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A Study On Roman Domination in Deg-centric Graphs

TIMMY TOMY THALAVAYALIL [1]

ABSTRACT. The deg-centric graph of a simple, connected graph G, denoted by G_d, is a graph constructed from G such that $V(G_d) = V(G)$ and $E(G_d) = \{v_i v_j : d_G(v_i, v_j) \leq deg_G(v_i)\}$. This paper presents the Roman domination number of deg-centric graphs. Also, investigate the properties and structural characteristics of this type of graph.

1. INTRODUCTION

For a basic terminology of graph theory, we refer to [4]. For further topics on graph classes, [9]. The number of edges of a graph G is denoted by $e(G)$. Recall that the distance between two distinct vertices v_i and v_j of G, denoted by $d_G(v_i, v_j)$, is the length of the shortest path joining them. The eccentricity of a vertex $v_i \in V(G)$, denoted by $e(v_i)$, is the farthest distance from v_i to some vertex of G. The diameter of a graph is the maximum eccentricity among all the vertices. A particular type of newly derived graphs based on the vertex degrees and distances in graphs called *deg-centric graphs* have been introduced in [5] as follows, The *degree centric graph* or *deg-centric graph* of a graph G is the graph G_d with $V(G_d) = V(G)$ and $E(G_d) = \{v_i v_j : d_G(v_i, v_j) \leq deg_G(v_i)\}$ [5]. Let G be a graph and G_d be the deg-centric graph of G. Then, the successive iteration *deg-centric graph* of G, denoted by G_{d^k}, is defined as the derived graph obtained by taking the deg-centric graph successively k times, that is, $G_{d^k} = ((G_d)_d \ldots)_d$, ($k$-times). This process is known as *deg-centrication process* [5].

Roman dominating functions and their variants have been in the literature for over more than two decades [2, 1, 3]. Cockayne et al.[1] were the first to mathematically formulate the concept of Roman dominating functions in graphs based on the defence strategy of Roman Emperor Constantine that was mentioned in the work of Ian Stewart (see[13]). A *Roman Dominating Function* (RDF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The value $\omega(f) = \sum_{v \in V} f(v)$ is called the weight of f. The least value of $\omega(f)$ among all the Roman dominating functions f on S is called the *Roman domination number* of S, denoted by $\gamma_R(S)$. A Roman dominating function f with $\omega(f) = \gamma_R(S)$ is called a γ_R-function of S [1].

The functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. There is always a one-one correspondence between these functions and the ordered partitions induced by them and thus these functions can be written as $f = (V_0, V_1, V_2)$.

Motivated by the above-mentioned studies, we investigate the Roman domination and some properties of deg-centric graphs.

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Definition 1.1. [5] *The degree centric graph or deg-centric graph of simple graph G, denoted by G_d, is the graph with $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i, v_j) \leq \max\{\deg_G(v_i), \deg_G(v_j)\}\}$. This graph transformation is called* deg-centrication *of the graph.*

Definition 1.2. [5] The *iterated deg-centric graph* of a graph G, denoted by G_{d^k}, is defined as the graph obtained by applying *deg-centrication* successively k-times. That is, $G_{d^k} = ((G_d)_{d\ldots})_d$, ($k$-times).

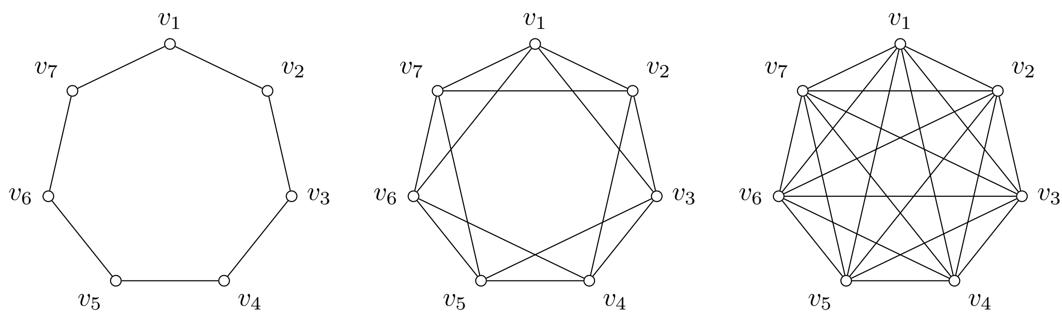
Theorem 1.1. [5] *The deg-centric graph of a non-star graph G with $\delta(G) \geq diam(G)$ is complete.*

Corollary 1.1. [5] *The deg-centric graph G_d of a non-star graph G with $deg_G(v_i) \geq e(v_i)$ is complete.*

Proposition 1.1. [1] *For any graph G of order n, $\gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K_n}$.*

Proposition 1.2. [1] *If G is a graph of order n which contains a vertex of degree $n-1$, then $\gamma(G) = 1$ and $\gamma_R(G) = 2$.*

An illustration of iterated deg-centrication of a cycle graph on 7 vertices is given in Figure 1.



(A) C_7 (B) C_{7_d} (C) $C_{7_{d^2}}$

FIGURE 1 Example of iterated deg-centrication of C_7.

2. RESULTS

Proposition 2.3. *For a connected graph G of order n, if $deg_G(v_i) \geq e_G(v_i)$, for all $v_i \in V(G)$, then, $\gamma_R(G_d) = 2$.*

Proof. Connected graph G of order n, if $deg_G(v_i) \geq e_G(v_i)$, then,in view of Proposition 1.1, $G_d \cong K_n$. In Roman domination, the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. The deg-centric graph is a complete graph with n vertices so that we can assign value two to any vertex v_i. That is, $f(v_i) = 2$, all other vertices are adjacent with v_i, we can assign value zero to these vertices. Then, after summation, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R(G_d) = 2$. □

Proposition 2.4. *If G_d is a graph of order $n \geq 3$ which contains a vertex of degree $n-1$, then $\gamma_R(G_d) = 2$.*

Proof. The result is a direct consequence of Proposition 1.2. □

Proposition 2.5. *For any deg-centric graph G_d of order n, $G_d = \overline{K_n}$, then, $\gamma_R(G_d) = n$.*

Proof. The result is a direct consequence of the Definition of Roman domination, and we can assign each vertex a separate value. □

3. Roman Domination Number of Deg-centric Graphs

This section will address the Roman domination number of the deg-centric graphs.

Proposition 3.6. *For a complete graph K_n, $n \geq 3$, $\gamma_R((K_n)_d) = 2$.*

Proof. For a complete graph K_n, $\delta(K_n) \geq e_G(v_i)$, the deg-centric graph of a complete graph K_n of order $n \geq 3$ is always isomorphic to the complete graph K_n. In view of Proposition 2.3, $\gamma_R((K_n)_d) = 2$. □

For convenience, a path P_n is depicted on a horizontal line, and the vertices are labelled from left to right as v_1, v_2, v_3,..., v_n.

Proposition 3.7. *For a path P_n,*

$$\gamma_R((P_n)_d) = \begin{cases} 1; & \text{if } n = 1 \\ 2\lceil \frac{n}{5} \rceil, & \text{if } n \equiv 0,2,3,4 (\mod 5). \\ 2\lfloor \frac{n}{5} \rfloor + 1, & \text{if } n \equiv 1 (\mod 5). \end{cases}$$

.

Proof. The path P_n is depicted on a horizontal line, and the vertices are labelled from left to right as v_1, v_2, v_3,..., v_n. In Roman domination, the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. In view of Definition1.1, the vertices v_1, v_n have a degree of two, vertices v_2, v_{n_1} have a degree of three and the vertices $v_3, v_4, \ldots, v_{n-3}, v_{n-2}$ have a degree of four in $(p_n)_d$. Consider the deg-centric graph P_1, clearly $\gamma_R((P_1)_d) = 1$. For $n = 2, 3, 4,$, clearly v_2 added to the least domination set. Hence, $\gamma_R((P_n)_d) = 2$. Similarly, in P_5, vertex v_3 is adjacent to all other vertices in the deg-centric graph, $\gamma_R((P_5)_d) = 2$. Hence, $\gamma_R((P_5)_d) = 2\lceil \frac{n}{5} \rceil$.

If $n \geq 6$, with consecutively labeled vertices $\{v_1, v_2, v_3, \ldots, v_n\}$. In view of Definition 1.1, the path P_6, in Roman domination of $(p_6)_d$, we can assign value two to vertex v_3. That is, $f(v_3) = 2$, assign all 4 adjacent vertices value as zero, that is $f(v_1) = 0, f(v_2) = 0, f(v_4) = 0$ and $f(v_5) = 0$. Then, the remaining vertex v_6 adds value as $f(v_6) = 1$. Hence$\gamma_R((P_6)_d) = 3$. Hence, $\gamma_R((P_n)_d) = 2\lfloor \frac{n}{5} \rfloor$. Similarly, in the Roman domination of $(p_7)_d$, we can assign value two to vertices v_3 and v_7. That is, $f(v_3) = 2$ and $f(v_7) = 2$, assign all other adjacent vertices value as zero, $\gamma_R((P_7)_d) = 4$. Hence, $\gamma_R((P_7)_d) = 2\lceil \frac{n}{5} \rceil$.

If $n \equiv 0, 2, 3, 4 (\mod 5)$, in Roman domination of $(p_n)_d$, we can assign value 2 to the vertices v_{3+5i}, all other vertices are adjacent with these vertices, add value zero to these vertices, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2\lceil \frac{n}{5} \rceil$.

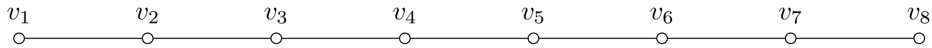
If $n \equiv 1 (\mod 5)$, in Roman domination of $(p_n)_d$, we can assign value 2 to vertices v_{3+5i}, all other vertices are adjacent with these vertices except vertex v_{5i+1}. Then, the remaining vertex v_{5i+1} adds value as $f(v_{5i+1}) = 1$. Hence, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2\lfloor \frac{n}{5} \rfloor + 1$. □

An illustration of proposition 3.7 is given in Figure 2.

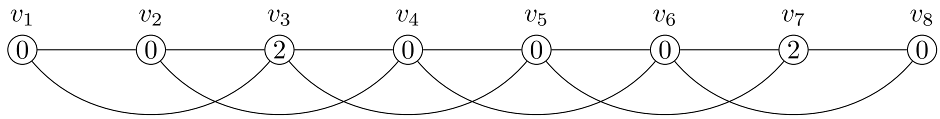
A *star graph*, denoted by $k_{1,n}$, $n \geq 0$, is obtained by attaching n pendant vertices (also called leaves) to a central vertex v_0.

Proposition 3.8. *For $n \geq 1$, $\gamma_R((K_{1,n})_d) = 2$.*

Proof. In view of Definition 1.1, the deg-centric graph of a star graph $k_{1,n}$, $n \geq 0$, is always isomorphic to the star graph. If $n \geq 0$, in Roman domination, in the central vertex, assign value as two, and all other vertices as zero. Hence, $\gamma_R((K_{1,n})_d) = 2$. □



(A) P_8



(B) $(P_8)_d$

FIGURE 2 $\gamma_R((P_8)_d) = 4$.

Proposition 3.9. *For a cycle C_n, $n \geq 3$,*

$$\gamma_R((C_n)_d) = \begin{cases} 2\lceil \frac{n}{5} \rceil, & \textit{if } n \equiv 0,2,3,4(\mod 5). \\ 2\lfloor \frac{n}{5} \rfloor + 1, & \textit{if } n \equiv 1(\mod 5). \end{cases}$$

.

Proof. In Roman domination, the functions $f : V \to \{0,1,2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0,1,2$. Consider the deg-centric graph C_n, $n \leq 5$, clearly $(C_n)_d$ is complete, $\gamma_R((C_n)_d) = 2$. Hence, $\gamma_R((C_n)_d) = 2\lceil \frac{n}{5} \rceil$. If $n \geq 6$, with consecutively labeled vertices $\{v_1, v_2, v_3, \ldots, v_n\}$. In view of Definition 1.1, $\deg_{C_n}(v_i) = 2$, for all $v_i \in V(C_n)$, any vertex v_i in $(C_n)_d$ is adjacent to all vertices $d_{C_n}(v_i, v_j) \leq 2$, the deg-centric graph, $(c_n)_d$ is always a 4-regular graph.

In view of Definition 1.1, the cycle C_6, in Roman domination of $(C_6)_d$, we can assign value two to vertex v_3. That is, $f(v_3) = 2$, assign all 4 adjacent vertices value as zero, that is $f(v_1) = 0, f(v_2) = 0, f(v_4) = 0$ and $f(v_5) = 0$. Then, the remaining vertex v_6 adds value as $f(v_6) = 1$, $\gamma_R((C_6)_d) = 3$. If $n \equiv 1(\mod 5)$, in Roman domination of $(C_n)_d$, we can assign value 2 to vertices v_{3+5i}, all other vertices are adjacent with these vertices except vertex v_{5i+1}. Then, the remaining vertex v_{5i+1} adds value as $f(v_{5i+1}) = 1$. Hence, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2\lfloor \frac{n}{5} \rfloor + 1$. Hence, $\gamma_R((C_n)_d) = 2\lfloor \frac{n}{5} \rfloor$.

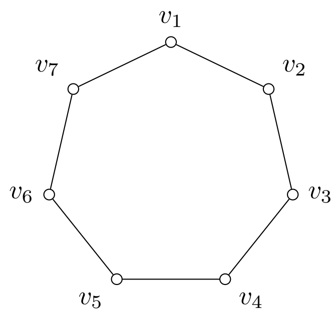
Similarly, the Roman domination of $(C_7)_d$, we can assign value two to vertices v_3 and v_7. That is, $f(v_3) = 2$ and $f(v_7) = 2$, assign all other adjacent vertices value as zero, $\gamma_R((C_7)_d) = 4$. Hence, $\gamma_R((C_7)_d) = 2\lceil \frac{n}{5} \rceil$. If $n \equiv 0,2,3,4(\mod 5)$, in Roman domination of $(C_n)_d$, we can assign value 2 to the vertices v_{3+5i}, all other vertices are adjacent with these vertices, add value zero to these vertices, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2\lceil \frac{n}{5} \rceil$. □

An illustration of Proposition 3.9 is given in Figure 3.

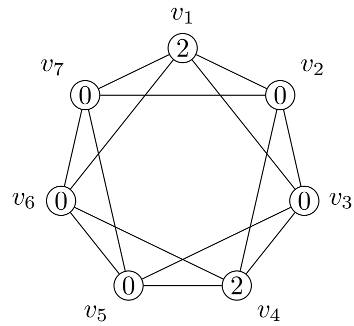
A non-trivial *bistar graph*, denoted by $S_{a,b}$, is a graph obtained by joining the centers of two non-trivial star graphs $k_{1,a}$, $a \geq 1$ and $k_{1,b}$, $b \geq 1$ with the edge v_0u_0.

Proposition 3.10. *For $a, b \geq 2$, $\gamma_R((S_{a,b})_d) = 2$.*

Proof. The bistar graph $S_{a,b}$; $a, b \geq 1$, let the pendant vertices of $k_{1,a}$ be the set $X = \{v_1, v_2, \ldots, v_a\}$ and let the pendant vertices of $k_{1,b}$ be the set $Y = \{u_1, u_2, \ldots, u_b\}$. Finally, let $W = \{v_0, u_0\}$ be center vertices. By Definition 1.1, it follows that both v_0, u_0 are adjacent with all other $a+b+1$ vertices. In Roman domination, we can assign value two to any one of these vertices. That is, $f(v_0) = 2$ 0r $f(u_0) = 2$, assign all $a+b+1$ adjacent vertices value as zero. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((S_{a,b})_d) = 2$. □



(A) C_7



(B) $(C_7)_d$

FIGURE 3 $\gamma_R((C_7)_d) = 4$.

Note that, the complete bipartite graph $K_{n,m}$, $n, m \geq 2$, by Definition 1.1, all the vertices of $k_{n,m}$ are adjacent in $(K_{n,m})_d$; that is, the deg-centric graph is complete. Hence, $\gamma_R((K_{n,m})_d) = 2$.

A *wheel graph* denoted by $W_{1,n}$, $n \geq 3$ is obtained by taking a cycle C_n, $n \geq 3$ (the rim with rim-vertices) and adding the central vertex v_0 with *spokes* namely, edges v_0v_i, $1 \leq i \leq n$.

Proposition 3.11. *For $n \geq 3$, $\gamma_R((W_{1,n})_d) = 2$.*

Proof. For a wheel graph $W_{1,n}$, $n \geq 3$, note that, $deg(v_i) \geq e(v_i)$ in wheel graph, for all $v_i \in V(W_{1,n})$. In view of Definition 1.1, $(W_{1,n})_d$ is isomorphic to K_{n+1}. In views of Proposition 2.4, $\gamma_R((W_{1,n})_d) = 2$. □

A *helm graph*, denoted by $H_{1,n,}$, $n \geq 3$ is a graph obtained from a wheel graph $W_{1,n}$ by attaching a pendant vertex u_i to the corresponding *rim vertex* v_i [9].

Proposition 3.12. *For $n \geq 3$, $\gamma_R((H_{1,n})_d) = 2$.*

Proof. The helm graph $H_{1,n,}$, $n \geq 3$ is of the order $2n+1$. Let $V(H_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$.In view of Definition 1.1, there are $2n$ edge incident to v_0 and v_i in $(H_{1,n})_d$. In Roman domination, we can assign value two to any of these vertices v_0 0r v_i. That is, $f(v_0) = 2$ 0r $f(v_i) = 2$, assign all $2n$ adjacent vertices value as zero. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((H_{1,n})_d) = 2$. □

A *closed helm graph* denoted by $CH_{1,n}$, $n \geq 3$ is the graph obtained from a helm graph $H_{1,n}$ by cyclically joining the pendant vertices to form an outer rim.

Proposition 3.13. *For $n \geq 3$, $\gamma_R((CH_{1,n})_d) = 2$.*

Proof. Consider a closed helm graph $CH_{1,n}$ $n \geq 3$, is of the order $2n + 1$. Let $V(CH_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. For all $CH_{1,n}$, $n < 8$, $\delta(CH_{1,n}) = 3$. If $n = 3, 4, 5, 6, 7$ the diameter is bounded by $diam(CH_{1,n}) \leq 3$. Since $\delta(CH_{1,n}) \geq diam(CH_{1,n})\delta(CH_{1,n}) = 3$. Then, the deg-centric graph of a closed helm graph $CH_{1,n}$ of order $n < 8$ is the complete graph. Finally, by Proposition 2.4, $\gamma_R((CH_{1,n})_d) = 2$. If $n \geq 8$, we have $\delta(CH_{1,n}) = 3$ and $diam(CH_{1,n}) = 4$, in $CH_{1,n}$ center vertex v_0, $deg(v_0) = n$. In view of Definition 1.1, $deg(v_0) = 2n$ in deg-centric graph. Now we can assign value as two in Roman domination, $f(v_0) = 2$, all other $2n$ vertices are adjacent to v_0, and assign value zero to all these values. Finally, $\gamma_R((CH_{1,n})_d) = 2$. □

A *sunlet graph* , denoted by Sl_n, $n \geq 3$, is a graph obtained by attaching a pendant vertex to every vertex of a cycle graph c_n, $n \geq 3$. In other words, a sunlet graph on $2n$ vertices is obtained by taking the corona product $C_n \circ K_1$. Recall that the corona between G of order n and H is denoted by $G \circ H$. It is obtained by taking n copies of H and joining a copy of H to each vertex of G.

Proposition 3.14. *For $n \geq 3$,*

$$\gamma_R((Sl_n)_d) = \begin{cases} 2\lceil \frac{n}{5} \rceil, & \text{if } n \equiv 0, 2, 3, 4 (\mod 5). \\ 2\lfloor \frac{n}{5} \rfloor + 1, & \text{if } n \equiv 1 (\mod 5). \end{cases}$$

Proof. The sunlet graph Sl_n, $n \geq 3$ is of the order $2n$. Let $V(Sl_n) = \{v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. If $3 \leq n \leq 5$, in view of Definition 1.1, $\deg_{Sl_n}(v_i) = 3 > e(v_i) = 2$ then all v_i vertices are adjacent with other $2n - 1$ vertices. In Roman domination, the functions $f : V \rightarrow \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. In the Roman domination set, add any one of the vertex v_i to the value 2. Hence, $\gamma_R((Sl_n)_d) = 2$.

If $n = 6$, by Definition 1.1, then all v_i vertices are adjacent with other $2n - 2$ vertices. However, since all u_i are pendant vertices, in view of Definition 1.1, no edge forms from a u_i in $(Sl_n)_d$. In the Roman domination set, add value 2 to the vertex v_3, $f(v_3) = 2$, and one non-adjacent vertex u_6 add value 1, $f(u_6) = 1$. Hence, $\gamma_R((Sl_6)_d) = 3$. If $n \equiv 1 (\mod 5)$, in Roman domination of $(Sl_n)_d$, we can assign value 2 to vertices v_{3+5i}, all other vertices are adjacent with these vertices except vertex u_{5i+1}. Then, the remaining vertex u_{5i+1} adds value as $f(u_{5i+1}) = 1$. Hence, the least value of $\omega(f) = \sum_{v,u \in V} f(v) + f(u) = 2\lfloor \frac{n}{5} \rfloor + 1$. Hence, $\gamma_R((Sl_n)_d) = 2\lfloor \frac{n}{5} \rfloor + 1$.

If $n \geq 7$, by Definition 1.1, then all v_i vertices are adjacent with eleven vertices. However, since all u_i are pendant vertices, no edge forms from a u_i in $(Sl_n)_d$. Then, all u_i have degree five in $(Sl_n)_d$. The deg-centric graph of (Sl_n), $n = 7$, with consecutively labeled rim vertices $\{v_1, v_2, v_3, \ldots, v_n\}$. If $n \equiv 0, 2, 3, 4 (\mod 5)$, in Roman domination of $(Sl_n)_d$, we can assign value 2 to the vertices v_{3+5i}, all other vertices are adjacent with these vertices, add value zero to these vertices, the least value of $\omega(f) = \sum_{u,v \in V} f(v) + f(u) = 2\lceil \frac{n}{5} \rceil$. □

An illustration of a proposition 3.14 is given in Figure 4.

A *djembe graph*, denoted by $D_{1,n}$, is obtained by joining the vertices u_i's; $1 \leq i \leq n$ of a closed helm graph $CH_{1,n}$ to its central vertex v_0.

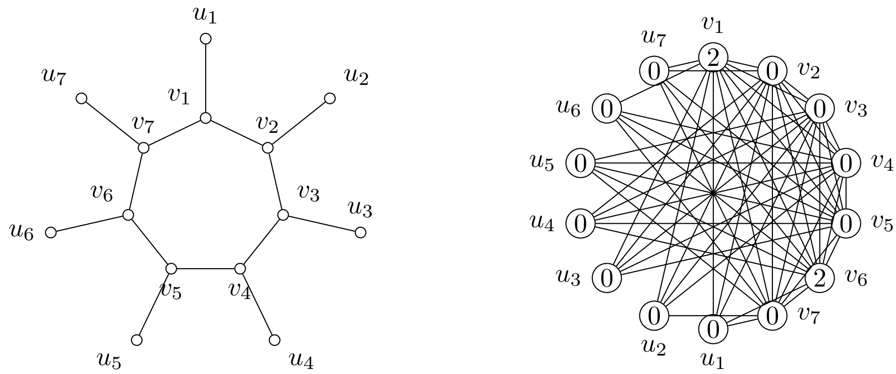
Proposition 3.15. *For $n \geq 3$, $\gamma_R((D_{1,n})_d) = 2$.*

Proof. The djembe graph $D_{1,n}$, $n \geq 3$, is of the order $2n+1$. Let $V(D_{1,n}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $\delta(D_{1,n}) = 4 > diam(D_{1,n}) = 2$, by Definition 1.1, $2n$ edge incident at all $2n + 1$ vertices in $(D_{1,n})_d$. That is, $(D_{1,n})_d \cong K_{2n+1}$. In views of Proposition 2.4, $\gamma_R((D_{1,n})_d) = 2$. □

A *double wheel* DW_n is obtained by taking two copies of a wheel W_n $n \geq 3$ and *merging* the two central vertices.

Proposition 3.16. *For $n \geq 3$, $\gamma_R((DW_n)_d) = 2$.*

Proof. The double wheel graph DW_n, $n \geq 3$, is of the order $2n + 1$. Let $V(DW_n) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $\deg(v_0) = 2n > e(v_0) = 1$ and $\deg(v_i) = \deg_G(u_i) = 3 > e(v_0) = 1$ in DW_n, by Definition 1.1, $2n$ edge incident from all $2n + 1$ vertices in $(DW_n)_d$. That is, $(DW_n)_d \cong K_{2n+1}$. In views of Proposition 2.4, $\gamma_R((DW_n)_d) = 2$. □



(A) Sl_7 (B) $(Sl_7)_d$

FIGURE 4 $\gamma_R(Sl_7)_d) = 4.$

A *gear graph*, denoted by G_n,$n \geq 3$, is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the rim of a wheel graph W_n.

Proposition 3.17. *For* $n \geq 3$, $\gamma_R((G_n)_d) = 2$.

Proof. The gear graph, $n \geq 4$, is of the order $2n + 1$. Let $V(G_n) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $deg_G(v_0) = n > e(v_0) = 2$, by Definition1.1, $2n$ edge forms from v_0 in $(G_n)_d$, $f(v_0) = 2$, and assign a value zero to all other adjacent vertices. Finally, $\gamma_R((G_n)_d) = 2$. □

A *web graph*, denoted by $Wb_{1,n}$, $n \geq 3$ is the graph obtained by attaching a pendant edge to each vertex of the outer cycle (or rim) of the closed helm graph $CH_{1,n}$.

Proposition 3.18. *For* $n \geq 3$, $\gamma_R((Wb_{1,n})_d) = 2$.

Proof. The web graph $Wb_{1,n}$, $n \geq 3$, is of the order $3n+1$. Let $V(Wb_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_{n-1}, v_n, u_1, u_2, u_3, \ldots, u_n, \underbrace{w_1, w_2, w_3, \ldots, w_n}_{\text{pendant vertices}}\}$. If $n \geq 3$, Since $deg(v_0) = n \geq e(v_0) = 2$ in $Wb_{1,n}$, by Definition 1.1, all other vertices are incident to v_0 in $(Wb_{1,n})_{ld}$, $f(v_0) = 2$, and assign value zero to all other vertices. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((Wb_{1,n})_d) = 2$. □

A *flower graph*, $F_{1,n}$, $n \geq 3$ is a graph obtained from a helm graph $H_{1,n,}$ by joining each of its pendant vertices u_i's to its central vertex v_0.

Proposition 3.19. *For* $n \geq 3$, $\gamma_R((F_{1,n})_d) = 2$.

Proof. The flower graph $F_{1,n,}$, $n \geq 3$ is of the order $2n + 1$. Let $V(F_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $\delta(F_{1,n,}) = 2 = diam(F_{1,n,}) = 2$, Since $\deg(v_0) = 2n \geq \deg(v_i) = n \geq e(v_0) = 2$ in $F_{1,n,}$, by Definition 1.1, $2n$ edge incident at v_0 and v_i in $(F_{1,n,})_d$. In Roman domination, we can assign value two to vertex v_0. That is, $f(v_0) = 2$ assign all $2n$ adjacent vertices value as zero. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((F_{1,n})_d) = 2$. □

The *sunflower graph*, denoted by $SF_{1,n}$, $n \geq 3$ is obtained from the wheel $W_{1,n}$ by attaching n vertices u_i, $1 \leq i \leq n$ such that each u_i is adjacent to v_i and v_{i+1} and count the suffix is taken modulo n.

Proposition 3.20. *For $n \geq 3$, $\gamma_R((SF_{1,n})_d) = 2$.*

Proof. For a sunflower graph $SF_{1,n,}$, $n \geq 3$, is of the order $2n+1$. Let $V(SF_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $\deg(v_0) = n \geq e(v_0) = 2$ and $deg(v_i) = n+1 \geq e(v_i) = 2$ in $SF_{1,n}$. Then by Definition 1.1, $2n$ edge incident at v_0 and v_i in $(SF_{1,n,})_d$. In Roman domination, we can assign value two to any of these vertices v_0 0r v_i. That is, $f(v_0) = 2$ 0r $f(v_i) = 2$, assign all $2n$ adjacent vertices value as zero. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((SF_{1,n})_d) = 2$. □

A *closed sunflower graph* $CSF_{1,n}$ is obtained by adding the edge $u_i u_{i+1}$ of the sunflower graph.

Proposition 3.21. *For $n \geq 3$, $\gamma_R((CSF_{1,n})_d) = 2$.*

Proof. The closed sunflower graph $CSF_{1,n,}$, is of the order $2n+1$. Let $V(CSF_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. For $n \geq 3$, $\delta(CSF_{1,n,}) \geq diam(CSF_{1,n,})$, By Definition 1.1, $(CSF_{1,n})_d$ is complete. In view of Proposition 2.4, $\gamma_R((CSF_{1,n})_d) = 2$. □

Consider a complete graph K_n with the vertex set $V = v_1, v_2, v_3, \ldots, v_n$. Let $U = u_1, u_2, u_3, \ldots, u_n$ be a copy of $V(G)$ such that u_i corresponds to v_i. The *sun graph*, denoted by S_n, is a graph with vertex set $V \cup U$ and two vertices x and y are adjacent in S_n if $x \sim y$ in K_n and $x = u_i$, $y \in v_i, v_{i+1}$.

Proposition 3.22. *For $n \geq 3$, $\gamma_R((S_n)_d) = 2$.*

Proof. The sun graph S_n, $n \geq 3$, is of the order $2n$. Let $V(S_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Since $deg_{S_n}(v_i) = n+1 > e(v_i) = 2$, by Definition 1.1, $(2n-1)$ edges form from v_i in $(S_n)_d$, assign any one of v_i value as two, $f(v_i) = 2$, and assign a value of zero to all other $2n-1$ vertices. Finally, $\gamma_R((S_n)_d) = 2$. □

A *closed sun graph* CS_n is the graph obtained from adding the edges $u_i u_{i+1}$ in the sun graph. In view of Definition 1.1, the deg-centric graph of a closed sun graph CS_n, $n \geq 3$, is complete which implies $\varepsilon((CS_n)_d) = \varepsilon(K_{2n})$. That is, $\gamma_R((CS_n)_d) = 2$.

A *friendship graph*, denoted by $F_n, n \geq 1$, is obtained by joining n copies of the complete graph K_3 with a common vertex. In view of Definition 1.1, $\gamma_R((F_n)_d) = 2$.

An *antiprism graph*, denoted by A_n, $n \geq 3$ is a graph obtained two cycles C_n and C'_n of order n with vertex sets $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and $U = \{u_1, u_2, u_3, \ldots, u_n\}$ respectively. Join the vertices $u_i v_i$ and $u_i v_{i+1}$ to form the additional edges.

Proposition 3.23. *For $n \geq 3$,*

$$\gamma_R((A_n)_d) = \begin{cases} 2\lceil \frac{n}{8} \rceil, & \text{if } n \equiv 0,2,3,4,5,6,7 (\mod 8). \\ 2\lfloor \frac{n}{8} \rfloor + 1, & \text{if } n \equiv 1 (\mod 8). \end{cases}$$

Proof. Consider an antiprism graph A_n, $n \geq 3$, is of the order $2n$. Let $V(A_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. In Roman domination, the functions $f : V \to \{0,1,2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0,1,2$. If $3 \leq n \leq 8$, $\deg(v_i) = \deg(u_i) = 4 > e(v_i) = e(u_i)$ in A_n then by Definition 1.1, $(A_n)_d \cong K_{2n}$,$\gamma_R((A_n)_d) = 2$. Hence, $\gamma_R((A_n)_d) = 2\lceil \frac{n}{8} \rceil$.

If $n \geq 9$, $\deg(v_i) = \deg(u_i) = 4$ in A_n then by Definition 1.1, $\deg(v_i) = \deg(u_i) = 16$ in $(A_n)_d$. The deg-centric graph of (A_n), $n \geq 9$, with consecutively labeled rim vertices $\{v_1, v_2, v_3, \ldots, v_n\}$. If $n \equiv 0,2,3,4,5,6,7 (\mod 8)$, the Roman domination set of $(A_n)_d$, of vertex v_i, where $1 \leq i \leq n$, the vertex v_i, added to the value 2, $f(v_i) = 2$, the vertex v_i adjacent with eight vertices in rim , $v_{i-4}, v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$,$v_{i+3}$ and v_{i+4} in $(A_n)_d$

added the value zero. Then we can add value 2 to the vertices v_{8i+1}, $i = 1, 2, 3 \dots n$, $f(v_{8i+1}) = 2$. Hence, $\gamma_R((A_n)_d) = 2\lceil \frac{n}{8} \rceil$.

If $n \equiv 1(\mod 8)$, in Roman domination of $(A_n)_d$, we can assign value 2 to vertices v_{4+8i}, all other vertices are adjacent with these vertices except vertex u_{8i+1}. Then, the remaining vertex u_{8i+1} adds value as $f(u_{8i+1}) = 1$. Hence, the least value of $\omega(f) = \sum_{v,u \in V} f(v) + f(u) = 2\lfloor \frac{n}{8} \rfloor + 1$. Hence, $\gamma_R((A_n)_d) = 2\lfloor \frac{n}{8} \rfloor + 1$. □

4. Conclusion

The Roman domination of deg-centric graphs has been discussed, and the Roman domination number of deg-centrication of some graph classes also. Various exploratory results have been presented to establish some foundation for further research. As a scope of the study, the researchers can extend the study on graph theoretical parameters to deg-centric graphs of various class graphs and obtain fruitful results. New researchers can also study different types of graph domination.

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[1] Department of Mathematics, Kristu Jayanti University, Bengaluru -560077, INDIA.
Email address: timmy@kristujayanti.com