, L^2(0,1))$.

Declarations

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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