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# On Brousseau Sums of Tetranacci Numbers

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ABSTRACT. In this paper, we show how to find the Brousseau sum $\sum_{k=0}^{n} k^m T_k$ and the shifted Brousseau sum $\sum_{k=1}^{n} k^m T_{k+r}$ for all integers $m, r$ with $m \geq 0$. Here, $(T_k)_{k\geq 0}$ is the Tetranacci sequence, which is defined by the fourth-order linear recurrence $T_k = T_{k-1} + T_{k-2} + T_{k-3} + T_{k-4}$ for $k \geq 4$ with $T_0 = 0$, $T_1 = T_2 = 1$, and $T_3 = 2$. These numbers are extended to negative indices by $T_k = T_{k+4} - T_{k+3} - T_{k+2} - T_{k+1}$ for $k < 0$. Furthermore, we find an exact formula for the Brousseau sum $\sum_{k=1}^{n} k^m T'_k$, where $T'_k$ is the $k^{\text{th}}$ "generic" Tetranacci number defined by $T'_k = T'_{k-1} + T'_{k-2} + T'_{k-3} + T'_{k-4}$ for $k > 1$ with arbitrary initial values $T'_{-2}, T'_{-1}, T'_0$, and $T'_1$.

## 1. INTRODUCTION

The Tetranacci sequence (see [6]) $(T_k)_{k\geq 0}$ is defined by the fourth-order linear recurrence

$$T_k = T_{k-1} + T_{k-2} + T_{k-3} + T_{k-4}, \tag{1.1}$$

for $k \geq 4$ with initial terms $T_0 = 0, T_1 = T_2 = 1$ and $T_3 = 2$. This sequence is A000078 in the OEIS [10] with an alternate indexing convention of $T_0 = T_1 = T_2 = 0$ and $T_3 = 1$. We can extend the definition of Tetranacci numbers to negative indices by

$$T_k = T_{k+4} - T_{k+3} - T_{k+2} - T_{k+1},$$

for $k < 0$. Many researchers have studied the properties of Tetranacci numbers (see [12, 14, 16, 17]).

The Tetranacci numbers are regarded as a generalization of the classical Fibonacci numbers (A000045 in OEIS). Brousseau [2, 3], Erbacher and Fuchs [7], Ledin [9], Zeitlin [18] and recently Ollerton and Shannon [11], Shannon and Ollerton [13], and Dresden [5] have developed various methods to find expressions for the Brousseau sums of the form

$$\sum_{k=0}^{n} k^m F_k,$$

where $F_k$ is the $k^{\text{th}}$ Fibonacci number and $m$ is a non-negative integer. In this paper, we are interested in finding polynomial forms of the Brousseau sums

$$\sum_{k=0}^{n} k^m T_k$$

of the Tetranacci numbers. Waddill [16] proved that

$$3 \cdot \sum_{k=0}^{n} T_k = T_{n+2} + 2T_n + T_{n-1} - 1. \tag{1.2}$$

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We can rewrite (1.2) as

$$3 \cdot \sum_{k=0}^{n} T_k = T_{n+1} + 3T_n + 2T_{n-1} + T_{n-2} - 1. \tag{1.3}$$

Further, Schumacher [12] proved that

$$3 \cdot \sum_{k=0}^{n} kT_k = \left(n - \frac{7}{3}\right)T_{n+1} + \left(3n - 1\right)T_n + \left(2n - \frac{5}{3}\right)T_{n-1} + \left(n - \frac{4}{3}\right)T_{n-2} + \frac{7}{3}. \tag{1.4}$$

Using (1.3) and (1.4), we can show that

$$\begin{aligned} 3 \cdot \sum_{k=0}^{n} k^2 T_k = &\left(n^2 - \frac{14}{3}n + \frac{59}{9}\right)T_{n+1} + \left(3n^2 - 2n + \frac{17}{3}\right)T_n \\ &+ \left(2n^2 - \frac{10}{3}n + \frac{37}{9}\right)T_{n-1} + \left(n^2 - \frac{8}{3}n + \frac{26}{9}\right)T_{n-2} - \frac{59}{9}. \end{aligned} \tag{1.5}$$

These identities (1.3), (1.4), and (1.5) suggest the following generalization:

$$3 \cdot \sum_{k=0}^{n} k^m T_k = \psi_0^{(m)}(n)T_{n+1} + \psi_1^{(m)}(n)T_n + \psi_2^{(m)}(n)T_{n-1} + \psi_3^{(m)}(n)T_{n-2} - \psi_0^{(m)}(0), \tag{1.6}$$

where $\psi_i^{(m)}(n)$ are some "coefficient polynomials" in $n$ of degree $m$. We utilize basic recursion techniques that include only the binomial coefficients to obtain expressions for these polynomials. We further extend (1.6) to the shifted Brousseau sums

$$\sum_{k=1}^{n} k^m T_{k+r},$$

where $r$ is an integer. This is our Theorem 2.4, which involves the identity

$$\begin{aligned} 3 \cdot \sum_{k=1}^{n} k^m T_{k+r} = &\psi_0^{(m)}(n)T_{n+r+1} + \psi_1^{(m)}(n)T_{n+r} + \psi_2^{(m)}(n)T_{n+r-1} + \psi_3^{(m)}(n)T_{n+r-2} \\ &- \left[\psi_0^{(m)}(0)T_{r+1} + \psi_1^{(m)}(0)T_r + \psi_2^{(m)}(0)T_{r-1} + \psi_3^{(m)}(0)T_{r-2}\right]. \end{aligned} \tag{1.7}$$

If we define the "generic" Tetranacci numbers $T'_k$ by $T'_k = T'_{k-1} + T'_{k-2} + T'_{k-3} + T'_{k-4}$ for $k > 1$ with arbitrary initial values $T'_{-2}, T'_{-1}, T'_0$, and $T'_1$, then we can derive an improved version of (1.7) as

$$\begin{aligned} 3 \cdot \sum_{k=1}^{n} k^m T'_k = &\psi_0^{(m)}(n)T'_{n+1} + \psi_1^{(m)}(n)T'_n + \psi_2^{(m)}(n)T'_{n-1} + \psi_3^{(m)}(n)T'_{n-2} \\ &- \left[\psi_0^{(m)}(0)T'_1 + \psi_1^{(m)}(0)T'_0 + \psi_2^{(m)}(0)T'_{-1} + \psi_3^{(m)}(0)T'_{-2}\right]. \end{aligned}$$

Throughout this paper, we assume that $\binom{0}{0} = 1$ and $0^0 = 1$.

## 2. Main result

2.1. **Tetranacci numbers and Powers.** In this part, we will derive a recursive formula for the convolution of the Tetranacci numbers and powers expressed as

$$\sum_{k=0}^{n} k^m T_{n-k}.$$

This will be accomplished by utilizing the lemma provided.

**Lemma 2.1.** *For all integers $m, n \geq 1$, the following identity holds:*

$$T_n = n^m - \big[(-1)^m + (-2)^m\big]T_{n-1} - (-1)^m T_{n-2} + \sum_{k=1}^{n}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k}, \tag{2.8}$$

*where $\alpha_j = 1 - (-1)^j - (-2)^j - (-3)^j$.*

*Proof.* The proof is by induction on $n$. The cases $n = 1, n = 2, n = 3$, and $n = 4$ can be confirmed by direct substitution. Note that $T_{-1} = T_{-2} = 0$. Fix $n > 4$ and assume that (2.8) holds for all positive integers less than $n$. Then

$$\begin{aligned} T_{n-1} &= (n-1)^m - \big[(-1)^m + (-2)^m\big]T_{n-2} - (-1)^m T_{n-3} \\ &\quad + \sum_{k=1}^{n-1}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k-1}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} T_{n-2} &= (n-2)^m - \big[(-1)^m + (-2)^m\big]T_{n-3} - (-1)^m T_{n-4} \\ &\quad + \sum_{k=1}^{n-2}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k-2}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} T_{n-3} &= (n-3)^m - \big[(-1)^m + (-2)^m\big]T_{n-4} - (-1)^m T_{n-5} \\ &\quad + \sum_{k=1}^{n-3}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k-3}, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} T_{n-4} &= (n-4)^m - \big[(-1)^m + (-2)^m\big]T_{n-5} - (-1)^m T_{n-6} \\ &\quad + \sum_{k=1}^{n-4}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k-4}. \end{aligned} \tag{2.12}$$

Note that $T_{n-k-1} = 0$ at $k = n-1$. So we may drop the corresponding term from the outer sum in (2.9) and obtain

$$\begin{aligned} T_{n-1} &= (n-1)^m - \big[(-1)^m + (-2)^m\big]T_{n-2} - (-1)^m T_{n-3} \\ &\quad + \sum_{k=1}^{n-2}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k-1}. \end{aligned} \tag{2.13}$$

Since $T_{n-k-3} = T_{n-k-4} = 0$ at $k = n-2$ and $T_{n-k-4} = 0$ at $k = n-3$, we may rewrite (2.11) and (2.12) respectively as

$$\begin{aligned} T_{n-3} &= (n-3)^m - \big[(-1)^m + (-2)^m\big]T_{n-4} - (-1)^m T_{n-5} \\ &\quad + \sum_{k=1}^{n-2}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k-3}, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} T_{n-4} &= (n-4)^m - \big[(-1)^m + (-2)^m\big]T_{n-5} - (-1)^m T_{n-6} \\ &\quad + \sum_{k=1}^{n-2}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k-4}, \end{aligned} \tag{2.15}$$

Now, adding (2.10), (2.13), (2.14), (2.15) and then using the Tetranacci recurrence (1.1), we obtain

$$\begin{aligned} T_n &= (n-1)^m + (n-2)^m + (n-3)^m + (n-4)^m - \big[(-1)^m + (-2)^m\big]T_{n-1} \\ &\quad - (-1)^m T_{n-2} + \sum_{k=1}^{n-2}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k}. \end{aligned} \tag{2.16}$$

At $k = n-1$, we have $T_{n-k} = 1$ and

$$\begin{aligned} 3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j} &= 3(n-1)^m - \sum_{j=1}^{m}\left[1-(-1)^j-(-2)^j-(-3)^j\right]\binom{m}{j}(n-1)^{m-j} \\ &= 3(n-1)^m - \big[n^m - (n-1)^m - (n-2)^m + (n-1)^m \\ &\quad - (n-3)^m + (n-1)^m - (n-4)^m + (n-1)^m\big] \\ &= (n-1)^m + (n-2)^m + (n-3)^m + (n-4)^m - n^m. \end{aligned}$$

Hence by adding the term corresponding to $k = n-1$ to the outer sum and then subtracting it from the right-hand side of (2.16), we obtain

$$\begin{aligned} T_n &= (n-1)^m + (n-2)^m + (n-3)^m + (n-4)^m \\ &\quad - \big[(-1)^m + (-2)^m\big]T_{n-1} - (-1)^m T_{n-2} + \sum_{k=1}^{n-1}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k} \\ &\quad - \left[(n-1)^m + (n-2)^m + (n-3)^m + (n-4)^m - n^m\right] \\ &= n^m - \big[(-1)^m + (-2)^m\big]T_{n-1} - (-1)^m T_{n-2} + \sum_{k=1}^{n-1}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k}. \end{aligned}$$

Since $T_{n-k} = 0$ at $k = n$, we may simply add the corresponding term to the outer sum in the above equation. Thus,

$$T_n = n^m - \big[(-1)^m + (-2)^m\big]T_{n-1} - (-1)^m T_{n-2} + \sum_{k=1}^{n}\left[3k^m - \sum_{j=1}^{m}\alpha_j\binom{m}{j}k^{m-j}\right]T_{n-k},$$

and the proof is complete by induction. □

**Theorem 2.1.** *For all integers $m \geq 1$ and $n \geq 0$, the following identity holds:*

$$\begin{aligned} 3\cdot\sum_{k=0}^{n} k^m T_{n-k} &= \big(1-\alpha_m\big)T_n + \big[(-1)^m + (-2)^m\big]T_{n-1} + (-1)^m T_{n-2} - n^m \\ &\quad + \sum_{j=1}^{m}\alpha_j\binom{m}{j}\left[\sum_{k=0}^{n}k^{m-j}T_{n-k}\right]. \end{aligned} \tag{2.17}$$

*Proof.* The case where $n = 0$ is trivial. Let $n \geq 1$. Then we may rewrite (2.8) in Lemma 2.1 as

$$\begin{aligned} T_n &= n^m - \big[(-1)^m + (-2)^m\big]T_{n-1} - (-1)^m T_{n-2} + 3\cdot\sum_{k=1}^{n}k^m T_{n-k} \\ &\quad - \sum_{j=1}^{m}\alpha_j\binom{m}{j}\left[\sum_{k=1}^{n}k^{m-j}T_{n-k}\right]. \end{aligned} \tag{2.18}$$

Since $k^m T_{n-k} = 0$ at $k = 0$, we may start the first summation on the right-hand side of (2.18) at $k = 0$ instead of at $k = 1$. Thus, we get

$$\begin{aligned} T_n &= n^m - \big[(-1)^m + (-2)^m\big]T_{n-1} - (-1)^m T_{n-2} + 3 \cdot \sum_{k=0}^{n} k^m T_{n-k} \\ &\quad - \sum_{j=1}^{m} \alpha_j \binom{m}{j} \left[\sum_{k=1}^{n} k^{m-j} T_{n-k}\right]. \end{aligned} \tag{2.19}$$

Since the term corresponding to $k = 0$ in the last summation on the right-hand side of (2.19) is $0^{m-j}T_n = 0$ for $j \neq m$ and $1 \cdot T_n$ for $j = m$, we may rewrite (2.19) in the form

$$\begin{aligned} T_n &= n^m - \big[(-1)^m + (-2)^m\big]T_{n-1} - (-1)^m T_{n-2} + 3 \cdot \sum_{k=0}^{n} k^m T_{n-k} \\ &\quad - \sum_{j=1}^{m} \alpha_j \binom{m}{j} \left[\sum_{k=0}^{n} k^{m-j} T_{n-k}\right] + \alpha_m T_n, \end{aligned}$$

and hence

$$\begin{aligned} 3 \cdot \sum_{k=0}^{n} k^m T_{n-k} &= \big(1 - \alpha_m\big)T_n + \big((-1)^m + (-2)^m\big)T_{n-1} + (-1)^m T_{n-2} - n^m \\ &\quad + \sum_{j=1}^{m} \alpha_j \binom{m}{j} \left[\sum_{k=0}^{n} k^{m-j} T_{n-k}\right], \end{aligned}$$

as desired. □

2.2. **Convolutions.** Using (1.3) and Theorem 2.1, we can find the sums $\sum_{k=0}^{n} k^m T_{n-k}$ for $m = 1, 2, 3, \ldots$ in a recursive manner. For example, setting $m = 1$ in (2.17) yields

$$3 \cdot \sum_{k=0}^{n} k T_{n-k} = -6T_n - 3T_{n-1} - T_{n-2} - n + 7 \cdot \sum_{k=0}^{n} T_{n-k}.$$

But it follows from(1.3) that

$$3 \cdot \sum_{k=0}^{n} T_{n-k} = T_{n+1} + 3T_n + 2T_{n-1} + T_{n-2} - 1.$$

Therefore,

$$\begin{aligned} 3 \cdot \sum_{k=0}^{n} k T_{n-k} &= -6T_n - 3T_{n-1} - T_{n-2} - n + \frac{7}{3}\big(T_{n+1} + 3T_n + 2T_{n-1} + T_{n-2} - 1\big) \\ &= \frac{7}{3}T_{n+1} + T_n + \frac{5}{3}T_{n-1} + \frac{4}{3}T_{n-2} - \left(n + \frac{7}{3}\right). \end{aligned}$$

Proceeding like this, we obtain the following set of identities:

$$3\cdot\sum_{k=0}^{n}T_{n-k}=1T_{n+1}+3T_n+2T_{n-1}+1T_{n-2}-1,$$

(2.20)
$$3\cdot\sum_{k=0}^{n}kT_{n-k}=\frac{7}{3}T_{n+1}+1T_n+\frac{5}{3}T_{n-1}+\frac{4}{3}T_{n-2}-\left(n+\frac{7}{3}\right),$$

$$3\cdot\sum_{k=0}^{n}k^2T_{n-k}=\frac{59}{9}T_{n+1}+\frac{17}{3}T_n+\frac{37}{9}T_{n-1}+\frac{26}{9}T_{n-2}-\left(n^2+\frac{14}{3}n+\frac{59}{9}\right),$$

$$3\cdot\sum_{k=0}^{n}k^3T_{n-k}=\frac{251}{9}T_{n+1}+\frac{83}{3}T_n+\frac{205}{9}T_{n-1}+\frac{128}{9}T_{n-2}-\left(n^3+7n^2+\frac{59}{3}n+\frac{251}{9}\right).$$

A pattern is evident in these convolution identities (2.20). To identify the rule of formation of the coefficients and the polynomials in these identities, we need to define the following four sequences of numbers.

**Definition 2.1.** *For all integers $m\geq 0$, we define the sequences $\left(A_i^{(m)}\right)_{m\geq 0}$ for $i=0,1,2,3$, as follows:*

$$A_0^{(m)}=\begin{cases}1, & \text{if } m=0\,;\\ \frac{1}{3}\sum_{j=1}^{m}\alpha_j\binom{m}{j}A_0^{(m-j)}, & \text{if } m\geq 1,\end{cases}$$

$$A_1^{(m)}=\begin{cases}3, & \text{if } m=0\,;\\ (-1)^m+(-2)^m+(-3)^m+\frac{1}{3}\sum_{j=1}^{m}\alpha_j\binom{m}{j}A_1^{(m-j)}, & \text{if } m\geq 1,\end{cases}$$

$$A_2^{(m)}=\begin{cases}2, & \text{if } m=0\,;\\ (-1)^m+(-2)^m+\frac{1}{3}\sum_{j=1}^{m}\alpha_j\binom{m}{j}A_2^{(m-j)}, & \text{if } m\geq 1,\end{cases}$$

$$A_3^{(m)}=\begin{cases}1, & \text{if } m=0\,;\\ (-1)^m+\frac{1}{3}\sum_{j=1}^{m}\alpha_j\binom{m}{j}A_3^{(m-j)}, & \text{if } m\geq 1,\end{cases}$$

*where* $\alpha_j=1-(-1)^j-(-2)^j-(-3)^j$.

The first few terms of these sequences are given in Table 1.

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $A_0^{(m)}$ | 1 | $\frac{7}{3}$ | $\frac{59}{9}$ | $\frac{251}{9}$ | $\frac{4661}{27}$ | $\frac{107659}{81}$ | $\frac{981311}{81}$ | $\frac{31359377}{243}$ | $\cdots$ |
| $A_1^{(m)}$ | 3 | 1 | $\frac{17}{3}$ | $\frac{83}{3}$ | $\frac{1451}{9}$ | $\frac{32947}{27}$ | $\frac{303353}{27}$ | $\frac{9711041}{81}$ | $\cdots$ |
| $A_2^{(m)}$ | 2 | $\frac{5}{3}$ | $\frac{37}{9}$ | $\frac{205}{9}$ | $\frac{3787}{27}$ | $\frac{84125}{81}$ | $\frac{770113}{81}$ | $\frac{24749815}{243}$ | $\cdots$ |
| $A_3^{(m)}$ | 1 | $\frac{4}{3}$ | $\frac{26}{9}$ | $\frac{128}{9}$ | $\frac{2486}{27}$ | $\frac{55888}{81}$ | $\frac{506834}{81}$ | $\frac{16273616}{243}$ | $\cdots$ |

TABLE 1. First few terms of $A_0^{(m)}, A_1^{(m)}, A_2^{(m)}$, and $A_3^{(m)}$

Now, we generalize the convolution identities given in (2.20).

**Theorem 2.2.** *For all integers $m,n\geq 0$, the following identity holds:*

(2.21)
$$3\cdot\sum_{k=0}^{n}k^mT_{n-k}=A_0^{(m)}T_{n+1}+A_1^{(m)}T_n+A_2^{(m)}T_{n-1}+A_3^{(m)}T_{n-2}-\sum_{r=0}^{m}\binom{m}{r}A_0^{(r)}n^{m-r}.$$

*Proof.* To show that (2.21) holds for all integers $m \geq 0$, we use induction on $m$. When $m = 0$, it follows from (1.3) that the left-hand side of (2.21) is

$$3 \cdot \sum_{k=0}^{n} T_{n-k} = 3 \cdot \sum_{k=0}^{n} T_k = T_{n+1} + 3T_n + 2T_{n-1} + T_{n-2} - 1,$$

and the right-hand side is

$$A_0^{(0)} T_{n+1} + A_1^{(0)} T_n + A_2^{(0)} T_{n-1} + A_3^{(0)} T_{n-2} - A_0^{(0)} = T_{n+1} + 3T_n + 2T_{n-1} + T_{n-2} - 1.$$

Thus, the base case of $m = 0$ is verified. Now, set $m \geq 1$ and assume that (2.21) holds for all non-negative integers less than $m$. Using Theorem 2.1, we have

$$\begin{aligned} 3 \cdot \sum_{k=0}^{n} k^m T_{n-k} = &\big(1 - \alpha_m\big) T_n + \big[(-1)^m + (-2)^m\big] T_{n-1} + (-1)^m T_{n-2} - n^m \\ &+ \sum_{j=1}^{m} \alpha_j \binom{m}{j} \Bigg[\sum_{k=0}^{n} k^{m-j} T_{n-k}\Bigg]. \end{aligned} \tag{2.22}$$

Applying the induction hypothesis, we obtain

$$\begin{aligned} 3 \cdot \sum_{k=0}^{n} k^{m-j} T_{n-k} =& A_0^{(m-j)} T_{n+1} + A_1^{(m-j)} T_n + A_2^{(m-j)} T_{n-1} + A_3^{(m-j)} T_{n-2} \\ &- \sum_{r=0}^{m-j} \binom{m-j}{r} A_0^{(r)} n^{m-j-r}, \end{aligned}$$

for $j = 1, 2, \ldots, m$. Therefore,

$$\begin{aligned} &\sum_{j=1}^{m} \alpha_j \binom{m}{j} \Bigg[\sum_{k=0}^{n} k^{m-j} T_{n-k}\Bigg] \\ &= \frac{1}{3} \Bigg\{ \Bigg[\sum_{j=1}^{m} \alpha_j \binom{m}{j} A_0^{(m-j)}\Bigg] T_{n+1} + \Bigg[\sum_{j=1}^{m} \alpha_j \binom{m}{j} A_1^{(m-j)}\Bigg] T_n \\ &\quad + \Bigg[\sum_{j=1}^{m} \alpha_j \binom{m}{j} A_2^{(m-j)}\Bigg] T_{n-1} + \Bigg[\sum_{j=1}^{m} \alpha_j \binom{m}{j} A_3^{(m-j)}\Bigg] T_{n-2} \\ &\quad - \sum_{j=1}^{m} \alpha_j \binom{m}{j} \sum_{r=0}^{m-j} \binom{m-j}{r} A_0^{(r)} n^{m-j-r} \Bigg\} \\ &= A_0^{(m)} T_{n+1} + \big[A_1^{(m)} + \alpha_m - 1\big] T_n + \big[A_2^{(m)} - (-1)^m - (-2)^m\big] T_{n-1} \\ &\quad + \big[A_3^{(m)} - (-1)^m\big] T_{n-2} - \frac{1}{3} \sum_{j=1}^{m} \sum_{r=0}^{m-j} \alpha_j \binom{m}{j} \binom{m-j}{r} A_0^{(r)} n^{m-j-r}, \end{aligned}$$

where the last equality follows from Definition 2.1.
Substituting this into (2.22), we get

$$\begin{aligned} 3 \cdot \sum_{k=0}^{n} k^m T_{n-k} = &A_0^{(m)} T_{n+1} + A_1^{(m)} T_n + A_2^{(m)} T_{n-1} + A_3^{(m)} T_{n-2} - n^m \\ &- \frac{1}{3} \sum_{j=1}^{m} \sum_{r=0}^{m-j} \alpha_j \binom{m}{j} \binom{m-j}{r} A_0^{(r)} n^{m-j-r}. \end{aligned}$$

Thus,

$$3 \cdot \sum_{k=0}^{n} k^m T_{n-k} = A_0^{(m)} T_{n+1} + A_1^{(m)} T_n + A_2^{(m)} T_{n-1} + A_3^{(m)} T_{n-2} - \Phi^{(m)}(n),$$

where

$$\Phi^{(m)}(n) = n^m + \frac{1}{3} \sum_{j=1}^{m} \sum_{r=0}^{m-j} \alpha_j \binom{m}{j} \binom{m-j}{r} A_0^{(r)} n^{m-j-r}. \tag{2.23}$$

To conclude the proof, we must show that

$$\Phi^{(m)}(n) = \sum_{r=0}^{m} \binom{m}{r} A_0^{(r)} n^{m-r}.$$

Now, keeping $j$ fixed and changing $r$ to $r-j$ in (2.23), we get

$$\Phi^{(m)}(n) = n^m + \frac{1}{3} \sum_{j=1}^{m} \sum_{r=j}^{m} \alpha_j \binom{m}{j} \binom{m-j}{r-j} A_0^{(r-j)} n^{m-r}.$$

If we switch the order of summation and use the binomial identity (see [1, Identity 134]) $\binom{m}{j}\binom{m-j}{r-j} = \binom{m}{r}\binom{r}{j}$, this becomes

$$\begin{aligned}\Phi^{(m)}(n) &= n^m + \sum_{r=1}^{m} \binom{m}{r} n^{m-r} \left[ \frac{1}{3} \sum_{j=1}^{r} \alpha_j \binom{r}{j} A_0^{(r-j)} \right] \\ &= n^m + \sum_{r=1}^{m} \binom{m}{r} n^{m-r} A_0^{(r)}.\end{aligned}$$

Since $A_0^{(r)} = 1$ at $r = 0$, we have

$$\Phi^{(m)}(n) = \sum_{r=0}^{m} \binom{m}{r} A_0^{(r)} n^{m-r},$$

as desired. □

2.3. **Brousseau Sums.** The convolution identity in Theorem 2.2 can be used to find the Brousseau sums

$$\sum_{k=0}^{n} k^m T_k$$

of the Tetranacci numbers, for $m \geq 0$.

As an illustration, consider the case of $m = 2$. We have,

$$\begin{aligned}3 \cdot \sum_{k=0}^{n} k^2 T_k &= 3 \cdot \sum_{k=0}^{n} (n-k)^2 T_{n-k} \\ &= 3n^2 \cdot \sum_{k=0}^{n} T_{n-k} - 6n \cdot \sum_{k=0}^{n} k T_{n-k} + 3 \cdot \sum_{k=0}^{n} k^2 T_{n-k}.\end{aligned}$$

Now, using the first three convolution identities in (2.20), we obtain

$$\begin{aligned}
3\cdot\sum_{k=0}^{n}k^2T_k &= n^2\left[T_{n+1}+3T_n+2T_{n-1}+T_{n-2}-1\right]\\
&\quad-2n\left[\frac{7}{3}T_{n+1}+T_n+\frac{5}{3}T_{n-1}+\frac{4}{3}T_{n-2}-\left(n+\frac{7}{3}\right)\right]\\
&\quad+\left[\frac{59}{9}T_{n+1}+\frac{17}{3}T_n+\frac{37}{9}T_{n-1}+\frac{26}{9}T_{n-2}-\left(n^2+\frac{14}{3}n+\frac{59}{9}\right)\right]\\
&=\left(n^2-\frac{14}{3}n+\frac{59}{9}\right)T_{n+1}+\left(3n^2-2n+\frac{17}{3}\right)T_n\\
&\quad+\left(2n^2-\frac{10}{3}n+\frac{37}{9}\right)T_{n-1}+\left(n^2-\frac{8}{3}n+\frac{26}{9}\right)T_{n-2}-\frac{59}{9}.
\end{aligned}$$

The general formula is given here.

**Theorem 2.3.** *For all integers $m, n \geq 0$, the following identity holds:*

$$3\cdot\sum_{k=0}^{n}k^mT_k=\psi_0^{(m)}(n)T_{n+1}+\psi_1^{(m)}(n)T_n+\psi_2^{(m)}(n)T_{n-1}+\psi_3^{(m)}(n)T_{n-2}-\psi_0^{(m)}(0), \tag{2.24}$$

*where*

$$\psi_i^{(m)}(n)=\sum_{j=0}^{m}(-1)^j\binom{m}{j}A_i^{(j)}n^{m-j}, \tag{2.25}$$

*for $i \in \{0, 1, 2, 3\}$.*

*Proof.* It is easy to check the case for $m = 0$. Now, set $m \geq 1$. Then, using the binomial expansion, we have

$$\begin{aligned}
\sum_{k=0}^{n}k^mT_k &= \sum_{k=0}^{n}(n-k)^nT_{n-k}\\
&=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{m}{j}n^{m-j}(-k)^jT_{n-k}.
\end{aligned}$$

Switching the order of summation, we get

$$\sum_{k=0}^{n}k^mT_k=\sum_{j=0}^{m}(-1)^j\binom{m}{j}n^{m-j}\left[\sum_{k=0}^{n}k^jT_{n-k}\right]. \tag{2.26}$$

Replacing $m$ by $j$ in (2.21), we have

$$3\cdot\sum_{k=0}^{n}k^jT_{n-k}=A_0^{(j)}T_{n+1}+A_1^{(j)}T_n+A_2^{(j)}T_{n-1}+A_3^{(j)}T_{n-2}-\sum_{r=0}^{j}\binom{j}{r}A_0^{(r)}n^{j-r}. \tag{2.27}$$

Combining (2.26) and (2.27), we obtain

$$3\cdot\sum_{k=0}^{n}k^mT_k=\left[\sum_{j=0}^{m}(-1)^j\binom{m}{j}A_0^{(j)}n^{m-j}\right]T_{n+1}+\left[\sum_{j=0}^{m}(-1)^j\binom{m}{j}A_1^{(j)}n^{m-j}\right]T_n$$
$$+\left[\sum_{j=0}^{m}(-1)^j\binom{m}{j}A_2^{(j)}n^{m-j}\right]T_{n-1}+\left[\sum_{j=0}^{m}(-1)^j\binom{m}{j}A_3^{(j)}n^{m-j}\right]T_{n-2}$$
$$-\sum_{j=0}^{m}\sum_{r=0}^{j}(-1)^j\binom{m}{j}\binom{j}{r}A_0^{(r)}n^{m-r}.$$

If we write

$$\psi_i^{(m)}(n)=\sum_{j=0}^{m}(-1)^j\binom{m}{j}A_i^{(j)}n^{m-j},$$

for $i\in\{0,1,2,3\}$, then

$$3\cdot\sum_{k=0}^{n}k^mT_k=\psi_0^{(m)}(n)T_{n+1}+\psi_1^{(m)}(n)T_n+\psi_2^{(m)}(n)T_{n-1}+\psi_3^{(m)}(n)T_{n-2}$$
$$-\sum_{j=0}^{m}\sum_{r=0}^{j}(-1)^j\binom{m}{j}\binom{j}{r}A_0^{(r)}n^{m-r}.$$

Now, by switching the order of summation, this becomes

$$3\cdot\sum_{k=0}^{n}k^mT_k=\psi_0^{(m)}(n)T_{n+1}+\psi_1^{(m)}(n)T_n+\psi_2^{(m)}(n)T_{n-1}+\psi_3^{(m)}(n)T_{n-2}$$
$$-\sum_{r=0}^{m}A_0^{(r)}n^{m-r}\left[\sum_{j=r}^{m}(-1)^j\binom{m}{j}\binom{j}{r}\right].$$

Finally, applying the binomial identity

$$\sum_{j=r}^{m}(-1)^j\binom{m}{j}\binom{j}{r}=\begin{cases}0, & \text{if } r\neq m\,;\\ (-1)^m, & \text{if } r=m,\end{cases}$$

from Gould's collection (see [8, Identity 3.119]), we obtain

$$3\cdot\sum_{k=0}^{n}k^mT_k=\psi_0^{(m)}(n)T_{n+1}+\psi_1{}^{(m)}(n)T_n+\psi_2^{(m)}(n)T_{n-1}+\psi_3^{(m)}(n)T_{n-2}-(-1)^mA_0^{(m)}.$$

Since $\psi_0^{(m)}(0)=(-1)^mA_0^{(m)}$, the proof is complete. □

**Theorem 2.4.** *For all $m,n,r\in\mathbb{Z}$ with $m\geq 0$ and $n\geq 1$, the following identity holds:*

$$3\cdot\sum_{k=1}^{n}k^mT_{k+r}=\sum_{i=0}^{3}\left[\psi_i^{(m)}(n)T_{n+r+1-i}-\psi_i^{(m)}(0)T_{r+1-i}\right], \tag{2.28}$$

*where $\psi_i^{(m)}(n)$ is as defined in (2.25).*

*Proof.* The case of $r = 0$ follows from (2.24). Note that $T_0 = T_{-1} = T_{-2} = 0$ and $T_1 = 1$. Now, assume that $r \geq 1$. Then, using the binomial expansion, we have

$$\begin{aligned}\sum_{k=1}^{n} k^m T_{k+r} &= \sum_{k=r+1}^{n+r} (k-r)^m T_k \\ &= \sum_{k=r+1}^{n+r} \sum_{j=0}^{m} \binom{m}{j} k^{m-j} (-r)^j T_k \\ &= \sum_{j=0}^{m} \binom{m}{j} (-r)^j \left[ \sum_{k=r+1}^{n+r} k^{m-j} T_k \right].\end{aligned}$$

Thus,

$$\sum_{k=1}^{n} k^m T_{k+r} = \sum_{j=0}^{m} \binom{m}{j} (-r)^j \left[ \sum_{k=0}^{n+r} k^{m-j} T_k - \sum_{k=0}^{r} k^{m-j} T_k \right].$$

Now, multiplying both the sides by 3 and applying Theorem 2.3, we obtain

$$\begin{aligned}3 \cdot \sum_{k=1}^{n} k^m T_{k+r} &= \sum_{j=0}^{m} \binom{m}{j} (-r)^j \left[ \sum_{i=0}^{3} \psi_i^{(m-j)}(n+r) T_{n+r+1-i} - \sum_{i=0}^{3} \psi_i^{(m-j)}(r) T_{r+1-i} \right] \\ &= \sum_{j=0}^{m} \sum_{i=0}^{3} \binom{m}{j} (-r)^j \left[ \psi_i^{(m-j)}(n+r) T_{n+r+1-i} - \psi_i^{(m-j)}(r) T_{r+1-i} \right].\end{aligned}$$

By switching the order of summation, this becomes

$$\begin{aligned}3 \cdot \sum_{k=1}^{n} k^m T_{k+r} = \sum_{i=0}^{3} \Bigg\{ \left[ \sum_{j=0}^{m} \binom{m}{j} (-r)^j \psi_i^{(m-j)}(n+r) \right] T_{n+r+1-i} \\ - \left[ \sum_{j=0}^{m} \binom{m}{j} (-r)^j \psi_i^{(m-j)}(r) \right] T_{r+1-i} \Bigg\}.\end{aligned} \tag{2.29}$$

Now, for $x \in \mathbb{Z}$, write

$$\sum\nolimits_i (x) = \sum_{j=0}^{m} \binom{m}{j} (-r)^j \psi_i^{(m-j)}(x).$$

Then,

$$\sum\nolimits_i (x) = \sum_{j=0}^{m} \binom{m}{j} (-r)^j \left[ \sum_{k=0}^{m-j} (-1)^k A_i^{(k)} \binom{m-j}{k} x^{m-j-k} \right].$$

By switching the order of summation and using the binomial identity (see [1, Identity 134]) $\binom{m}{j}\binom{m-j}{k} = \binom{m}{k}\binom{m-k}{j}$, this becomes

$$\sum\nolimits_i (x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} A_i^{(k)} \left[ \sum_{j=0}^{m-k} \binom{m-k}{j} x^{m-k-j} (-r)^j \right].$$

Thus,

$$\sum\nolimits_i (x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} A_i^{(k)} (x-r)^{m-k}. \tag{2.30}$$

Setting $x = n + r$ in (2.30) yields

(2.31) $$\sum\nolimits_i (n+r) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} A_i^{(k)} n^{m-k} = \psi_i^{(m)}(n).$$

Setting $x = r$ in (2.30) yields

(2.32) $$\sum\nolimits_i (r) = (-1)^m A_i^{(m)} = \psi_i^{(m)}(0),$$

since

$$(x-r)^{m-k} = \begin{cases} 1, & \text{if } k = m \text{ and } x = r\,; \\ 0, & \text{if } 0 \le k < m \text{ and } x = r. \end{cases}$$

Substituting (2.31) and (2.32) in (2.29), we obtain

$$3 \cdot \sum_{k=1}^{n} k^m T_{k+r} = \sum_{i=0}^{3} \Big[ \psi_i^{(m)}(n) T_{n+r+1-i} - \psi_i^{(m)}(0) T_{r+1-i} \Big].$$

The case $r < 0$ follows from the above case and the identity [15, Identitity (5.5)]

$$T_{r+k} = T_{r-2}T_{k+3} + \big(T_{r-3} + T_{r-4} + T_{r-5}\big)T_{k+2} + \big(T_{r-3} + T_{r-4}\big)T_{k+1} + T_{r-3}T_k.$$

This completes the proof. □

## 3. Extension to Generic Tetranacci Numbers

In this section, we find an exact formula for the Brousseau sums

$$\sum_{k=1}^{n} k^m T'_k$$

of the generic Tetranacci numbers. The usual Tetranacci numbers $T_k$ and the generic Tetranacci numbers $T'_k$ are connected by the relation [15, Identitity (5.4)]

(3.33) $$T'_k = T'_1 T_k + \big(T'_0 + T'_{-1} + T'_{-2}\big)T_{k-1} + \big(T'_0 + T'_{-1}\big)T_{k-2} + T'_0 T_{k-3}.$$

**Theorem 3.5.** *For all integers $m \ge 0$ and $n \ge 1$, the following identity holds:*

(3.34) $$3 \cdot \sum_{k=1}^{n} k^m T'_k = \sum_{i=0}^{3} \Big[ \psi_i^{(m)}(n) T'_{n+1-i} - \psi_i^{(m)}(0) T'_{1-i} \Big],$$

*where $\psi_i^{(m)}(n)$ is as defined in (2.25).*

*Proof.* Using the identity (3.33), we have

(3.35) $$\begin{aligned} \sum_{k=1}^{n} k^m \mathfrak{T}'_k = T'_1 \Big[ \sum_{k=1}^{n} k^m T_k \Big] &+ \big(T'_0 + T'_{-1} + T'_{-2}\big) \Big[ \sum_{k=1}^{n} k^m T_{k-1} \Big] \\ &+ \big(T'_0 + T'_{-1}\big) \Big[ \sum_{k=1}^{n} k^m T_{k-2} \Big] + T'_0 \Big[ \sum_{k=1}^{n} k^m T_{k-3} \Big]. \end{aligned}$$

Replacing $r$ by $0, -1, -2$, and $-3$ successively in (2.28) yields

(3.36) $$3 \cdot \sum_{k=1}^{n} k^m T_k = \sum_{i=0}^{3} \Big[ \psi_i^{(m)}(n) T_{n+1-i} - \psi_i^{(m)}(0) T_{1-i} \Big],$$

(3.37) $$3 \cdot \sum_{k=1}^{n} k^m T_{k-1} = \sum_{i=0}^{3} \Big[ \psi_i^{(m)}(n) T_{n-i} - \psi_i^{(m)}(0) T_{-i} \Big],$$

$$3 \cdot \sum_{k=1}^{n} k^m T_{k-2} = \sum_{i=0}^{3} \left[ \psi_i^{(m)}(n) T_{n-1-i} - \psi_i^{(m)}(0) T_{-1-i} \right], \tag{3.38}$$

and

$$3 \cdot \sum_{k=1}^{n} k^m T_{k-3} = \sum_{i=0}^{3} \left[ \psi_i^{(m)}(n) T_{n-2-i} - \psi_i^{(m)}(0) T_{-2-i} \right]. \tag{3.39}$$

Now, (3.34) follows by substituting (3.36)-(3.39) in (3.35) and applying the identity (3.33) four times. □

## 4. Numerical examples

As an illustration, we find the weighted sum

$$\sum_{k=1}^{n} k L_k^{(4)},$$

where $L_k^{(4)}$ is the $k^{\text{th}}$ 4-Lucas number (see [4]) defined by $L_k^{(4)} = L_{k-1}^{(4)} + L_{k-2}^{(4)} + L_{k-3}^{(4)} + L_{k-4}^{(4)}$ for $k > 3$ with initial values $L_0^{(4)} = 4, L_1^{(4)} = 1, L_2^{(4)} = 3$, and $L_3^{(4)} = 7$. Note that $L_{-1}^{(4)} = L_{-2}^{(4)} = -1$.

Setting $m = 1$ and $T'_k = L_k^{(4)}$ in (3.34) yields

$$3 \cdot \sum_{k=1}^{n} k L_k^{(4)} = \sum_{i=0}^{3} \left[ \psi_i^{(1)}(n) L_{n+1-i}^{(4)} - \psi_i^{(1)}(0) L_{1-i}^{(4)} \right],$$

where

$$\psi_i^{(1)}(n) = A_i^{(0)} n - A_i^{(1)}.$$

Thus, we have

$$\begin{aligned} \sum_{k=1}^{n} k L_k^{(4)} &= \frac{1}{3} \Big[ \psi_0^{(1)}(n) L_{n+1}^{(4)} + \psi_1^{(1)}(n) L_n^{(4)} + \psi_2^{(1)}(n) L_{n-1}^{(4)} + \psi_3^{(1)}(n) L_{n-2}^{(4)} \\ &\quad - \psi_0^{(1)}(0) L_1^{(4)} - \psi_1^{(1)}(0) L_0^{(4)} - \psi_2^{(1)}(0) L_{-1}^{(4)} - \psi_3^{(1)}(0) L_{-2}^{(4)} \Big] \\ &= \frac{1}{3} \left[ \left( n - \frac{7}{3} \right) L_{n+1}^{(4)} + (3n-1) L_n^{(4)} + \left( 2n - \frac{5}{3} \right) L_{n-1}^{(4)} + \left( n - \frac{4}{3} \right) L_{n-2}^{(4)} + \frac{10}{3} \right]. \end{aligned}$$

Likewise, for $m = 2$, we get the following lovely formula:

$$\begin{aligned} \sum_{k=1}^{n} k^2 L_k^{(4)} = \frac{1}{3} \Bigg[ &\left( n^2 - \frac{14}{3} n + \frac{59}{9} \right) L_{n+1}^{(4)} + \left( 3n^2 - 2n + \frac{17}{3} \right) L_n^{(4)} \\ &+ \left( 2n^2 - \frac{10}{3} n + \frac{37}{9} \right) L_{n-1}^{(4)} + \left( n^2 - \frac{8}{3} n + \frac{26}{9} \right) L_{n-2}^{(4)} - \frac{200}{9} \Bigg]. \end{aligned}$$

## 5. Conclusions

In this study, we determined the polynomial forms of the Brousseau sums of Tetranacci numbers. We introduced a novel identity that builds upon the findings from studies conducted since 1963. The proof is straightforward because all we need to establish our new identity are the binomial coefficients. We have expanded the analysis to include generalized Tetranacci numbers. The new approach we have adopted is also applicable to a wider range of related problems. Future studies may focus on developing techniques to derive polynomial expressions for the Brousseau sums associated with the general fourth-order linear recurrences given by

$$R_k = pR_{k-1} + qR_{k-2} + rR_{k-3} + sR_{k-4},$$

for $k \geq 1$ with arbitrary initial values $R_{-3}, R_{-2}, R_{-1}$, and $R_0$, where $p, q, r$, and $s$ are real numbers. Exploring the relationships between the coefficient polynomials $\psi_i^{(m)}$ and representing the Brousseau sums with a single polynomial could also be an avenue for future research.

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