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On minimal and mean spaces

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ABSTRACT. The minimal and mean open sets are relatively new concepts in the literature of topological spaces. We use these two notions to introduce and study of the minimal and mean spaces which are generated by bases consisting of minimal and mean open sets respectively. We see that every minimal space is of zero-dimensional. We also obtain an equivalence relation between a T_1 cut-point space and a mean space.

1. INTRODUCTION AND PRELIMINARY

We agree to write $(Y, \mathscr{S})$ for a topological space on a nonempty set Y endowed with a topology $\mathscr{S}(\neq \{\emptyset, Y\})$. The term proper open set is used for an open set $G \in \mathscr{S} - \{\emptyset, Y\}$ of Y. Likewise, we use the term proper closed set for a closed set E in Y such that $Y - E \in \mathscr{S} - \{\emptyset, Y\}$. For $P \subset Y$, $Cl(P)$ stands for the closure of P and $Int(P)$, for the interior of P in Y. The cardinality of $P \subset Y$ is represented by $|P|$.

Undoubtedly, the concept of open sets plays the key role in studying topological spaces. There in the literature of topological spaces, we find several generalizations of open sets, e.g., semi-open set [4], pre-open sets [5] and many more. In recent years, Nakaoka and Oda [11, 10, 9] introduced and studied two new varieties of open sets called maximal and minimal open sets. Following the research works of Nakaoka and Oda [11, 10, 9], another variety of open sets called mean open set [7] in topological spaces is posed. It reveals that the study of open sets in topological spaces from different angles bear some importance to the researchers. In the present study, we use the mean and minimal open sets to give some new twists to the topological spaces after the name minimal and mean spaces.

Here is mainly three sections except introductory section in the article. Section 2 is about the motivation of introducing notions such as minimal (Definition 3.7) and mean spaces (Definition 4.8). In Section 3, the notion of minimal spaces is introduced and some properties of such spaces are investigated. Theorem 3.8 ensures the existence of minimal spaces whereas Theorem 3.9 gives necessary and sufficient conditions for existence of minimal spaces. Theorem 3.12 shows that a minimal space is disconnected. We focus on mean spaces in Section 4. Natural existence of mean spaces are shown in Example 4.2. Theorem 4.14 is regarding the existence of mean spaces. A necessary and sufficient condition for existence of mean spaces is established in Theorem 4.15. Three equivalent statements on the connected mean spaces are proved in Theorem 4.18. Some examples are also framed to substantiate results as and when necessity arises.

Some established notions and results are hereby recalled so that the readers can go through the contents easily as far as practical without consulting the cited articles.

Definition 1.1 (Nakaoka and Oda [11, 9]). *An open set $W \in \mathscr{S} - \{\emptyset, Y\}$ of Y is called minimal open in Y if $\emptyset$ and W are the only open sets on Y contained in W.*

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Definition 1.2 (Nakaoka and Oda [9]). *A closed set $E(\neq \emptyset, Y)$ is called minimal closed in Y if $\emptyset$ and E are the only closed sets in Y contained in E.*

Definition 1.3 (Nakaoka and Oda [10]). *An open set $W \in \mathscr{S} - \{\emptyset, Y\}$ of Y is called maximal open in Y if Y and W are the only open sets in Y containing W.*

Definition 1.4 (Mukharjee and Bagchi [7]). *An open set W of Y is called mean open in Y if there exist two different open sets $G, V(\neq \emptyset, W, Y)$ in Y such that $G \subset W \subset V$.*

Definition 1.5 (Honari and Bahrampour [2]). *A point $t \in Y$ in a connected topological space $(Y, \mathscr{S})$ is called a cut point in Y if $Y - \{t\}$ is a disconnected subspace of Y. The connected topological space Y is called a cut-point space if each point of Y is a cut point of Y.*

Definition 1.6 (Howard and Tachtsis [3]). *A cover $\mathscr{W}$ of a topological space $(Y, \mathscr{S})$ is called a minimal cover if for each $W \in \mathscr{W}$, $\mathscr{W} - \{W\}$ does not form a cover of Y.*

If each member of a cover $\mathscr{W}$ of Y is minimal open (resp. minimal closed) then $\mathscr{W}$ is called a minimal open (resp. minimal closed) cover of Y.

Theorem 1.1 (Nakaoka and Oda [11]). *If W and V are minimal open set and open set respectively in a topological space Y, then either $W \cap V = \emptyset$ or $W \subset V$ holds. If $V(\neq W)$ is also a minimal open set in Y, then $W \cap V = \emptyset$.*

Theorem 1.2 (Nakaoka and Oda [10]). *If W and V are maximal open set and open set respectively in a topological space Y, then either $W \cup V = Y$ or $V \subset W$ holds. If $V(\neq W)$ is also a maximal open set in Y, then $W \cup V = Y$.*

Theorem 1.3 (Nakaoka and Oda [9]). *A closed set F in Y is minimal closed in Y iff $Y - F$ is maximal open in Y.*

Theorem 1.4 (Mukharjee [6]). *If W is maximal as well as minimal open in Y, then either (i) W is the only proper open set in Y or (ii) W and $Y - W$ are the only proper open sets in Y.*

Lemma 1.1 (Bagchi and Mukharjee [1]). *The nonempty open sets in a T_1 connected space Y are infinite and they are not minimal open in Y.*

Theorem 1.5 (Bagchi and Mukharjee [1]). *A proper open set W in a T_1 connected space Y is mean open if and only if $W \neq Y - \{s\}, s \in Y$.*

Theorem 1.6 (Bagchi and Mukharjee [1]). *Suppose $(Y, \mathscr{S})$ is a T_1 connected space and $\mathscr{S}_{mo}$ is the collection of all mean open sets in Y. The collection $\{\emptyset\} \cup \mathscr{S}_{mo}$ constitutes a basis of a topology $\mathscr{S}$ on Y.*

Theorem 1.7 (Honari and Bahrampour [2]). *Let $b \in Y$ be a cut point of a connected space Y such that $Y - \{b\} = M \mid N$. The set $\{b\}$ is either open or closed in Y. Also M and N are closed (resp. open) in Y if $\{b\}$ is open (resp. closed) in Y.*

2. Motivation

We observe that $\{w\}$ is a minimal open set (Definition 1.1) for any $w \in Y$ in the discrete topological space Y and so $\{\emptyset\} \cup \{\{w\} \mid w \in Y\}$ constitutes a basis of the discrete topological space Y. It means that the discrete topological space Y has a basis forms by minimal open sets and empty set only. It is well-known that the collection $\{(a, b) | a, b \in \mathbb{R}\}$ of open intervals along with $\{\emptyset\}$ forms a basis for the usual topological space where $\mathbb{R}$ is the set of real numbers. Since (a, b) is a mean open set (Definition 1.4) in $\mathbb{R}$, we conclude that $\mathbb{R}$ has a basis consisting of mean open sets and empty set only. These two facts lead us to introduce and to investigate the minimal spaces (Definition 3.7) and mean spaces (Definition 4.8).

3. Minimal Spaces

We introduce below the notion of minimal spaces and obtain some characterizations on such spaces.

Definition 3.7. *A topological space $(Y, \mathscr{S})$ is said to be a minimal space if there exists a basis $\{\emptyset\} \cup \mathscr{C}$ of $\mathscr{S}$ where $\mathscr{C}$ is a family of minimal open sets of Y. The family $\{\emptyset\} \cup \mathscr{C}$ is said to be a minimal basis for $\mathscr{S}$ or Y when no confusion arises on $\mathscr{S}$.*

We see that a basis $\mathscr{C}$ is minimal for Y if the elements of $\mathscr{C}$ are pairwise disjoint.

We agree to write $\mathscr{M}$ for the family of all minimal open sets in Y. We also write $\mathscr{N} = \{\emptyset\} \cup \mathscr{M}$. Clearly, if $\mathscr{C}$ is a minimal basis for Y then $\mathscr{C} \subset \mathscr{N}$.

Theorem 3.8 (Existence of minimal spaces)**.** *Suppose that Y is a nonempty set with $|Y| \geq 3$. There exists a topology $\mathscr{S}$ in Y different from the discrete topology in Y such that $(Y, \mathscr{S})$ is a minimal space.*

Proof. Suppose $\mathscr{B} = \{\emptyset\} \cup \{B_\gamma \mid \gamma \in \Gamma\}$ is a family of subsets of Y such that $B_\nu \cap B_\mu = \emptyset$ for any two $\nu, \mu \in \Gamma, \nu \neq \mu$, $\bigcup_{\gamma \in \Gamma} B_\gamma = Y$ and $|B_\gamma| \geq 2$ for some $\gamma \in \Gamma$. Then $\mathscr{B}$ forms a basis for some topology $\mathscr{S}$ on Y. Suppose $B_{\gamma_1} \in \mathscr{B}$ with $|B_{\gamma_1}| \geq 2$ and $s, y \in B_{\gamma_1}$. It then follows that $\{s\}, \{y\}$ are not the members of $\mathscr{B}$. Hence $(Y, \mathscr{S})$ is not the discrete space. Again since any two distinct members of $\mathscr{B}$ are disjoint, $\mathscr{B}$ is a minimal basis for a topology $\mathscr{S}$ on Y. Hence $(Y, \mathscr{S})$ is a minimal space. □

Theorem 3.9. *A topological space Y is a minimal space iff $\mathscr{M}$ forms an open cover of Y. Further, $\mathscr{M}$ forms a minimal open cover of Y.*

Proof. Firstly, let Y be a minimal space and $\mathscr{D}$ be a minimal basis for Y. The collection $\mathscr{D}$ being a basis for Y, $\bigcup_{B \in \mathscr{D}} B = Y$. Since $\mathscr{D} \subset \mathscr{N}$, $\mathscr{M} = \mathscr{N} - \{\emptyset\}$ constitutes an open cover of Y.

Conversely, suppose $\mathscr{M}$ is an open cover of Y. As two distinct elements of $\mathscr{M}$ are disjoint by Theorem 1.1 and $\mathscr{M} \subset \mathscr{N}$, $\mathscr{N}$ is a minimal basis for Y. It means that Y is a minimal space.

$\mathscr{M}$ is a minimal open cover of Y follows from the fact that any two distinct members of $\mathscr{M}$ are disjoint. □

Remark 3.1. *We see that $\mathscr{M}$ forms a partition on Y when Y is a minimal space. Hence there exists an equivalence relation on Y.*

Theorem 3.10. *If $\mathscr{B}$ is a basis for a topological space Y and M is a minimal open set in Y, then $M \in \mathscr{B}$.*

Proof. If possible, let $M \notin \mathscr{B}$. There is a subcollection $\mathscr{B}_1$ of $\mathscr{B}$ containing at least two nonempty basic open sets distinct from M such that $M = \bigcup_{B \in \mathscr{B}_1} B$ which implies that $B \subset M$ for some $B(\neq \emptyset, M) \in \mathscr{B}_1$. But this contradicts the fact that M is a minimal open set in Y. □

Corollary 3.1. *The minimal basis for a minimal space is unique.*

Proof. Let $\mathscr{H}, \mathscr{B}$ be two minimal bases for a minimal space Y. Suppose $M \in \mathscr{H} - \{\emptyset\}$. Since M is minimal open in Y and $\mathscr{B}$ is a basis for Y, we have $M \in \mathscr{B}$ by Theorem 3.10 and hence $\mathscr{H} \subset \mathscr{B}$. So we obtain $\mathscr{H} = \mathscr{B}$. □

Corollary 3.2. *If $\mathscr{B}$ is a minimal basis of a minimal space Y, then $\mathscr{B} = \mathscr{N}$.*

Proof. We have $\mathscr{B} \subset \mathscr{N}$. Also by Theorem 3.10, $\mathscr{N} \subset \mathscr{B}$. Hence $\mathscr{B} = \mathscr{N}$. □

Theorem 3.11 (Existence of maximal open sets). *A minimal space Y contains a maximal open set. Further, if $\mathscr{M}^*$ denotes the collection of all maximal open sets on Y, then $\mathscr{M}^* = \{\bigcup_{G\in\mathscr{M}-\{P\}} G \mid P \in \mathscr{M}\}$.*

Proof. Given Y is a minimal space. So $\mathscr{M}$ is a minimal open cover of Y by Theorem 3.9. For any $M \in \mathscr{M}$, $\bigcup_{G\in\mathscr{M}-\{M\}} G$ is a proper open set in Y. If possible, suppose that N is an open set in Y such that $\bigcup_{G\in\mathscr{M}-\{M\}} G \subset N$. Since M is a minimal open set in Y, we conclude that either $M \subset N$ or $M \cap N = \emptyset$ by Theorem 1.1. If $M \subset N$, then $\bigcup_{G\in\mathscr{M}} G \subset N$ which shows $Y \subset N$ and so we get $N = Y$. If $M \cap N = \emptyset$, then we have $N \subset Y - M = \bigcup_{G\in\mathscr{M}-\{M\}} G$. So $\bigcup_{G\in\mathscr{M}-\{M\}} G = N$. So it follows that $\bigcup_{G\in\mathscr{M}-\{M\}} G$ is a maximal open set in Y.

From the above discussion, it is clear that $\{\bigcup_{G\in\mathscr{M}-\{P\}} G \mid P \in \mathscr{M}\} \subset \mathscr{M}^*$. Now let $G \in \mathscr{M}^*$. If possible, let $G \neq \bigcup_{H\in\mathscr{M}-\{M\}} H$ for any $M \in \mathscr{M}$. Since $\mathscr{N}$ is the minimal basis for Y by Corollary 3.2 and any two distinct members of $\mathscr{M}$ are disjoint by Theorem 1.1, there exist at least two disjoint members O, N in $\mathscr{M}$ such that $G = \bigcup_{H\in\mathscr{H}} H$ where $\mathscr{H}$ is a subfamily of $\mathscr{M} - \{M, N, O\}$. But this contradicts the fact of the maximality of G because G is a subset of the proper open sets $\bigcup_{H\in\mathscr{M}-\{M,N\}} H$ and $\bigcup_{H\in\mathscr{M}-\{M,O\}} H$. Hence $\mathscr{M}^* \subset \{\bigcup_{G\in\mathscr{M}-\{P\}} G \mid P \in \mathscr{M}\}$ and thus $\mathscr{M}^* = \{\bigcup_{G\in\mathscr{M}-\{P\}} G \mid P \in \mathscr{M}\}$. □

Remark 3.2. *From Theorem 3.11, it is clear that if $\mathscr{M}^*$ denotes the family of all maximal open sets in Y, then $|\mathscr{M}| = |\mathscr{M}^*|$.*

Corollary 3.3. *A minimal space Y is T_1 iff Y is discrete.*

Proof. Firstly, suppose that Y is T_1. Then for each $z \in Y$, $\{z\}$ is minimal closed in Y. So $Y - \{z\}$ is maximal open in Y by Theorem 1.3. Hence by Theorem 3.11, $Y - \{z\} = \bigcup_{G\in\mathscr{M}-\{P\}} G$ for some $P \in \mathscr{M}$ which gives $P = \{z\}$. Therefore for each $z \in Y$, $\{z\}$ is open in Y and so Y is a discrete space.

The converse part is straightforward. □

Corollary 3.4. *A minimal space Y is metrizable iff Y is a discrete space.*

Proof. The proof follows by Corollary 3.3 with the fact that every metric space is T_1. □

Theorem 3.12. *A minimal space Y is disconnected.*

Proof. As Y is a minimal space, $\mathscr{M}$ forms a minimal open cover of Y by Theorem 3.9. By Theorem 3.11, it follows that $\bigcup_{G\in\mathscr{M}-\{M\}} G$ is a maximal open set in Y for any $M \in \mathscr{M}$. Hence by Theorem 1.2, $\{\bigcup_{G\in\mathscr{M}-\{M\}} G, M\}$ makes a separation on Y for each $M \in \mathscr{M}$. □

Example 3.1 (Mukharjee [6]). *On the set R of real numbers, we define $\mathscr{T} = \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a], [a, \infty)\}$. The topological space $(R, \mathscr{T})$ is disconnected but it is not a minimal space. Hence we conclude that the converse of Theorem 3.12 need not be true.*

It is well known that a topological space Y is of zero-dimensional iff Y has a basis consisting of clopen sets only.

Corollary 3.5. *A minimal space Y is of zero-dimensional.*

Proof. From Theorem 3.12, it is clear that each $M \in \mathscr{M}$ is closed in Y. Thus we have a basis for Y consisting of clopen sets only and so Y is of zero-dimensional. □

Theorem 3.13. *A minimal space Y is compact iff $|\mathscr{M}| = n$, where $n(\neq 1)$ is a finite natural number.*

Proof. Firstly, let Y be a compact space. Since Y is a minimal space, $\mathscr{M}$ is a minimal open cover of Y by Theorem 3.9. If possible, let $|\mathscr{M}|$ be infinite. As Y is compact, $\mathscr{M}$ has a finite subcover. But this contradicts the fact that $\mathscr{M}$ is a minimal open cover of Y. Since a minimal open set is a proper open set, $|\mathscr{M}| \neq 1$.

Conversely, suppose $|\mathscr{M}| = n$, where $n(\neq 1)$ is a finite natural number. Since Y is a minimal space, $\mathscr{M}$ is a finite basis for Y by Corollary 3.2. As we know that a topological space Y is compact if and only if there is a basis $\mathscr{C}$ for Y such that any open cover of Y by members of $\mathscr{C}$ has a finite subcover, Y is a compact topological space. □

Remark 3.3. *Recall that a basis $\mathscr{C}$ of a topological space Y is uniform if for each point $z \in Y$ and any open set M with $z \in M$, there are at most finitely many members C of $\mathscr{C}$ for which $z \in C \nsubseteq M$. So by Theorem 3.13, if Y is a compact minimal space then $\mathscr{M}$ forms a uniform basis for Y and in this case Y is metacompact as well as developable* [8, p. 262].

4. Mean Spaces

This section deals with mean topological spaces.

Definition 4.8. *A topological space Y is said to be a mean space if there is a basis $\{\emptyset\} \cup \mathscr{B}$ for Y where $\mathscr{B}$ consists of mean open sets of Y only. In this case, $\mathscr{B}$ is said to be a mean basis for Y.*

We see that a mean topological space has no minimal open set.

Example 4.2. *(i) Every T_1 connected topological space is a mean space (by Theorem 1.6).*
(ii) Sorgenfrey line is a disconnected mean space.

Remark 4.4. *From the above examples, it is clear that two mean spaces may not be homeomorphic.*

It may appears that a disconnected space may be a mean space but it is not true.

Example 4.3. *Let R be the set of real numbers and $\mathscr{T} = \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a], (a, \infty), [a, \infty), R - \{a\}\}$. The topological space $(R, \mathscr{T})$ is disconnected but it is not a mean space.*

Theorem 4.14. *There exists a mean space on a linear ordered set Y if for $p, q \in Y$ with $p < q$, we have $t \in Y$ such that $p < t < q$.*

Proof. We have to prove that there is a mean basis $\mathscr{B}$ for some topology $\mathscr{S}$ on Y. For this, we consider the following four cases:
Case (i): Let Y has neither the least nor the greatest element. Then $\mathscr{B} = \{\emptyset\} \cup \{(s, t) \mid s, t \in Y\}$ gives a mean basis for Y.
Case (ii): Let u be the least element of Y but Y has no greatest element. Then $\mathscr{B} = \{\emptyset\} \cup \{[u, d) \mid d \in Y\}$ gives a mean basis for Y.
Case (iii): Let v be the greatest element of Y but Y has no least element. Then $\mathscr{B} = \{\emptyset\} \cup \{(c, v] \mid c \in Y\}$ gives a mean basis for Y.
Case (iv): Let u be the least element and v be the greatest element of Y. Since $u < v$, there exists $c \in Y$ such that $u < c < v$. The subbasis $\{\emptyset\} \cup \{[u, d) \mid d \in Y, u < d\} \cup \{(e, v] \mid e \in Y, e < v\}$ gives a mean basis for Y. □

Corollary 4.6. *There exists a compact, connected and non-metrizable mean space.*

Proof. We consider a linearly ordered set Y having the following properties:
(i) whenever $p, q \in Y$ with $p < q$ there is a $z \in Y$ such that $p < z < q$, and
(ii) Y has both least and greatest elements, namely u and v respectively.
Let $\mathscr{C}$ be the basis for Y generated by the subbasis constructed in Case (iv) of Theorem 4.14 and $\mathscr{P}$ be the topology generated by the basis $\mathscr{C}$. Clearly, $\mathscr{C}$ is a mean basis for $\mathscr{P}$. As any two distinct open sets of $(Y, \mathscr{P})$ have nonempty intersection, $(Y, \mathscr{P})$ is connected. Let

$\mathscr{U}$ be an open cover of Y. We choose $U_1, U_2 \in \mathscr{U}$ such that $u \in U_1, v \in U_2$. Then $\{U_1, U_2\}$ is a finite subcover of $\mathscr{U}$. So $(Y, \mathscr{P})$ is compact. Since $(Y, \mathscr{P})$ is not a Hausdorff space, it is non-metrizable. □

Theorem 4.15. *A cut-point space Y is a mean space iff Y is T_1.*

Proof. Necessity: Any $t \in Y$ is a cut point of Y and so $\{t\}$ is either open or closed in Y follows by Theorem 1.7. As Y is a mean space, there has no minimal open set in Y. Therefore $\{t\}$ is closed in Y. It then follows that Y is a T_1-space.
Sufficiency: As Y is a T_1 connected topological space, we get a mean basis for Y by Theorem 1.6. □

Theorem 4.16. *Let Y, Z be two topological spaces and there is a homeomorphism $f : Y \to Z$. If Y is a mean space, Z is so.*

Proof. Let $\mathscr{D}$ be a mean basis for Y. We prove that $\{f(B) \mid B \in \mathscr{D}\}$ forms a mean basis for Z. As each $B \in \mathscr{D}$ is mean open in Y and f is bijective, $f(B)$ is mean open in Z. Suppose that M is open in Z. Then $f^{-1}(M)$ is open in Y. It means that there have a subfamily $\mathscr{D}_1$ of $\mathscr{D}$ such that $f^{-1}(M) = \bigcup_{B\in\mathscr{D}_1} B$. Now $f^{-1}(M) = \bigcup_{B\in\mathscr{D}_1} B \Rightarrow M = f(\bigcup_{B\in\mathscr{D}_1} B) = \bigcup_{B\in\mathscr{D}_1} f(B)$. As $\{f(C) \mid C \in \mathscr{D}_1\}$ is a subcollection of $\{f(C) \mid C \in \mathscr{D}\}$, it follows that $\{f(C) \mid C \in \mathscr{D}\}$ is a basis for Y. □

Lemma 4.2. *Suppose Y is a mean space and G_1, G_2 be distinct open sets in Y with $G_1 \cap G_2 \neq \emptyset$. Then $G_1 \cap G_2$ is a mean open set in Y.*

Proof. The following two cases may arise:
Case (i): $G_i \subsetneq G_j, i, j \in \{1, 2\}, i \neq j$. Then $G_1 \cap G_2$ is a proper subset of G_2 or G_1 which gives that $G_1 \cap G_2$ is not a maximal open set in Y.
Case (ii): $G_i \nsubseteq G_j, i, j \in \{1, 2\}, i \neq j$. Then $G_1 \cap G_2$ is a proper subset of both G_1, G_2 and so $G_1 \cap G_2$ is not a maximal open set in Y.

In either of above two cases, as Y has no minimal open set, $G_1 \cap G_2$ is a mean open set in Y. □

For any subset A of the topological space Y, we express the derived set of A by A'.

Theorem 4.17. *For any open set G in a mean space Y, $G \subset G'$.*

Proof. As G is open in the mean space Y, G is an infinite subset of Y. Let $g \in G$. To show that $g \in G'$, let M be an open set with $g \in M$. Then the following two cases may arise:
Case (i): $G = M$. Then M contains a point of G other than g.
Case (ii): $G \neq M$. Then $G \cap M \neq \emptyset$ and so by Lemma 4.2, $G \cap M$ is a mean open set in Y. Thus $G \cap M$ is infinite and so M contains a point of G other than g.
For both of the above two cases, we see that any for open set M with $g \in M$ consists of a point of G different from g which means $g \in G'$. Hence we have $G \subset G'$. □

Corollary 4.7. *An open set M in a mean space Y is also closed in Y iff $M = M'$.*

Proof. If $M = M'$, then $Cl(M) = M \cup M' = M$ and so M is closed in Y.

Conversely, let M be closed in Y. Then $Cl(M) = M$. It yields $M \cup M' = M \Rightarrow M' \subset M$. Therefore by the Theorem 4.17, $M = M'$. □

Corollary 4.8. *For any open set M in a connected mean space Y, we have $M \subsetneq M'$.*

Proof. By Theorem 4.17, $M \subset M'$ which yields either $M \subsetneq M'$ or $M = M'$. The open set M in Y would also be closed in Y if $M = M'$ by Corollary 4.7 which in turns would implies that Y is disconnected. Hence $M = M'$ is not possible as Y is connected. □

Theorem 4.18. *Three statements given below are equivalent in a connected mean space Y:*

(i) M *is a proper regularly open set.*
(ii) $Int(M') = M$.
(iii) $M' - M \subset (Y - M')'$.

Proof. $(i) \Rightarrow (ii)$: Given M is open in a connected mean space Y. So we have $M \subsetneq M'$ by Corollary 4.8. Then $Cl(M) = M \cup M' = M'$. As M is a proper regularly open set in Y, $Cl(M) \neq Y$. Since $M = Int(Cl(M))$, we see that $M = Int(M')$.

$(ii) \Rightarrow (iii)$: Let $t \in M' - M$. Then $t \notin M$ and by (ii) t is not an interior point of M'. Hence any open set V with $t \in V$ consists of a point $z(\neq t)$ such that $z \notin M'$ which in turns implies that $z \in Y - M'$. Therefore $t \in (Y - M')'$ and so $M' - M \subset (Y - M')'$.

$(iii) \Rightarrow (i)$: We have $M \subset M'$ according to Theorem 4.17. So $M \subset Int(M')$. Then we only require to show that $Int(M') \subset M$. Let $p \in Int(M')$. So we have an open set N in Y such that $p \in N \subset M'$. If $p \notin M$ then by (iii) N contains a point of $Y - M'$ but this contradicts the fact that $N \subset M'$. Thus $p \in M$ and so $Int(M') \subset M$. Hence $Int(Cl(M)) = Int(M') = M$. □

Theorem 4.19. *For any topological space Z, a function $h : Z \to \mathbb{R}$ is continuous iff $h^{-1}(M)$ is open in Z for any mean open set M in $\mathbb{R}$.*

Proof. Firstly, suppose $h^{-1}(M)$ is open in Z for each mean open set M in $\mathbb{R}$. Then $h^{-1}(d, \infty)$ and $h^{-1}(-\infty, d)$ are open sets in Z for each $d \in \mathbb{R}$. Therefore h is upper as well as lower semicontinuous. Thus h is continuous.

The converse part is obvious. □

5. Conclusion

A basis of a topological space Y is sufficient to find all characterizations of that topological space Y and it is easier to study a topological space Y by a basis compare to taking into consideration of the topological structure. Theorem 3.8 and Theorem 4.14 ensure the existence of minimal spaces and mean spaces respectively. An endorsement to the minimal spaces is due to Theorem 3.9. Thus we expect that the article would make it convenient to get findings of topological spaces. A natural example of a cut-point space is the real line R endowed with the usual topology. Theorem 4.15 glorifies the mean spaces by the cut-point spaces. Theorem 4.19 reveals that a continuous function of the type $f : Y \to \mathbb{R}$ can be characterized by the mean open sets in $\mathbb{R}$. In overall, we conclude that the present work may further enhance and promote the research in the fields like the cut-point spaces, connectedness, compactness etc.

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