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Fixed points and convergence results in ordered hyperbolic spaces for monotone Suzuki mean nonexpansive mappings

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ABSTRACT. This paper is related to the fixed point results of monotone Suzuki mean nonexpansive mappings and convergence and Δ-convergence of a sequence in ordered hyperbolic space defined by an iteration scheme introduced by Abbas and Nazir.

1. INTRODUCTION

The geometric properties of space play an important role in metric fixed point theory. Since Banach spaces have convex structures, it is studied extensively in the literature. However, metric spaces do not have this structure. Therefore, there is a need to introduce and define convex structures.

Hyperbolic spaces are rich in geometrical structure, and they are suitable to obtain new results in topology, graph theory, multi-valued analysis and metric fixed point theory. The study of fixed point theory for nonexpansive mappings in the framework of hyperbolic spaces was initiated by Takahashi [22].

Goebel and Kirk [15] used hyperbolic-type spaces, which contain hyperbolic metric spaces. To study, existence and approximations of fixed points for nonexpansive mappings, Reich and Shafrir [20] introduced hyperbolic metric spaces. After that, Kohlenbach [16] introduced a more general definition of hyperbolic metric spaces, and he proved existence of fixed points for nonexpansive mappings. Leustean [17] showed that $CAT(0)$ spaces are uniformly convex hyperbolic metric spaces.

The study of monotone nonexpansive mappings increases rapidly in few years. The existence of approximate fixed points for semi-groups of nonlinear monotone mappings acting in a Banach vector space endowed with a partial order was proved in 2015 by Bachar and Khamsi [3]. Dehaish and Khamsi [12] gave an analogues result to Browder and Gohde's fixed point theorem for monotone nonexpansive mappings in uniformly convex hyperbolic metric spaces in 2016. Recently, Shukla et al. [21] proved existence and convergence results for a monotone mappings satisfying some conditions in partially ordered hyperbolic metric spaces.

A generalization of nonexpansive mapping in Banach space was introduced by Zhang [26] in 1975, namely mean nonexpansive mapping, as follows:
A mapping $T : \mathcal{K} \to \mathcal{K}$ is called mean nonexpansive if

$$||Tx - Ty|| \leq \sigma||x - y|| + \tau||x - Ty||,$$

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where $\sigma, \tau \geq 0$ and $\sigma + \tau \leq 1$.

The point is here that, for $\sigma = 1$ and $\tau = 0$, a nonexpansive mapping is a mean nonexpansive, but converse need not be always true. The following example shows that a mean nonexpansive mapping is not necessarily nonexpansive mapping.

Example 1.1. [2] *Suppose that* $T : [0, 1] \to [0, 1]$ *is a mapping defined by*

$$Tx = \begin{cases} \frac{x}{5} + \frac{5}{12}, \ x \in [0, \frac{1}{2}]; \\ \frac{x}{6} + \frac{5}{12}, \ x \in [\frac{1}{2}, 1]. \end{cases}$$

Here T *is mean nonexpansive mapping with* $\sigma = \frac{1}{3}$, $\tau = \frac{2}{3}$, *but not continuous at* $x = \frac{1}{2}$. *Thus* T *is not a nonexpansive mapping.*

The concept of generalized nonexpansive mappings, which are called Suzuki generalized nonexpansive mappings or the condition (C), was introduced by Suzuki [23], and obtained some fixed point results and convergence results for such mappings in Banach spaces. In 2021, Mebawondu et al. [19] introduced a new class of monotone generalized nonexpansive mappings, namely monotone Suzuki mean nonexpansive mappings, which are wider than the class of nonexpansive mappings, mean nonexpansive mappings, and mappings satisfying condition (C). Mebawondu et al. [19] established some weak and strong convergence theorem for their proposed iterative scheme in the framework of an ordered Banach space.

Uddin et al. [25], introduced the concept of partial order in the setting of $CAT(0)$ spaces as follows:
Let $\mathcal{X}$ be a complete $CAT(0)$ space endowed with partial order " $\preceq$ ". An order interval is any of the subsets

$$[\sigma, \to) = \{x \in \mathcal{X} : \sigma \preceq x\} \ or \ (\leftarrow, \sigma] = \{x \in \mathcal{X} : x \preceq \sigma\},$$

for any $\sigma \in \mathcal{X}$. So an order interval $[x, y]$ for all $x, y \in \mathcal{X}$ is given by

$$[x, y] = \{\vartheta \in \mathcal{X} : x \preceq \vartheta \preceq y\}.$$

Following iteration scheme is introduced by Abbas and Nazir [1] in 2014, as follows:
For $x_1 \in \mathcal{K}$, the sequence $\{x_k\}$ is defined by

$$(1.1) \qquad \begin{cases} z_k = (1 - \gamma_k)x_k + \gamma_k T x_k, \\ y_k = (1 - \beta_k)T x_k + \beta_k T z_k, \\ x_{k+1} = (1 - \alpha_k)T y_k + \alpha_k T z_k, k \in \mathbb{N}, \end{cases}$$

where $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ are sequences in $(0, 1)$.

Iteration scheme (1.1) can be expressed in hyperbolic space as follows:

$$(1.2) \qquad \begin{cases} z_k = W(x_k, T x_k, \gamma_k), \\ y_k = W(T x_k, T z_k, \beta_k), \\ x_{k+1} = W(T y_k, T z_k, \alpha_k), k \in \mathbb{N}, \end{cases}$$

Recently, lots of work has been done to establish existence or approximate fixed points of nonexpansive mappings in hyperbolic metric spaces (refer [6, 8, 10, 11]). In this paper, convergence and Δ-convergence of a sequence defined by (1.2) is proved for monotone Suzuki mean nonexpansive mapping in the framework of ordered hyperbolic space.

2. PRELIMINARY

Let $\mathcal{X}$ be a non-empty set. A point $x \in \mathcal{X}$ is called a fixed point of a mapping $T : \mathcal{X} \to \mathcal{X}$ if $Tx = x$. Through-out the literature $F(T)$ denotes set of fixed points of T, i.e., $F(T) = \{x \in \mathcal{X} : Tx = x\}$.

According to [9], a mapping $T : \mathcal{X} \to \mathcal{X}$ is called

(i) Lipschitz mapping if $d(Tx, Ty) \leq Ld(x, y), \forall\, x, y \in \mathcal{X}$, where $L > 0$.
(ii) Contraction mapping if $d(Tx, Ty) \leq Ld(x, y), \forall\, x, y \in \mathcal{X}$, where $0 < L < 1$.
(iii) Nonexpansive if $d(Tx, Ty) \leq d(x, y), \forall\, x, y \in \mathcal{X}$ and $L = 1$.

Definition 2.1. [18] *A mapping T defined on a closed convex subset $\mathcal{K}$ of a metric space space $\mathcal{X}$ is said to satisfy condition (C), if*

$$\frac{1}{2}d(x, y) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y),$$

for $x, y \in \mathcal{K}$.

Remark 2.1. *Note that T is generalization of nonexpansive mapping in the sense of Suzuki. It is obvious that every nonexpansive mapping satisfies condition (C), but the converse is not true. Consider the following examples:*

Example 2.2. *Let $T : [0, 2] \to [0, 2]$ defined by*

$$Tx = \begin{cases} 0, & x \neq 2, \\ 2, & x = 2. \end{cases} \tag{2.3}$$

It is clear that T is Suzuki nonexpansive mapping and also nonexpansive.

Example 2.3. *Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{K} = [0, \frac{5}{2}]$ is subset of $\mathcal{X}$. Let $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $d(x, y) = |x - y|$. Clearly $(\mathcal{X}, d)$ is metric space. Let T be a mapping defined on $\mathcal{K}$ such that*

$$Tx = \begin{cases} 0, & x \in [0, 2], \\ 4x - 12, & x \in [0, \frac{5}{2}]. \end{cases} \tag{2.4}$$

Then T is Suzuki nonexpansive mapping. However it is not a nonexpansive mapping.

Definition 2.2. [25] *Let $\mathcal{K}$ be a non-empty subset of an ordered metric space $\mathcal{X}$. A mapping $T : \mathcal{K} \to \mathcal{K}$ is said to be:*

(i) *monotone if $Tx \preceq Ty\ \forall\, x, y \in \mathcal{K}$ with $x \preceq y$,*
(ii) *monotone nonexpansive if T is monotone and*

$$d(Tx, Ty) \leq d(x, y),$$

$\forall\, x, y \in \mathcal{K}$ with $x \preceq y$,

(iii) *monotone quasi-nonexpansive if T is monotone and*

$$d(Tx, p) \leq d(x, p)$$

$\forall\, x \in \mathcal{K}$, $p \in F(T)$.

Definition 2.3. [16] *A hyperbolic space $(\mathcal{X}, d, W)$ is a metric space $(\mathcal{X}, d)$ together with a convexity mapping $W : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ such that $\forall\, x, y, z \in \mathcal{X}$ and $\alpha, \beta \in [0, 1]$,*

(i) $d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, y) + \alpha d(u, x)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
(iii) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(z, w) + \alpha d(x, y)$.

Example 2.4. [14] *Let $\mathcal{X} = \mathbb{R}$ and $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a mapping defined by*

$$d(x, y) = ||x - y||.$$

It is clear that d is metric on $\mathcal{X}$. Let $\mathcal{K} = [0, 1]$ be a subset of $\mathcal{X}$. Further define a mapping $W : \mathcal{X} \times \mathcal{X} \times [0, 1]$ by

$$W(x, y, \alpha) = \alpha x + (1 - \alpha)y,$$

$\forall\, x, y \in \mathcal{X}$ and $\alpha \in [0, 1]$. Then $(\mathcal{X}, d, W)$ is hyperbolic space.

Remark 2.2. *In Example 2.4, if $T : \mathcal{K} \to \mathcal{K}$ defined by*

$$Tx = \begin{cases} 1 - x, & x \in [0, \frac{1}{5}); \\ \frac{x+4}{5}, & x \in [\frac{1}{5}, 1]. \end{cases}$$

Then T is a Suzuki mean nonexpansive mapping in hyperbolic space $(\mathcal{X}, d, W)$, but not mean nonexpansive.(mapping T is chosen as given in [19]*)*

Definition 2.4. [7] *A non-empty subset $\mathcal{K}$ of a hyperbolic space $\mathcal{X}$ is said to be convex, if $W(x, y, \alpha) \in \mathcal{K}$, $\forall\, x, y \in \mathcal{K}$ and $\alpha \in [0, 1]$.*

Definition 2.5. [24] *A hyperbolic space $\mathcal{X}$ is said to be uniformly convex if for any $r > 0$ and $\varepsilon \in (0, 2]$, $\exists$ a $\delta \in (0, 1]$ such that $\forall\, x, y, z \in \mathcal{X}$,*

$$d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r,$$

provided $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq \varepsilon r$.

Definition 2.6. [5] *Let $\mathcal{K}$ be a non-empty subset of a metric space $\mathcal{X}$ and $\{x_k\}$ be any bounded sequence in $\mathcal{K}$. For $x \in \mathcal{X}$, there is a continuous functional $r(., \{x_k\}) : \mathcal{X} \to [0, \infty)$ defined by*

$$r(x, \{x_k\}) = \limsup_{k \to \infty} d(x_k, x).$$

The asymptotic radius $r(\mathcal{K}, \{x_k\})$ of $\{x_k\}$ with respect to $\mathcal{K}$ is given by

$$r(\mathcal{K}, \{x_k\}) = \inf\{r(x, \{x_k\}) : x \in \mathcal{K}\}.$$

A point $x \in \mathcal{K}$ is said to be an asymptotic center of the sequence $\{x_k\}$ with respect to $\mathcal{K}$, if

$$r(x, \{x_k\}) = \inf\{r(y, \{x_k\}) : y \in \mathcal{K}\}.$$

The set of all asymptotic centers of $\{x_k\}$ with respect to $\mathcal{K}$ is denoted by $A(\mathcal{K}, \{x_k\})$.

Definition 2.7. [11] *In hyperbolic space $(\mathcal{X}, d, W)$ an order interval is any of the subsets*

$$[\sigma, \rightarrow) = \{x \in \mathcal{X} : \sigma \preceq x\} \text{ or } (\leftarrow, \sigma] = \{x \in \mathcal{X} : x \preceq \sigma\},$$

for any $\sigma \in \mathcal{X}$. So an order interval $[x, y]$ $\forall\, x, y \in \mathcal{X}$ is given by

$$[x, y] = \{z \in \mathcal{X} : x \preceq z \preceq y\}.$$

Remark 2.3. [11] *The order intervals are closed and convex .*

Definition 2.8. [5] *A sequence $\{x_k\}$ in $\mathcal{X}$ is said to be $\Delta-$ converges to $x \in \mathcal{X}$ if x is the unique asymptotic center of $\{x_{k_n}\}$ of $\{x_k\}$. In this case $\Delta - \lim_{k \to \infty} x_k = x$.*

Definition 2.9. [24] *Let $\mathcal{X}$ be a hyperbolic space. A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such $\delta = \eta(r, \varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0, 2]$ is known as a modulus of uniform convexity of $\mathcal{X}$. The mapping η is said to be monotone, if it decreases with r.*

Definition 2.10. [24] *Let $\mathcal{K}$ be a convex subset of a hyperbolic space $(\mathcal{X}, d, W)$. A mapping $T : \mathcal{K} \to \mathcal{K}$ with non-empty fixed point set $F(T)$ in $\mathcal{K}$ will be said to satisfy Condition (I), if there is a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ $\forall\, x \in \mathcal{K}$, where $d(x, F(T)) = \inf\{||x - z|| : z \in F(T)\}$.*

Lemma 2.1. [17] *Let $\mathcal{X}$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequences $\{x_k\}$ in $\mathcal{X}$ has a unique asymptotic center with respect to any non-empty closed convex subset $\mathcal{K}$ of $\mathcal{X}$.*

Definition 2.11. [5] *Let $\mathcal{K}$ be a non-empty closed subset of a complete hyperbolic space $\mathcal{X}$ and $\{x_k\}$ be a sequence in $\mathcal{K}$. Then $\{x_k\}$ is called Fejer monotone sequence with respect to $\mathcal{K}$ if $\forall\, x \in \mathcal{K}$ and $k \in \mathbb{N}$,*

$$d(x_{k+1}, x) \leq d(x_k, x).$$

Proposition 2.1. [5] *Let $\mathcal{K}$ be a non-empty closed subset of a complete hyperbolic space $\mathcal{X}$ and $\{x_k\}$ be a sequence in $\mathcal{K}$. Suppose $T : \mathcal{K} \to \mathcal{K}$ is any nonlinear mapping and the sequence $\{x_k\}$ is Fejer monotone with respect of $\mathcal{K}$, then we have the following:*

(i) *$\{x_k\}$ is bounded.*
(ii) *The sequence $\{d(x_k, x^*)\}$ is decreasing and converges $\forall\, x^* \in F(T)$.*
(iii) *$\lim_{k\to\infty} d(x_k, F(T))$ exists.*

Lemma 2.2. [13] *Let $(\mathcal{X}, d, W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in \mathcal{X}$ and $\{t_k\}$ be a sequence in $[a, b]$ for some $\sigma, \tau \in (0, 1)$. If $\{x_k\}$ and $\{y_k\}$ are sequences in $\mathcal{X}$ such that $\limsup_{k\to\infty} d(x_k, x^*) \leq c$, $\limsup_{k\to\infty} d(y_k, x^*) \leq c$, and $\lim_{k\to\infty} d(W(x_k, y_k, t_k,)x^*) \leq c$, for some $c > 0$. Then $\lim_{k\to\infty} d(x_k, y_k) = 0$.*

Lemma 2.3. [4] *The following properties related to $\Delta-$ convergence on a $CAT(0)$ space $(\mathcal{X}, d)$ hold true:*

(i) *Every bounded sequence in $\mathcal{X}$ has a $\Delta-$ convergent subsequence.*
(ii) *Every $CAT(0)$ space satisfies the Opial's property, that is*

$$\limsup_{k\to\infty} d(x_k, x) < \limsup_{k\to\infty} d(x_k, y),$$

whenever a given sequence $\{x_k\}$ in $\mathcal{X}$ $\Delta-$ converges to x and $y \neq x$.

3. Main Results

The following results are generalizations of the results obtained by Mebawondu et al. [19] from ordered Banach space to ordered hyperbolic space.

Proposition 3.2. *Let $\mathcal{K}$ be a non-empty closed convex subset of an ordered hyperbolic space $(\mathcal{X}, d, W)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping with a fixed point $x^* \in \mathcal{K}$ and $x^* \preceq y$ for $y \in \mathcal{K}$. Then T is monotone quasi-nonexpansive mapping.*

Proof. Let $x^* \in F(T)$. Since T is Suzuki nonexpansive mapping,

$$\frac{1}{2} d(x^*, Tx^*) = 0 \leq d(x^*, y).$$

Also T is mean nonexpansive mapping,

$$\begin{aligned} d(x^*, Ty) &= d(Tx^*, Ty) \\ &\leq \sigma(x^*, y) + \tau(x^*, Ty) \\ (1-\tau) d(x^*, Ty) &\leq (1-\tau) d(x^*, y). \end{aligned}$$

This implies that $d(Tx^*, Ty) \leq d(x^*, y)$. Hence T is quasi-nonexpansive mapping. □

Lemma 3.4. *Let $\mathcal{K}$ be a non-empty closed convex subset of an ordered hyperbolic space $(\mathcal{X}, d, W)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping. Then $F(T)$ is closed and convex.*

Proof. To show that $F(T)$ is closed, let $\{x_k\}$ be a sequence in $F(T)$ such that $\{x_k\}$ converges to some $y \in \mathcal{K}$. Using mean nonexpansiveness of Ω,

$$\begin{aligned} d(x_k, Ty) &= d(Tx_k, Ty) \\ &\leq d(x_k, y) + \tau d(x_k, Ty) \\ (1-\tau) d(x_k, Ty) &\leq (1-\tau) d(x_k, y) \\ &\leq d(x_k, y). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides,

$$\lim_{k\to\infty} d(x_k, Ty) = 0.$$

By uniqueness of limit, $y = Ty$. Hence $F(T)$ is closed.

Now, to show that $F(T)$ is convex, let $x, y \in F(T)$ and $\alpha \in [0, 1]$. Then

$$\begin{aligned} d(x, T(W(x, y, \alpha))) &= d(Tx, TW(x, y, \alpha)) \\ &\leq \sigma d(x, W(x, y, \alpha)) + \tau d(x, T(W(x, y, \alpha))) \\ (1-\tau) d(x, T(W(x, y, \alpha))) &\leq (1-\tau) d(x, W(x, y, \alpha)) \\ &\leq d(x, W(x, y, \alpha)). \end{aligned}$$

Hence,

$$d(x, T(W(x, y, \alpha))) \leq d(x, W(x, y, \alpha)). \tag{3.5}$$

Using similar argument,

$$d(y, T(W(x, y, \alpha))) \leq d(y, W(x, y, \alpha)). \tag{3.6}$$

Observe that

$$\begin{aligned} d(x, y) &\leq d(x, T(W(x, y, \alpha))) + d(T(W(x, y, \alpha)), y) \\ &= d(Tx, T(W(x, y, \alpha))) + d(T(W(x, y, \alpha)), Ty) \\ &\leq \sigma(d(x, W(x, y, \alpha)) + \tau d(x, T(W(x, y, \alpha))) + \sigma d(y, W(x, y, \alpha))) \\ &+ \tau d(T(W(x, y, \alpha)), \wp) \\ &\leq \sigma[d(x, W(x, y, \alpha)) + d(y, W(x, y, \alpha)))] \\ &+ \tau[d(x, T(W(x, y, \alpha)) + d(T(W(x, y, \alpha)), y)] \\ &= \sigma d(x, y) + \tau d(x, y) \\ &\leq d(x, y). \end{aligned}$$

Therefore

$$d(x, y) \leq d(x, y). \tag{3.7}$$

From Equation (3.5) and (3.6), $d(x, T(W(x, y, \alpha))) = d(x, W(x, y, \alpha))$ and $d(y, T(W(x, y, \alpha))) = d(y, W(x, y, \alpha))$ respectively, because if strictly less than sign $<$ is used, then from Equation (3.7), there is a contradiction that $d(x, y) < d(x, y)$. Therefore

$$T(W(x, y, \alpha)) = W(x, y, \alpha),$$

$\forall\, x, y \in F(T)$ and $\alpha \in [0, 1]$. Thus $W(x, y, \alpha) \in F(T)$ which implies that $F(T)$ is convex. □

Lemma 3.5. *Let $\mathcal{K}$ be a non-empty closed convex subset of an ordered hyperbolic space $(\mathcal{X}, d, W)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping. Then $\forall\, x, y \in \mathcal{K}$ with $x \preceq y$,*

(i) $d(T^2x, Tx) \leq d(Tx, x)$.
(ii) *Either* $\frac{1}{2} d(x, Tx) \leq d(x, y)$ *or* $\frac{1}{2} d(Tx, T^2x) \leq d(Tx, y)$.

(iii) *Either* $d(Tx, Ty) \leq \sigma d(x, y) + \tau d(x, Ty)$ *or* $d(Tx, Ty) \leq \sigma d(Tx, y) + \tau d(Tx, Tx)$.

Proof. (i) Since $\frac{1}{2}d(x, Tx) < d(x, Tx)$ and T is mean nonexpansive mapping,

$$\begin{aligned} d(T^2x, Tx) &= d(T(Tx, Tx) \\ &\leq \sigma d(Tx, x) + \tau d(Tx, Tx) \\ &\leq d(Tx, x). \end{aligned}$$

(ii) On contrary, suppose that $\frac{1}{2}d(x, Tx) > d(x, y)$ or $\frac{1}{2}d(Tx, T^2x) > d(Tx, y)$. Observe that

$$\begin{aligned} d(x, Tx) &\leq d(x, y) + d(y, Tx) \\ &< \frac{1}{2}d(x, Tx) + \frac{1}{2}d(x, Tx) \\ &= d(x, Tx), \end{aligned}$$

which is a contradiction. Therefore either $\frac{1}{2}d(x, Tx) \leq d(x, y)$ or $\frac{1}{2}d(Tx, T^2x) \leq d(Tx, y)$.

(iii) The proof follows from (ii).

□

Theorem 3.1. *Let $\mathcal{K}$ be a non-empty closed convex subset of a uniformly convex ordered hyperbolic space $(\mathcal{X}, d, W)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping. Then $F(T) \neq \emptyset$ if and only if $\{T^k x\}$ is bounded sequence for some $x \in \mathcal{K}$ provided that $T^k x \preceq y$ for some $y \in \mathcal{K}$ with $x \preceq Tx$.*

Proof. Suppose that $\{x_k\} = \{T^k x\}$ is a bounded sequence and for some $x \in \mathcal{K}$, $x \preceq Tx$. Using monotonicity of T, $Tx \preceq T^2x \preceq T^3x \preceq \ldots$. By asymptotic center of $\{x_k\}$, $A(\mathcal{K}, \{x_k\}) = \{x^*\}$ such that $x_k \preceq x^*\ \forall\, k \in \mathbb{N}$. Since

$$\begin{aligned} \frac{1}{2}d(x_k, Tx_k) &= \frac{1}{2}d(T^k x, Tx_k) \\ &= \frac{1}{2}d(T^k x, T^{k+1}x) \\ &< d(T^k x, T^{k+1}x) \\ &= d(x_k, x_{k+1}). \end{aligned}$$

Now

$$\begin{aligned} d(x_{k+2}, x_{k+1}) &= d(T^{k+2}x, T^{k+1}x) \\ &= d(T(T^{k+1}x, T^k x)) \\ &= d(Tx_{k+1}, Tx_k) \\ &\leq \sigma d(x_{k+1}, x_k) + \tau d(x_{k+1}, x_{k+1}) \\ &\leq d(x_{k+1}, x_k). \end{aligned}$$

We claim that $d(x_{k+1}, x_k) \leq 2d(x_k, x^*)$ or $d(x_{k+2}, x_{k+1}) \leq d(x_{k+1}, x^*)\ \forall\, k \in \mathbb{N}$.
On contrary, suppose that $d(x_{k+1}, x_k) > 2d(x_k, x^*)$ or $d(x_{k+2}, x_{k+1}) > d(x_{k+1}, x^*)$.

$$\begin{aligned} d(x_{k+1}, x_k) &\leq d(x_{k+1}, x^*) + d(x^*, x_x) \\ &\leq \frac{1}{2}d(x_{k+2}, x_{k+1}) + \frac{1}{2}d(x_{k+1}, x_k) \\ &\leq \frac{1}{2}d(x_{k+1}, x_k) + \frac{1}{2}d(x_{k+1}, x_k) \\ &\leq d(x_{k+1}, x_k), \end{aligned}$$

which is a contradiction. Therefore $d(x_{k+1}, x_k) \leq 2d(x_k, x^*)$ or $d(x_{k+2}, x_{k+1}) \leq d(x_{k+1}, x^*)$ $\forall\, k \in \mathbb{N}$.

Now, using the fact that $\frac{1}{2}d(x_{k+1}, x_k) = \frac{1}{2}d(Tx_k, x_k) \leq d(x_k, x^*)$.

$$\begin{aligned} d(Tx_k, Tx^*) &\leq \sigma d(x_k, x^*) + \tau d(x_k, Tx^*) \\ &\leq d(x_k, x^*) \\ \limsup_{k\to\infty} d(Tx_k, Tx^*) &\leq \limsup_{k\to\infty} d(x_k, x^*). \end{aligned}$$

This implies that $Tx^* \in A(\mathcal{K}, \{x_k\})$. By uniqueness of limit, $x^* = Tx^*$.
Similar results will be followed by considering the fact that $\frac{1}{2}d(Tx, T^2x) \leq d(Tx, x^*)$. Hence $F(T) \neq \emptyset$.

Now suppose that $F(T) \neq \emptyset$ with $x^* \in F(T)$. Then by mathematical induction, $T^k x^* = x^*$ $\forall\, k \in \mathbb{N}$. Since $\{T^k x^*\}$ is a constant sequence, it is bounded. □

Theorem 3.2. *Let $\mathcal{K}$ be a non-empty closed convex subset of a complete uniformly convex ordered hyperbolic space $(\mathcal{X}, d, W)$ with monotone modulus of convexity η. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping and $\{x_k\}$ be a bounded sequence in $\mathcal{K}$ such that* $\lim_{k\to\infty} d(x_k, Tx_k) = 0$ *and* $\Delta - \lim_{k\to\infty} x_k = x^*$. *Then $x^* \in F(T)$.*

Proof. Since $\{x_k\}$ is a bounded sequence in $\mathcal{K}$, from Lemma 2.1, $\{x_k\}$ has a unique asymptotic center in $\mathcal{K}$. Since $\Delta - \lim_{k\to\infty} x_k = x^*$, we have $A(\{x_k\}) = \{x^*\}$. Observe that

$$\begin{aligned} d(x_k, Tx^*) &\leq d(x_k, Tx_k) + d(Tx_k, Tx^*) \\ &\leq d(x_k, Tx_k) + \sigma d(x_k, x^*) + \tau(x_k, Tx^*) \\ (1-\tau)d(x_k, Tx^*) &\leq \sigma d(x_k, x^*) + d(x_k, Tx_k) \\ &\leq \frac{1}{1-\tau}[\sigma d(x_k, x^*) + d(x_k, Tx_k)]. \end{aligned}$$

Since

$$\begin{aligned} r(Tx^*, \{x_k\}) &= \limsup_{k\to\infty} d(x_k, Tx^*) \\ &\leq \frac{1}{1-\tau} \limsup_{k\to\infty} [\sigma d(x_k, x^*) + d(x_k, Tx_k)] \\ &\leq \limsup_{k\to\infty} d(x_k, x^*) \\ &= r(x^*, \{x_k\}). \end{aligned}$$

By uniqueness of asymptotic center of $\{x_k\}$, $Tx^* = x^*$. Hence $x^* \in F(T)$. □

4. Convergence Results

In 2021, Mewomo et al. [18] obtained common fixed point of two mean nonexpansive mappings in the framework of hyperbolic spaces by using iteration scheme (1.2). This section contains some convergence results that extend the work of Mewomo et al. [18] from mean nonexpansive mapping to monotone Suzuki mean nonexpansive mapping by using iteration scheme (1.2) in an ordered hyperbolic spaces.

Lemma 4.6. *Let $\mathcal{K}$ be a non-empty closed convex subset of an ordered hyperbolic space $(\mathcal{X}, d, W)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping. Let $x_1 \in \mathcal{K}$ such that $x_1 \preceq Tx_1$ and $\{x_k\}$ be a sequence in $\mathcal{K}$ defined by (1.2). Then $x_k \preceq Tx_k \preceq x_{k+1}$ $\forall\, k \in \mathbb{N}$.*

Proof. Applying induction on k. By assumption $x_1 \preceq \Omega x_1$. Since order intervals are convex, $x_1 \preceq (1-\gamma_1)x_1 + \gamma_1 T x_1 \preceq T x_1 \Rightarrow x_1 \preceq z_1 \preceq T x_1$.
Now $x_1 \preceq z_1$ and T is monotone, $Tx_1 \preceq Tz_1$. Therefore $Tx_1 \preceq (1-\beta_1)Tx_1 + \beta_1 T z_1 \preceq Tz_1 \Rightarrow Tx_1 \preceq y_1 \preceq Tz_1$. Since $x_1 \preceq z_1 \preceq Tx_1 \preceq y_1 \preceq Tz_1$, $Tz_1 \preceq Ty_1$. Therefore $Tz_1 \preceq (1-\alpha_1)Tz_1 + \alpha_1 T y_1 \preceq Ty_1 \Rightarrow Tz_1 \preceq x_2 \preceq Ty_1$. Since $x_1 \preceq Tz_1 \preceq x_2 \Rightarrow x_1 \preceq x_2$. Hence induction is true for $k = 1$.
Now assume that induction is true for $k \geq 2$. Since $x_k \preceq Tx_k$, $x_k \preceq (1-\gamma_k)x_k + \gamma_k \Omega x_k \preceq Tx_k \Rightarrow x_k \preceq z_k \preceq Tx_k$ and by monotonicity of T, $Tx_k \preceq Tz_k$. Therefore $Tx_k \preceq (1-\beta_k)Tx_k + \beta_k T z_k \preceq Tz_k \Rightarrow Tx_k \preceq y_k \preceq Tz_k$.
Since $Ty_k \preceq Tz_k$, therefore $Ty_k \preceq (1-\alpha_k)Ty_k + \alpha_k T z_k \preceq Tz_k \Rightarrow Ty_k \preceq x_{k+1} \preceq Tz_k$. Using similar approach as above, $x_{k+1} \preceq z_{k+1} \preceq Tx_{k+1}$.
Since T is monotone, we have $Tx_{k+1} \preceq (1-\beta_{k+1})Tx_{k+1} + \beta_{k+1} z_{k+1} \preceq z_{k+1} \Rightarrow Tx_{k+1} \preceq y_{k+1} \preceq z_{k+1}$. Since $Ty_{k+1} \preceq Tz_{k+1} \Rightarrow Ty_{k+1} \preceq (1-\alpha_{k+1})Ty_{k+1} + \alpha_{k+1} T z_k \preceq Tz_k$, i.e. $Ty_{k+1} \preceq x_{k+2} \preceq Tz_{k+1}$. Therefore $x_{k+1} \preceq Tx_{k+1} \preceq x_{k+2}$. □

Lemma 4.7. *Let $\mathcal{K}$ be a non-empty closed convex subset of a complete ordered hyperbolic space $(\mathcal{X}, \varrho, W)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping. Let $x_1 \in \mathcal{K}$ such that $x_1 \preceq Tx_1$ and $\{x_k\}$ be a sequence in $\mathcal{K}$ defined by (1.2). Then*

(i) *Sequence $\{x_k\}$ is bounded.*
(ii) $\lim_{k\to\infty} d(x_k, x^*)$ *exists* $\forall\, x^* \in F(T)$.
(iii) $\lim_{k\to\infty} d(x_k, F(T))$ *exists.*

Proof. Let $\{x_k\}$ be a sequence in $\mathcal{K}$ defined by (1.2) and $x^* \in F(T)$ with $x_1 \preceq x^*$. By monotonicity of T, $Tx_1 \preceq Tx^* = x^*$.
Observe that

$$\begin{aligned}
d(z_k, x^*) =& d(W(x_k, Tx_k, \gamma_k), x^*) \\
&\leq (1-\gamma_k)d(x_k, x^*) + \gamma_k d(Tx_k, x^*) \\
&\leq (1-\gamma_k)d(x_k, x^*) + \gamma_k[\sigma d(x_k, x^*) + \tau d(x_k, x^*)] \\
&\leq d(x_k, x^*), \\
d(y_k, x^*) =& d(W(Tx_k, Tz_k, \beta_k), x^*) \\
&\leq (1-\beta_k)d(Tx_k, x^*) + \beta_k d(Tz_k, x^*) \\
&\leq (1-\beta_k)[\sigma d(x_k, x^*) + \tau d(x_k, x^*)] + \beta_k[\sigma d(z_k, x^*) + \tau d(z_k, x^*)] \\
&\leq d(x_k, x^*), \\
d(x_{k+1}, x^*) =& d(W(Ty_k, Tz_k, \alpha_k), x^*) \\
&\leq (1-\alpha_k)d(Ty_k, x^*) + \alpha_k d(Tz_k, x^*) \\
&\leq d(x_k, x^*).
\end{aligned}$$

This shows that sequence $\{x_k\}$ is Fejer monotone with respect to $F(T)$, Hence by Proposition 2.1, the desire result holds. □

Lemma 4.8. *Let $\mathcal{K}$ be a non-empty closed convex subset of a complete ordered hyperbolic space $(\mathcal{X}, d, W)$ with monotone modulus of convexity η. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping such that $F(T) \neq \emptyset$. Let $x_1 \in \mathcal{K}$ such that $x_1 \preceq Tx_1$ and $\{x_k\}$ be a sequence in $\mathcal{K}$ defined by (1.2). Then* $\lim_{k\to\infty} d(x_k, Tx_k) = 0$.

Proof. From Lemma 4.7, $\lim_{k\to\infty} d(x_k, x^*)$ exists $\forall\, x^* \in F(T)$. Suppose that $\lim_{k\to\infty} d(x_k, x^*) = w$, where $w \geq 0$. The result is true for $w = 0$. So, suppose that $w > 0$. Since

$$d(z_k, x^*) \leq d(x_k, x^*), \;\; k \in \mathbb{N},$$

taking $\limsup_{k\to\infty}$ on both sides,

$$\limsup_{k\to\infty} d(z_k, x^*) \leq w.$$

Also

$$\begin{aligned} d(Tz_k, x^*) &\leq \sigma d(z_k, x^*) + \tau d(z_k, x^*) \\ &\leq d(z_k, x^*). \end{aligned}$$

Hence

$$\limsup_{k\to\infty} d(Tz_k, x^*) \leq w.$$

Similarly

$$d(y_k, x^*) \leq d(x_k, x^*),$$

hence

$$\limsup_{k\to\infty} d(y_k, x^*) \leq w,$$

and

$$\limsup_{k\to\infty} d(Ty_k, x^*) \leq w.$$

Observe that

$$d(x_{k+1}, x^*) = d(W(Ty_k, Tz_k, \alpha_k), x^*).$$

This implies that

$$\lim_{k\to\infty} d(W(Ty_k, Tz_k, \alpha_k), x^*) = w.$$

Hence from Lemma 2.2, $\lim_{k\to\infty} d(Ty_k, Tz_k) = 0$.
Now, using the fact that T is mean nonexpansive mapping,

$$\begin{aligned} d(Tx_k, x^*) &\leq \sigma d(x_k, x^*) + \tau d(x_k, x^*) \\ &\leq d(x_k, x^*). \end{aligned}$$

Taking $\limsup_{k\to\infty}$ on both sides,

$$\limsup_{k\to\infty} d(Tx_k, x^*) = w.$$

Since

$$\begin{aligned} d(x_{k+1}, x^*) &= d(W(Ty_k, Tz_k, \alpha_k), x^*) \\ &\leq (1-\alpha_k) d(Ty_k, x^*) + \alpha_k d(Tz_k, x^*) \\ &\leq (1-\alpha_k) d(Ty_k, x^*) + \alpha_k [d(Tz_k, Ty_k) + d(Ty_k, x^*)] \\ &\leq (\sigma+\tau) d(y_k, x^*) + \alpha_k d(Tz_k, Ty_K) \\ &\leq d(y_k, x^*) + \alpha_k d(Tz_k, Ty_k). \end{aligned}$$

Taking $\liminf_{k\to\infty}$ on both sides,

$$\liminf_{k\to\infty} d(y_k, x^*) \geq w.$$

Hence

$$\lim_{k\to\infty} d(y_k, x^*) = w \Rightarrow d(W(Tx_k, Tz_k, \beta_k), x^*) = w.$$

From Lemma 2.2,

$$\lim_{k\to\infty} d(Tx_k, Tz_k) = 0.$$

Since

$$\lim_{k\to\infty} d(Tx_k, Ty_k) \le \lim_{k\to\infty} d(Tx_k, Tz_k) + \lim_{k\to\infty} d(Tz_k, Ty_k)$$
$$\Rightarrow \lim_{k\to\infty} d(Tx_k, Ty_k) = 0.$$

Similarly, $\lim_{k\to\infty} d(z_k, x^*) = w$ and $\lim_{k\to\infty} d(W(x_k, Tx_k, \gamma_k), x^*) = w$ and from Lemma 2.2,

$$\lim_{k\to\infty} d(x_k, Tx_k) = 0.$$

□

Theorem 4.3. *Let $\mathcal{K}$ be a non-empty closed convex subset of a complete uniformly convex ordered hyperbolic space $(\mathcal{X}, d, W)$ with monotone modulus of uniform convexity η. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping. Let $x_1 \in \mathcal{K}$ such that $x_1 \preceq Tx_1$ and $\{x_k\}$ be a sequence in $\mathcal{K}$ defined by (1.2). Then $\{x_k\}$ converges strongly to $x^* \in F(T)$ if and only if $\liminf_{k\to\infty} d(x_k, F(T)) = 0$, where $d(x_k, F(T)) = \inf\{d(x_k, T^*) : T^* \in F(T)\}$.*

Proof. If $\{x_k\}$ converges strongly to $x^* \in F(T)$, then $\lim_{k\to\infty} d(x_k, x^*) = 0$. Since $0 \le d(x_k, F(T)) = \inf\{d(x_k, x^*) : x^* \in F(T)\}$, $\lim_{k\to\infty} d(x_k, F(T)) = 0$.
Conversely, suppose that $\lim_{k\to\infty} d(x_k, F(T)) = 0$. Since

$$d(x_{k+1}, x^*) \le d(x_k, x^*),$$

which implies that

$$d(x_{k+1}, F(T)) \le d(x_k, F(T)).$$

This implies that $\lim_{k\to\infty} d(x_k, F(T))$ exists. Therefore by assumption $\lim_{k\to\infty} d(x_k, F(T)) = 0$.
Next, to show that $\{x_k\}$ is a Cauchy sequence in $\mathcal{K}$. For $k > n$,

$$d(x_k, x_n) \le d(x_k, x^*) + d(x^*, x_n)$$
$$\le 2d(x_k, x^*).$$

Taking inf on right hand side,

$$d(x_k, x_n) \le 2d(x_k, F(T)).$$

Hence, $d(x_k, x_n) \to 0$ as $k, n \to \infty$. Hence $\{x_k\}$ is Cauchy sequence in $\mathcal{K}$, therefore it converges to some $q \in \mathcal{K}$.
Now, to show that $q \in F(T)$, since $d(x_k, F(T)) = \inf\{x^* \in F(T) : d(x_k, x^*)\}$. So for each $\varepsilon > 0$, $\exists \{p_k\} \in F(T)$ such that

$$d(x_k, p_k) < d(x_k, F(T)) + \frac{\varepsilon}{2}$$

Since $d(p_k, q) \le d(x_k, p_k) + d(x_k, q) \Rightarrow \lim_{k\to\infty} d(p_k, q) \le \frac{\varepsilon}{2}$. Hence,

$$d(Tq, q) \le d(Tq, p_k) + d(p_k, q)$$
$$\le \sigma d(p_k, q) + \tau d(q, Tq) + d(p_k, q)$$
$$\le 2d(p_k, q).$$

Which implies that $d(Tq, q) \le \varepsilon$. Hence $d(Tq, q) = 0$. Since $F(T)$ is closed, $q \in F(T)$. □

Theorem 4.4. *Let $\mathcal{K}$ be a non-empty closed convex subset of a complete uniformly convex ordered hyperbolic space $(\mathcal{X}, d, W)$ with monotone modulus of uniform convexity η. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping such that $F(T) \neq \emptyset$. Also suppose that T satisfies Condition (I). Let $x_1 \in \mathcal{K}$ such that $x_1 \preceq Tx_1$ and $\{x_k\}$ be a sequence in $\mathcal{K}$ defined by (1.2). Then T converges strongly to a point of $F(T)$.*

Proof. From Lemma 4.7, $\lim_{k\to\infty} d(x_k, F(T))$ exists and from Lemma 4.8, $\lim_{k\to\infty} d(x_k, \Omega x_k) = 0$. Using the fact that

$$0 \leq \lim_{k\to\infty} d(x_k, F(T)) \leq \lim_{k\to\infty} d(x_k, Tx_k) = 0, \ \forall \ x \in \mathcal{K},$$

$$\lim_{k\to\infty} f(d(x_k, F(T))) = 0.$$

Since T satisfies Condition (I), $\lim_{k\to\infty} d(x_k, F(T)) = 0$. Hence from Theorem 4.3, $\{x_k\}$ converges strongly to $x^* \in F(T)$. □

Theorem 4.5. *Let $\mathcal{K}$ be a non-empty closed convex subset of a complete uniformly convex ordered hyperbolic space $(\mathcal{X}, d, W)$ with monotone modulus of uniform convexity η. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping. Let $x_1 \in \mathcal{K}$ such that $x_1 \preceq Tx_1$. Suppose that $F(T) \neq \emptyset$ with $x^* \in F(T)$ such that $x_1 \preceq x^*$ and $\{x_k\}$ be a sequence in $\mathcal{K}$ defined by (1.2). Then $\{x_k\}$ converges weakly to a point of $F(T)$.*

Proof. From Lemma 4.7, $\{x_k\}$ is bounded and from Lemma 4.8, $\lim_{k\to\infty} d(x_k, Tx_k) = 0$. Since X is uniformly convex, $\exists$ subsequence $\{x_{k_n}\}$ of $\{x_k\}$ that converges weakly to $x^* \in \mathcal{K}$. Using Lemma 4.6, $x_1 \preceq x_{k_n} \preceq x^* \ \forall \ n \in \mathbb{N}$.

$$\begin{aligned} d(x_{k_n}, Tx^*) &= d(x_{k_n}, Tx_{k_n}) + d(Tx_{k_n}, Tx^*) \\ &\leq d(x_{k_n}, Tx_{k_n}) + \sigma(d(x_{k_n}, x^*) + \tau d(x_{k_n}, Tx^*). \end{aligned}$$

This implies that

$$\begin{aligned} \liminf_{k\to\infty} d(x_{k_n}, Tx^*) &\leq \liminf_{k\to\infty} d(x_{k_n}, x^*) \\ \lim_{k\to\infty} d(x_{k_n}, Tx^*) &= 0. \end{aligned}$$

Now, to how that $\{x_k\}$ has a unique weak subsequential limit in $F(T)$. Let x^* and u are two weak limits of the subsequence $\{x_{k_n}\}$ and $\{x_{k_m}\}$ of $\{x_k\}$. By similar approach, $u \in F(T)$. Now suppose that $x^* \neq u$. By Opial's property,

$$\begin{aligned} \lim_{k\to\infty} d(x_k, x^*) &= \lim_{k\to\infty} d(x_{k_n}, x^*) \\ &< \lim_{k\to\infty} d(x_{k_n}, u) \\ &= \lim_{k\to\infty} d(x_k, u) \\ &< \lim_{k\to\infty} d(x_{k_m}, u) \\ &= \lim_{k\to\infty} d(x_{k_m}, x^*) \\ &= \lim_{k\to\infty} d(x_k, x^*), \end{aligned}$$

which is a contradiction. Hence $x^* = u$. □

Theorem 4.6. *Let $\mathcal{K}$ be a non-empty closed convex subset of a complete uniformly convex ordered hyperbolic space $(\mathcal{X}, d, W)$ with monotone modulus of convexity η and satisfies Opial's property. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone Suzuki mean nonexpansive mapping. Let $x_1 \in \mathcal{K}$ such that $x_1 \preceq Tx_1$. Suppose that $F(T) \neq \emptyset$ with $x^* \in F(T)$ such that $x_1 \preceq x*$ and $\{x_k\}$ be a sequence in $\mathcal{K}$ defined by (1.2). Then $\{x_k\}$ $\Delta-$ converges to a fixed point x^* of T.*

Proof. From Lemma 4.7, $\lim_{k\to\infty} d(x_k, x^*)$ exists, and $\{x_k\}$ is bounded sequences and also from Lemma 4.8, $\lim_{k\to\infty} d(Tx_k, x_k) = 0$.
Let $B(\{x_k\}) = \cup X(\{y_k\})$, where union is taken over all subsequence $\{y_k\}$ of $\{x_k\}$. To prove that $\{x_k\}$ $\Delta-$ converges to a fixed point p of T, first we will show that $B(\{x_k\}) \subset F(T)$ and after-that $B(\{x_k\})$ is singleton set.
Let $v \in B(\{x_k\})$, then $\exists$ a subsequence $\{v_k\}$ of $\{x_k\}$ such that $X(\{v_k\}) = v$. Since $\mathcal{X}$ is

uniformly convex, $\exists$ a subsequence $\{w_k\}$ of $\{v_k\}$ such that $\Delta - \lim_{k\to\infty} w_k = w$ and $w \in \mathcal{K}$. Since $\{w_k\}$ is subsequence of $\{x_k\}$, we have $\lim_{k\to\infty} d(Tw_k, w_k) = 0$, $w = Tw$ and hence $w \in F(T)$.
Next to show show that $w = v$. Suppose not, i.e., $w \neq v$. Since X satisfies Opial's condition,

$$\begin{aligned}\limsup_{k\to\infty} d(w_k, w) &< \limsup_{k\to\infty} d(w_k, v)\\ &\leq \limsup_{k\to\infty} d(v_k, v)\\ &< \limsup_{k\to\infty} d(v_k, w)\\ &\leq \limsup_{k\to\infty} d(x_k, w)\\ &= \limsup_{k\to\infty} d(w_k, w)\end{aligned}$$

Which is a contradiction. Hence $w = v \in F(T)$. Now to show that $B(\{x_k\})$ is singleton set, let $X(\{v_k\}) = v$ and $X(\{x_k\}) = x$. Since we already proved that $v = w$, so it is sufficient to prove that $v = x$. If $v \neq x$, then doing the same procedure as above,

$$\begin{aligned}\limsup_{k\to\infty} d(w_k, v) &< \limsup_{k\to\infty} d(w_k, x)\\ &\leq \limsup_{k\to\infty} d(x_k, x)\\ &< \limsup_{k\to\infty} d(x_k, v)\\ &= \limsup_{k\to\infty} d(w_k, v)\end{aligned}$$

Which is a contradiction. Hence $v = x$, which proves that $B(\{x_k\})$ is singleton set and that particular element will be fixed point of T. □

5. Numerical Example

Example 5.5. *Let $\mathcal{X} = \mathbb{R}$ with metric d defined by $d(x, y) = |x-y|$. Define $W : \mathcal{X}\times\mathcal{X}\times[0, 2] \to \mathcal{X}$ by*

$$W(x, y, \zeta) = \zeta x + (1 - \zeta)y,$$

for $x, y \in \mathcal{X}$, $\zeta \in [0, 2]$. Then $(\mathcal{X}, d, W)$ is hyperbolic space. For this

(i)

$$\begin{aligned}d(u, W(x, y, \zeta)) &= |u - W(x, y, \zeta)|\\ &= |u - \zeta x - (1 - \zeta)y|\\ &= |(1 - \zeta)(u - y) + \zeta(u - x)|\\ &\leq (1 - \zeta)d(u, y) + \zeta d(u, x).\end{aligned}$$

(ii)

$$\begin{aligned}d(W(x, y, \zeta), W(x, y, \tau)) &= |W(x, y, \zeta) - W(x, y, \tau)|\\ &= |\zeta x - \zeta y - \tau x + \tau y|\\ &= |(\zeta - \tau)x - (\zeta - \tau)y|\\ &= |\zeta - \tau|d(x, y).\end{aligned}$$

(iii)

$$\begin{aligned} W(y,x,1-\zeta)) &= (1-\zeta)y + (1-(1-\zeta))x \\ &= W(x,y,\zeta). \end{aligned}$$

(iv)

$$\begin{aligned} d(W(x,z,\zeta),W(y,s,\zeta)) &= |(W(x,z,\zeta) - W(y,s,\zeta)| \\ &= |(1-\zeta)(z-s) + \zeta(x-y)| \\ &= (1-\zeta)d(z,s) + \zeta d(x,y). \end{aligned}$$

Also $(\mathcal{X}, d, W,)$ is a complete uniformly hyperbolic space with monotone modulus of uniform convexity and $\mathcal{K} = [0,2]$ is non-empty compact convex subset of $\mathcal{X}$. Now consider relation " $\preceq$ " on $\mathcal{X}$ by $x \preceq y \; \forall \; x, y \in \mathcal{X}$. Let $T : \mathcal{K} \to \mathcal{K}$ be a mapping defined by

$$Tx = \begin{cases} 1-x, \;\; x \in [0, \frac{1}{3}), \\ \frac{x+2}{3}, \;\; x \in (\frac{1}{3}, 2]. \end{cases}$$

Then T is a monotone Suzuki mean nonexpansive mapping, but not mean nonexpansive.

Proof. To show that T is not mean nonexpansive. Suppose that T is mean nonexpansive, so $\exists$ non-negative real numbers σ and τ with $\sigma + \tau \leq 1$ and $\varrho(Tx, Ty) \leq \sigma d(x,y) + \tau d(x, Ty)$, $\forall \; x, y \in [0,2]$. Now suppose that $x = 1$, $y = 0$. Then

$$\begin{aligned} d(Tx,Ty) &= |\frac{x+2}{3} - 1 + y| \\ &= 0 \\ &\leq \sigma d(x,y) + \tau d(x,Ty) \\ &= \sigma + \tau|1 - 1 + y| \\ &= \sigma. \end{aligned}$$

Clearly $d(Tx,Ty) \leq d(x,y)$ for $\sigma \leq 1$ and $\tau = 0$. So therefore, T is a nonexpansive mapping, but this contradicts the fact that T is not continuous. Hence T is not mean non-expansive.

Next, to show that T is monotone mapping. For this, consider the following cases:
Case I: when $x, y \in [0, \frac{1}{3})$. Then

$$\begin{aligned} d(Tx,Ty) &= |x+y| \\ &\leq |x| + |y| \\ &= d(x,y). \end{aligned}$$

Case II: when $x, y \in (\frac{1}{3}, 2]$. Then

$$\begin{aligned} d(Tx,Ty) &= |\frac{x+2}{3} - \frac{y+2}{3}| \\ &= \frac{1}{3}|x-y| \\ &< |x-y| = d(x,y). \end{aligned}$$

Hence T is monotone mapping.

To establish that T is a Suzuki mean nonexpansive mapping, consider the following cases.

Case I: when $x \in [0, \frac{1}{3})$.

$$\frac{1}{2}d(x,Tx) = \frac{1}{2}|x-1+x| \\ \leq |x| + \frac{1}{2}.$$

Suppose that

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \\ \Rightarrow \ |x| + \frac{1}{2} \leq |x| + |y| \\ \Rightarrow |y| \geq \frac{1}{2}.$$

Hence $\frac{1}{2}d(x,Tx) \leq d(x,y)$ for $|y| \geq \frac{1}{2}$, i.e., $y \in [\frac{1}{2}, 1] \subset [1,2]$.
Now

$$\begin{aligned} d(Tx,Ty) &= |1-x-\frac{y+2}{3}| \\ &= |\frac{1-3x-y}{3}| \\ &\leq \frac{1}{3} + |x| + \frac{|y|}{3}, \\ \sigma d(x,y) + \tau d(x,Ty) &= \sigma d(x,y) + \tau\frac{3x-y-2}{3} \\ &\leq \sigma|x| + \sigma|y| + \tau|x| + \tau\frac{|y|}{3} + \frac{2}{3}\tau. \end{aligned}$$

Suppose that

$$d(Tx,Ty) \leq \sigma d(x,y) + \tau d(x,Ty) \\ \Rightarrow \ \frac{1}{3} + |x| + \frac{|y|}{3} \leq (\sigma+\tau)|x| + (\sigma + \frac{\tau}{3})|y| + \frac{2}{3}\tau.$$

For $\sigma = \frac{1}{2}, \tau = \frac{1}{4}$,

$$\frac{1}{3} + |x| + \frac{|y|}{3} \leq (\frac{1}{2}+\frac{1}{4})|x| + (\frac{1}{2}+\frac{1}{12})|y| + \frac{1}{6} \\ \Rightarrow \ \frac{1}{3} + |x| + \frac{|y|}{3} \leq \frac{3}{4}|x| + \frac{7}{12}|y| + \frac{1}{6} \\ \Rightarrow \ |y| - \frac{2}{3} \geq |x|.$$

Since $x \in [0, \frac{1}{3})$, $|y| \geq \frac{2}{3}$, i.e., $y \in [\frac{2}{3}, 1] \subset [1,2]$. Hence T is Suzuki mean nonexpansive mapping.

Case II: when $x \in (\frac{1}{3}, 2]$. Then

$$\frac{1}{2}d(x,Tx) = \frac{1}{2}|x - \frac{x+2}{3}| \\ \leq |x| + 1.$$

Suppose that

$$\begin{aligned}\frac{1}{2}d(x,Tx) &\leq d(x,x)\\ \Rightarrow \ |x|+1 &\leq |x|+|y|\\ \Rightarrow |y| &\geq 1.\end{aligned}$$

Hence $\frac{1}{2}d(x,Tx) \leq d(x,x)$ for $|x| \geq 1$, i.e., $x \in [1,2] \subset [0,2]$.
Now

$$\begin{aligned}d(Tx,Ty) &= |\frac{x+2}{3} - \frac{y+2}{3}|\\ &\leq \frac{|x|}{3} + \frac{|y|}{3},\\ \sigma d(x,y) + \tau d(x,Ty) &= \sigma d(x,y) + \tau|\frac{3x-y-2}{3}|\\ &\leq \sigma|x| + \sigma|y| + \tau|x| + \tau|x| + \frac{\tau}{3}|x| + \frac{2}{3}\tau.\end{aligned}$$

Suppose that

$$\begin{aligned}d(Tx,Ty) &\leq \sigma d(x,y) + \tau d(x,Ty)\\ \Rightarrow \ \frac{|x|}{3} + \frac{|y|}{3} &\leq (\sigma+\tau)|x| + (\sigma + \frac{\tau}{3})|y| + \frac{2}{3}\tau.\end{aligned}$$

For $\sigma = \frac{1}{2}, \tau = \frac{1}{4}$,

$$\begin{aligned}\frac{|x|}{3} + \frac{|y|}{3} &\leq \frac{3}{4}|x| + \frac{7}{12}|y| + \frac{1}{6}\\ \Rightarrow \ \frac{-5}{12}|x| - \frac{3}{12}|y| &\leq \frac{1}{6}\\ \Rightarrow \ 5|x| + 3|y| &\geq -2,\end{aligned}$$

which is true as $y \in [\frac{1}{2}, 1]$ and $x \in (\frac{1}{3}, 2]$. Hence T is Suzuki mean nonexpansive mapping. □

CONFLICT OF INTEREST

The author declares that they have no conflict of interest.

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