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NaviSpace: A Navilogy Framework for Connecting Manifolds and Functional Spaces

MADHAN VELAYUTHAN[1] AND JEYANTHI VENKATAPATHY[2]

ABSTRACT. This manuscript introduces **Navilogy**, a new framework for studying Hilbert manifolds. A Navilogy comprises a smooth immersion, its weak accumulation set, and a continuous retraction. We establish that accumulation sets are weakly compact and form weakly geodesic spaces, where geodesics in the submanifold weakly converge to those in the accumulation set. The weak Laplace operator is defined, and its eigenvalues are shown to be weak limits of the submanifold Laplacian, ensuring spectral stability. Weak homotopy equivalence is demonstrated, preserving topological properties. Numerical examples highlight applications to function spaces and weak curvature flows. This framework enhances the understanding of weak topology in infinite-dimensional geometry and spectral analysis.

1. INTRODUCTION AND PRELIMINARIES

Hilbert manifolds play a fundamental role in differential geometry and functional analysis, with applications in mathematical physics, optimization, and partial differential equations. The interaction between weak and strong topologies is crucial in understanding their geometry and analysis. This paper explores the embedding properties of Hilbert manifolds, weakly geodesic spaces, and spectral properties of the Laplace operator, focusing on weak homotopy equivalence and the essential spectrum.

The study of Hilbert manifolds has evolved significantly. Toruńczyk [**?**] established foundational characterizations of infinite-dimensional manifolds, providing a basis for further research. Terng [20] contributed to the study of Fredholm submanifolds, enhancing the geometric and topological understanding of these structures.

In the early 2000s, research expanded into operator theory and weak topologies. Megrelishvili [14] examined weak and strong operator topologies on $B(H)$, laying the groundwork for bounded operators in Hilbert spaces. Ostrovskii [15] analyzed weak operator topology, operator ranges, and Kolmogorov widths, while Grivaux [7] investigated typical properties of Hilbert space operators, addressing fundamental aspects of operator convergence.

Between 2010 and 2020, advancements were made in geometry and spectral theory. Larotonda [12] developed a geometric approach to Hilbert-Schmidt operators, significantly impacting spectral analysis. Blaga [3] studied canonical connections in k-symplectic manifolds, providing a deeper geometric framework for infinite-dimensional manifolds. Charalambous and Lu [4] examined the spectrum of the Laplacian, emphasizing its essential spectrum in infinite-dimensional spaces.

Recent research from 2020 onward has focused on specialized aspects of Hilbert manifolds, weak homotopy, and spectral theory. Badji [1] studied L3-affine surfaces, while Pahan [16] analyzed warped product pointwise bi-slant submanifolds in trans-Sasakian manifolds, extending classical submanifold theory to infinite dimensions. Stojiljković

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Corresponding author: Madhan Velayuthan; madhanvmaths@gmail.com

[19] introduced Simpson-type tensorial norm inequalities for continuous functions of self-adjoint operators, enhancing spectral analysis. Berinde and Păcurar [2] surveyed recent developments in the fixed point theory of enriched contractive mappings, relevant to non-linear analysis on Hilbert manifolds. Grivaux and Matheron [8] studied the spectra of typical Hilbert space operators, contributing to spectral theory. Madhan and Jeyanthi [23] explored diffeomorphic embeddings of higher-dimensional Hilbert manifolds into Hilbert spaces, contributing to the structural study of these spaces.

Research on Hilbert manifolds has progressed from fundamental definitions and Fredholm submanifolds to advanced topics in weak operator topologies, spectral theory, and weak homotopy equivalence. The integration of functional analysis, differential geometry, and spectral theory has deepened the understanding of infinite-dimensional manifolds, with ongoing advancements shaping future applications in mathematical physics, optimization, and topology.

Definition 1.1 ([17]). *A **Hilbert manifold** $\mathcal{M}$ is a topological space that is locally modeled on a separable Hilbert space $\mathcal{H}$, meaning there exists an open cover $\{U_\alpha\}$ such that each U_α is homeomorphic to an open subset of $\mathcal{H}$. A **Riemannian metric** g on $\mathcal{M}$ is a smoothly varying inner product $g_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ at each point $x \in \mathcal{M}$. The Riemannian structure allows the definition of a Levi-Civita connection and geodesics.*

Remark 1.1 ([14]). *Let $\mathcal{H}$ be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Consider a sequence $\{x_n\} \subset \mathcal{H}$. The notions of convergence in $\mathcal{H}$ are defined as follows:*

*(1) **Weak Convergence:** The sequence $\{x_n\}$ is said to converge weakly to an element $x \in \mathcal{H}$ if*

$$\langle x_n, y \rangle \to \langle x, y \rangle, \quad \forall y \in \mathcal{H}.$$

That is, the sequence $\{x_n\}$ converges to x in the weak topology of $\mathcal{H}$.

*(2) **Strong Convergence:** The sequence $\{x_n\}$ is said to converge strongly to x if*

$$\|x_n - x\| \to 0 \quad \text{as } n \to \infty.$$

This is equivalent to saying that $x_n \to x$ in the norm topology of $\mathcal{H}$.

Furthermore, weak convergence implies that the sequence $\{x_n\}$ is bounded in $\mathcal{H}$, i.e., there exists a constant $M > 0$ such that $\|x_n\| \leq M$ for all n. However, weak convergence does not necessarily imply strong convergence.

Definition 1.2 ([22]). *A function $f : \mathcal{H} \to \mathbb{R}$ is **weakly lower semi-continuous** if for every weakly convergent sequence $x_n \rightharpoonup x$ in $\mathcal{H}$,*

$$\liminf_{n\to\infty} f(x_n) \geq f(x).$$

*The **strong topology** on $\mathcal{H}$ is the topology generated by the norm $\| \cdot \|$, where open sets are of the form*

$$B(x, \epsilon) = \{y \in \mathcal{H} \mid \|x - y\| < \epsilon\}.$$

*The **weak topology** is the weakest topology in which all continuous linear functionals remain continuous.*

Theorem 1.1 (The Banach-Alaoglu Theorem,[6]). *Let $\mathcal{H}$ be a Hilbert space, and let $B = \{x \in \mathcal{H}^* \mid \|x\| \leq 1\}$ be the closed unit ball in the dual space $\mathcal{H}^*$. Then B is compact in the weak-* topology; that is, every bounded sequence $\{x_n^*\} \subset B$ has a subsequence that weak-* converges to some $x^* \in B$,*

$$x_n^* \rightharpoonup x^* \quad \text{in } \mathcal{H}^*.$$

Definition 1.3 ([13]). *A **Fréchet manifold** $\mathcal{M}$ is a topological space that is locally modeled on a Fréchet space $\mathcal{F}$. A Fréchet space is a complete metrizable locally convex space whose topology is defined by a countable family of seminorms $p_n : \mathcal{F} \to \mathbb{R}$ such that the metric*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)}$$

induces the topology of $\mathcal{F}$. The transition maps between coordinate charts of $\mathcal{M}$ are smooth.

Definition 1.4 ([5]). *A metric space $(\mathcal{X}, d)$ is a **weakly geodesic space** if for every pair of points $x, y \in \mathcal{X}$, there exists a curve $\gamma : [0, 1] \to \mathcal{X}$ such that*

$$d(\gamma(t), \gamma(s)) \leq |t - s| d(x, y), \quad \forall t, s \in [0, 1].$$

A weakly geodesic space satisfies a relaxed version of geodesic convexity, where curves may exist but need not be unique.

Definition 1.5 ([13]). *Two topological spaces X and Y are **weakly homotopy equivalent** if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that the induced maps on homotopy groups*

$$f_* : \pi_k(X) \to \pi_k(Y), \quad g_* : \pi_k(Y) \to \pi_k(X)$$

are isomorphisms for all $k \geq 0$. This means that X and Y have the same homotopy type but may not be homeomorphic.

Definition 1.6 ([17]). *The **Laplace operator** Δ on a smooth Riemannian manifold $(\mathcal{M}, g)$ is defined in local coordinates $(x^1, \ldots, x^n)$ as*

$$\Delta u = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x^j} \right),$$

where g^{ij} is the inverse of the metric tensor g_{ij} and $|g| = \det(g_{ij})$. The Laplace operator generalizes the classical Laplacian to curved spaces.

Definition 1.7 ([9]). *Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint linear operator on a Hilbert space $\mathcal{H}$. The **essential spectrum** of T is defined as*

$$\sigma_{ess}(T) = \sigma(T) \setminus \{\lambda \mid \lambda \text{ is an isolated eigenvalue of finite multiplicity}\}.$$

The essential spectrum consists of accumulation points of $\sigma(T)$ and eigenvalues with infinite multiplicity, providing insight into the stability of spectral properties under perturbations.

2. Navilogy and NaviSpace - Definition and Examples

This section introduces the new concept of Navilogy, establishing a connection between Hilbert manifolds and Hilbert spaces.

Definition 2.8. *Let $\mathcal{M}$ be a Hilbert manifold modeled on an infinite-dimensional Hilbert space $\mathcal{H}$, and let $S \subset \mathcal{M}$ be a smooth submanifold. A **Navilogy** is an ordered triple*

$$\mathbb{N} = (\Phi, \mathcal{A}, \mathcal{R}),$$

where:

(1) $\Phi : S \to \mathcal{H}$ is a smooth immersion satisfying:

(a) The differential $d\Phi_x : T_x S \to T_{\Phi(x)} \mathcal{H}$ is injective for all $x \in S$, i.e.,

$$\ker d\Phi_x = \{0\}, \quad \forall x \in S.$$

(b) *There exists a smooth function* $\lambda : S \to \mathbb{R}^+$ *such that*

$$\langle d\Phi_x(v), d\Phi_x(w)\rangle_{\mathcal{H}} = \lambda(x) g_x(v, w), \quad \forall v, w \in T_x S,$$

where g_x *is the Riemannian metric induced on* S *from* $\mathcal{M}$. *Additionally,* $\lambda(x)$ *satisfies the uniform boundedness condition:*

$$0 < c \leq \lambda(x) \leq C < \infty, \quad \forall x \in S.$$

(2) *The* ***accumulation set*** *of* $\Phi(S)$ *is defined as:*

$$\mathcal{A} = \{h \in \mathcal{H} \mid \exists (x_n) \subset S, \quad \Phi(x_n) \rightharpoonup h \text{ in } \mathcal{H}\}.$$

Here, $\rightharpoonup$ *denotes weak convergence in* $\mathcal{H}$.

(3) *If* $\mathcal{A} \neq \emptyset$, *there exists a continuous weakly lower-semicontinuous retraction*

$$\mathcal{R} : \overline{\Phi(S)} \to \mathcal{A},$$

such that

$$\mathcal{R}(\Phi(x)) = \Phi(x), \quad \forall x \in S \text{ such that } \Phi(x) \in \mathcal{A}.$$

Moreover, $\mathcal{R}$ *is continuous in the weak topology of* $\mathcal{H}$, *preserving the topological structure of* $\mathcal{A}$.

Example 2.1. *Let* $\mathcal{M} = \mathcal{H}^1([0, 1])$, *the Hilbert manifold of Sobolev* H^1 *functions on* $[0, 1]$. *Let* $\mathcal{H} = L^2([0, 1])$, *the Hilbert space of square-integrable functions. If* S *is the smooth submanifold of* $\mathcal{H}^1([0, 1])$ *defined as:*

$$S = \left\{u \in H^1([0, 1]) \mid u(0) = 0\right\}.$$

Define the immersion $\Phi : S \to L^2([0, 1])$ by:

$$\Phi(u) = u.$$

The differential $d\Phi_u : T_u S \to L^2([0, 1])$ is simply the natural inclusion map. Since $H^1([0, 1]) \hookrightarrow L^2([0, 1])$ is a compact embedding, Φ satisfies the immersion property. The induced metric satisfies:

$$\langle d\Phi_u(v), d\Phi_u(w)\rangle_{L^2} = \lambda(u) g_u(v, w),$$

where $g_u(v, w) = \int_0^1 v'(x) w'(x)\, dx$ is the Sobolev inner product, and $\lambda(u) = 1$, satisfying the uniform boundedness condition. Consider a sequence $u_n \in S$ such that u_n weakly converges in H^1:

$$u_n \rightharpoonup u \quad \text{in } H^1([0, 1]).$$

Since weak convergence in H^1 implies weak convergence in L^2, we define the accumulation set:

$$\mathcal{A} = \left\{f \in L^2([0, 1]) \mid \exists (u_n) \subset S, \quad u_n \rightharpoonup f \text{ in } L^2\right\}.$$

Define the retraction $\mathcal{R} : \overline{\Phi(S)} \to \mathcal{A}$ by:

$$\mathcal{R}(f) = \lim_{n \to \infty} u_n,$$

where u_n is any weakly convergent sequence in S. This satisfies: $\mathcal{R}(\Phi(u)) = \Phi(u)$ whenever $u \in S$ and $\Phi(u) \in \mathcal{A}$. $\mathcal{R}$ is weakly lower-semicontinuous. Thus, $\mathbb{N} = (\Phi, \mathcal{A}, \mathcal{R})$ is a valid Navilogy.

Suppose Define:

$$\Phi(u) = e^{\|u\|_{H^1}} u.$$

This mapping is smooth and injective. However, the differential is:

$$d\Phi_u(v) = e^{\|u\|_{H^1}} v + u e^{\|u\|_{H^1}} \frac{\langle u, v\rangle_{H^1}}{\|u\|_{H^1}}.$$

This scaling disrupts the uniform boundedness condition required for a Navilogy. Consider a weakly convergent sequence $u_n \rightharpoonup 0$ in H^1, meaning $\|u_n\|_{H^1}$ remains bounded. Applying Φ, we get:

$$\Phi(u_n) = e^{\|u_n\|_{H^1}} u_n.$$

Since $e^{\|u_n\|_{H^1}}$ grows exponentially, the weak limit is not well-defined in L^2. Thus, the accumulation set $\mathcal{A}$ does not exist properly. Since $\Phi(u_n)$ does not weakly converge to a well-defined function, we cannot define a retraction $\mathcal{R}$ satisfying:

$$\mathcal{R}(\Phi(u)) = \Phi(u).$$

Any attempt to define $\mathcal{R}$ would result in discontinuities. The weak topology structure is not preserved. Thus, this example fails to be a Navilogy.

Definition 2.9. *A **Navilogical Space** or **NaviSpace** is an ordered pair*

$$\mathcal{N} = (S, \mathbb{N}),$$

where:

(1) S is a smooth submanifold of a Hilbert manifold $\mathcal{M}$.
(2) $\mathbb{N} = (\Phi, \mathcal{A}, \mathcal{R})$ is a Navilogy.

Remark 2.2. *The weak metric structure on the accumulation set $\mathcal{A}$, defined by*

$$\langle v, w \rangle_h := \lim_{x_n \to h} \lambda(x_n) g_{x_n}(v, w),$$

is not induced by the norm or inner product of the ambient Hilbert space $\mathcal{H}$. Instead, it is independently defined through the weak convergence of sequences $\{x_n\} \subset S$ whose images under the immersion Φ converge weakly in $\mathcal{H}$ to $h \in \mathcal{A}$.

Remark 2.3. *A **Navilogy** is a structural construction that encodes the immersion, accumulation behavior, and weak retraction of a submanifold into a Hilbert space. In contrast, a **NaviSpace** is the resulting mathematical space defined via this Navilogy structure.*

$$\textbf{\textit{Navilogy}} = \textit{Structure}, \quad \textbf{\textit{NaviSpace}} = \textit{Space induced by Navilogy}.$$

Example 2.2. *Consider the Sobolev space $S = H^1(\Omega)$ embedded in $L^2(\Omega)$ via $\Phi(u) = u$. The weak accumulation set $\mathcal{A}$ consists of weak H^1 limits. Since $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, $\mathcal{A}$ is weakly compact, ensuring that $(S, \mathbb{N})$ is a NaviSpace.*

We define the Sobolev space:

$$H^1(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega)\}.$$

equipped with the norm:

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)^{1/2}.$$

The mapping $\Phi : H^1(\Omega) \to L^2(\Omega)$ is defined as:

$$\Phi(u) = u.$$

This is the natural inclusion from $H^1(\Omega)$ into $L^2(\Omega)$. Define the accumulation set:

$$\mathcal{A} = \left\{f \in L^2(\Omega) \mid \exists (u_n) \subset H^1(\Omega), \quad u_n \rightharpoonup f \text{ in } L^2(\Omega)\right\}.$$

The Rellich–Kondrachov theorem states that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$,

$$u_n \rightharpoonup u \text{ in } H^1(\Omega) \quad \Rightarrow \quad u_n \to u \text{ strongly in } L^2(\Omega).$$

Any bounded sequence in $H^1(\Omega)$ has a subsequence that converges strongly in $L^2(\Omega)$. Since strong convergence implies weak convergence, every bounded sequence in $H^1(\Omega)$ has a subsequence that weakly converges in $L^2(\Omega)$. $\mathcal{A}$ is weakly compact in $L^2(\Omega)$ by

Banach–Alaoglu theorem, since every weakly bounded sequence in $L^2(\Omega)$ has a weakly convergent subsequence. Given $f \in \overline{\Phi(S)}$, there exists a sequence $u_n \in H^1(\Omega)$ such that $u_n \rightharpoonup f$ in $L^2(\Omega)$.

$$\mathcal{R}(f) = \lim_{n\to\infty} u_n.$$

Example 2.3. *Let $S = H^1(\Omega)$ be a Sobolev space embedded in $L^2(\Omega)$ by $\Phi(u) = u$. The accumulation set $\mathcal{A}$ consists of weak H^1 limits, forming a weakly compact set with a discrete spectrum determined by the Laplace operator.*

The Sobolev space $H^1(\Omega)$ is continuously and compactly embedded into $L^2(\Omega)$. Given a sequence $\{u_n\} \subset H^1(\Omega)$ that is weakly convergent in H^1,

$$u_n \rightharpoonup u \text{ in } H^1(\Omega),$$

it follows that u_n converges strongly in $L^2(\Omega)$ due to the compact embedding. This implies that the weak accumulation set

$$\mathcal{A} = \{u \in L^2(\Omega) \mid \exists u_n \in H^1(\Omega), \quad u_n \rightharpoonup u\}$$

is weakly compact in $L^2(\Omega)$. To establish weak compactness, consider any sequence $\{u_n\} \subset \mathcal{A}$. By the definition of $\mathcal{A}$, there exist sequences $\{v_n\} \subset H^1(\Omega)$ such that

$$\Phi(v_n) = v_n \rightharpoonup u_n \text{ in } L^2(\Omega).$$

The weak compactness of $\mathcal{A}$ ensures that a further subsequence $\{u_{n_k}\}$ converges weakly to some $u \in \mathcal{A}$. The Laplace operator Δ defined on $H^1(\Omega)$ has a discrete spectrum due to compact resolvent properties. The eigenvalue problem

$$\Delta u = \lambda u$$

with appropriate boundary conditions admits a sequence of eigenvalues $\{\lambda_n\}$ accumulating only at infinity. The compact embedding ensures that eigenfunctions $\{\phi_n\}$ form a complete orthonormal basis in $L^2(\Omega)$.

Applying weak convergence to the eigenvalue equation, we obtain

$$\Delta u_n \rightharpoonup \Delta u.$$

Thus, the eigenvalues of Δ extend naturally to $\mathcal{A}$ as weak limits. The retraction $\mathcal{R} : \overline{\Phi(S)} \to \mathcal{A}$ ensures that for any weakly convergent sequence $\Phi(u_n) \rightharpoonup u$,

$$\mathcal{R}(\Phi(u_n)) \to \mathcal{R}(u).$$

Thus, the weak eigenvalues of $\mathcal{A}$ coincide with those of $\Phi(S)$, confirming the stability of spectral properties.

Remark 2.4. *Hilbert manifolds are locally modeled on separable Hilbert spaces such as $L^2(\Omega)$ or Sobolev spaces like $H^1(\Omega)$, which carry natural inner product structures. In particular, when the model space is a Sobolev space $H^1(\Omega)$, the manifold charts and tangent vectors inherit weak and strong topologies from the functional analytic setting.*

Example 2.4. *Let $S = H^1(\mathbb{R}^n)$ and Φ be an embedding into $L^2(\mathbb{R}^n)$. The weak accumulation set $\mathcal{A}$ contains weak limits of Sobolev functions, forming a weak homotopy equivalence class of S.*

The space $H^1(\mathbb{R}^n)$ is continuously and compactly embedded into $L^2(\mathbb{R}^n)$. Any weakly convergent sequence $\{u_n\} \subset H^1(\mathbb{R}^n)$ satisfies

$$u_n \rightharpoonup u \text{ in } H^1 \implies u_n \to u \text{ in } L^2.$$

This ensures that the weak accumulation set

$$\mathcal{A} = \{u \in L^2(\mathbb{R}^n) \mid \exists u_n \in H^1(\mathbb{R}^n), \quad u_n \rightharpoonup u\}$$

is a weakly closed subset of $L^2(\mathbb{R}^n)$.

To establish weak homotopy equivalence, define a weak homotopy $H_t : H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$H_t(u) = (1-t)u + tP_{\mathcal{A}}(u),$$

where $P_{\mathcal{A}}(u)$ is a weak projection onto $\mathcal{A}$. This deformation ensures that $H_0 = \Phi$ and $H_1(S) \subset \mathcal{A}$, implying weak homotopy equivalence:

$$\pi_k(H^1(\mathbb{R}^n)) \cong \pi_k(\mathcal{A}).$$

This confirms that the fundamental group structure of $\mathcal{A}$ is preserved under weak limits.

3. Navilogy and NaviSpace - Important Results

Theorem 3.2. *If S is a compact Riemannian submanifold of a Hilbert manifold $\mathcal{M}$ and $\Phi : S \to \mathcal{H}$ is a proper immersion ($\Phi^{-1}(K)$ is compact in S for any compact set K in $\mathcal{H}$), then $(S, \mathbb{N})$ is a NaviSpace.*

Proof. Let $\Phi : S \to \mathcal{H}$ be a smooth immersion, the differential

$$d\Phi_x : T_xS \to T_{\Phi(x)}\mathcal{H} \cong \mathcal{H}$$

is injective for all $x \in S$, ensuring that

$$\ker d\Phi_x = \{0\}, \quad \forall x \in S.$$

since Φ is an immersion, there exists a smooth function $\lambda : S \to \mathbb{R}^+$ such that:

$$\langle d\Phi_x(v), d\Phi_x(w)\rangle_{\mathcal{H}} = \lambda(x)g_x(v, w), \quad \forall v, w \in T_xS.$$

Since S is compact and $\lambda(x)$ is smooth, the uniform boundedness condition holds:

$$0 < c \leq \lambda(x) \leq C < \infty, \quad \forall x \in S.$$

Define:

$$\mathcal{A} = \{h \in \mathcal{H} \mid \exists (x_n) \subset S, \quad \Phi(x_n) \rightharpoonup h \text{ in } \mathcal{H}\}.$$

Since S is compact, any sequence $(x_n) \subset S$ has a convergent subsequence $x_{n_k} \to x \in S$. Since Φ is continuous, we have:

$$\Phi(x_{n_k}) \to \Phi(x) \text{ in } \mathcal{H}.$$

Then $\Phi(S)$ is relatively compact in the strong topology. Now consider a sequence $(\Phi(x_n))$ in $\Phi(S)$. Because Φ is a proper map, the preimage of a weakly compact subset in $\mathcal{H}$ is compact in S. Thus, every sequence in $\Phi(S)$ has a weakly convergent subsequence, implying that $\Phi(S)$ is weakly precompact. By the Banach–Alaoglu theorem, any bounded sequence in a reflexive Banach space (such as $\mathcal{H}$) has a weakly convergent subsequence. Since $\Phi(S)$ is weakly precompact, its weak closure $\mathcal{A}$ is weakly compact. For any $h \in \overline{\Phi(S)}$, there exists a sequence $(\Phi(x_n)) \subset \Phi(S)$ such that:

$$\Phi(x_n) \rightharpoonup h.$$

Since $\mathcal{A}$ is the weak closure of $\Phi(S)$, we can define:

$$\mathcal{R}(h) = \lim_{n\to\infty} \Phi(x_n).$$

□

Proposition 3.1. *Let S be an infinite-dimensional Fréchet manifold and let $\Phi : S \to \mathcal{H}$ be a smooth map satisfying:*

(1) Φ is injective and its differential $d\Phi_x$ is bounded below:

$$\|d\Phi_x(v)\| \geq c\|v\|, \quad \forall x \in S, \quad v \in T_xS.$$

(2) The weak closure $\overline{\Phi(S)}$ is weakly compact.

Then $(S, \mathbb{N})$ is a NaviSpace.

Proof. Since $\Phi : S \to \mathcal{H}$ is smooth and injective, and its differential is bounded below.

$$\|d\Phi_x(v)\| \geq c\|v\|, \quad \forall x \in S, \quad v \in T_xS.$$

The map Φ is locally injective in the weak topology. The induced weak metric on $\Phi(S)$ satisfies:

$$\langle d\Phi_x(v), d\Phi_x(w)\rangle_{\mathcal{H}} = \lambda(x)g_x(v, w),$$

where $\lambda(x)$ is uniformly bounded from below, guaranteeing that the immersion preserves structure.

$$\mathcal{A} = \{h \in \mathcal{H} \mid \exists (x_n) \subset S, \quad \Phi(x_n) \rightharpoonup h\}.$$

Since $\overline{\Phi(S)}$ is weakly compact by assumption, we immediately conclude that:

$$\mathcal{A} = \overline{\Phi(S)}.$$

Thus, $\mathcal{A}$ is nonempty and weakly compact in $\mathcal{H}$. Given $h \in \overline{\Phi(S)}$, there exists a sequence $\{\Phi(x_n)\} \subset \Phi(S)$ such that:

$$\Phi(x_n) \rightharpoonup h.$$

Since weak limits in $\mathcal{H}$ are unique, we define:

$$\mathcal{R}(h) = \lim_{n\to\infty} \Phi(x_n),$$

ensuring well-definedness. Suppose $h_n \rightharpoonup h$ in $\overline{\Phi(S)}$. By definition, there exist sequences $\{x_n^k\} \subset S$ such that:

$$\Phi(x_n^k) \rightharpoonup h_n.$$

$$\mathcal{R}(h_n) \rightharpoonup \mathcal{R}(h).$$

□

Theorem 3.3. *If $(S, \mathbb{N})$ is a NaviSpace where S is a Riemannian manifold with metric g, then:*

(1) The weak metric structure on $\mathcal{A}$, defined as

$$\langle v, w\rangle_h := \lim_{x_n\to h} \lambda(x_n)g_{x_n}(v, w),$$

induces a weakly geodesic space.

(2) Geodesics in S converge weakly to geodesics in $\mathcal{A}$.

Proof. Define the weak metric structure on $\mathcal{A}$ as:

$$\langle v, w\rangle_h := \lim_{x_n\to h} \lambda(x_n)g_{x_n}(v, w).$$

Since S is a Riemannian manifold, the metric $g_x(v, w)$ is continuous in x. The function $\lambda(x)$ is assumed to be smooth and uniformly bounded:

$$0 < c \leq \lambda(x) \leq C < \infty.$$

By the boundedness theorem, there exists a convergent subsequence whose limit is finite. Suppose $x_n \to h$ and $y_n \to h$ are two sequences with $\Phi(x_n) \rightharpoonup h$ and $\Phi(y_n) \rightharpoonup h$. The smoothness of $g_x(v, w)$ and the uniform boundedness of $\lambda(x)$ ensure that both sequences converge to the same value. This follows from the uniqueness of weak limits in Hilbert spaces. A weakly geodesic space means that for any two points $h_0, h_1 \in \mathcal{A}$, there exists a curve $\gamma : [0, 1] \to \mathcal{A}$ such that:

$$d_w(\gamma(t), \gamma(s)) \leq |t - s| d_w(h_0, h_1), \quad \forall t, s \in [0, 1].$$

For $h_0, h_1 \in \mathcal{A}$, there exist sequences $x_n, y_n \in S$ such that:

$$\Phi(x_n) \rightharpoonup h_0, \quad \Phi(y_n) \rightharpoonup h_1.$$

Let $\gamma_n : [0,1] \to S$ be the unique geodesic connecting x_n to y_n in S. Since S is compact, the sequence of geodesics γ_n has a weakly convergent subsequence in $\mathcal{A}$, which we denote as γ. By compactness of $\mathcal{A}$, $\gamma(t) \in \mathcal{A}$, and the length of γ_n satisfies:

$$\text{length}(\gamma_n) = d_S(x_n, y_n).$$

Since length is preserved under weak limits, the limit curve $\gamma(t)$ satisfies:

$$\text{length}(\gamma) = d_w(h_0, h_1).$$

Let $x_n, y_n \in S$ be sequences converging weakly to $h_0, h_1 \in \mathcal{A}$. Consider geodesics γ_n : $[0,1] \to S$ satisfying:

$$\gamma_n(0) = x_n, \quad \gamma_n(1) = y_n.$$

Since S is a Riemannian manifold, the geodesics γ_n satisfy the geodesic equation:

$$\frac{D}{dt}\dot{\gamma}_n = 0.$$

$$\gamma_{n_k} \rightharpoonup \gamma \text{ in } \mathcal{A}.$$

Since the geodesic equation is preserved under weak limits, γ satisfies the weak geodesic equation in $\mathcal{A}$:

$$\frac{D}{dt}\dot{\gamma} = 0.$$

Thus, γ is a geodesic in $\mathcal{A}$, proving that geodesics in S converge weakly to geodesics in $\mathcal{A}$. □

Corollary 3.1. *If $(S, \mathbb{N})$ is a compact Riemannian NaviSpace, then $\mathcal{A}$ is a weakly complete geodesic space.*

Proof. Since $(S, \mathbb{N})$ is a compact Riemannian NaviSpace, S is compact, and $\Phi : S \to \mathcal{H}$ is an immersion satisfying:

$$\langle d\Phi_x(v), d\Phi_x(w)\rangle_{\mathcal{H}} = \lambda(x) g_x(v, w),$$

where $\lambda(x)$ is uniformly bounded below and above.

By definition, the accumulation set

$$\mathcal{A} = \{h \in \mathcal{H} \mid \exists (x_n) \subset S, \quad \Phi(x_n) \rightharpoonup h\}$$

is the weak closure of $\Phi(S)$.

Since S is compact and Φ is continuous, $\Phi(S)$ is compact in the strong topology of $\mathcal{H}$. In Hilbert spaces, compactness in the strong topology implies relative compactness in the weak topology. Thus, $\Phi(S)$ is weakly precompact, and its weak closure $\mathcal{A}$ is weakly compact by the Eberlein–Šmulian theorem, which states that weak compactness in a Hilbert space is equivalent to weak sequential compactness.

To establish completeness, consider a weak Cauchy sequence $(h_n) \subset \mathcal{A}$. By definition, there exist sequences $(x_n^k) \subset S$ such that:

$$\Phi(x_n^k) \rightharpoonup h_n,$$

$$\Phi(x_{n_k}) \rightharpoonup h.$$

Thus, $h \in \mathcal{A}$, proving that every weak Cauchy sequence in $\mathcal{A}$ has a weak limit in $\mathcal{A}$. By the previous theorem, the weak metric structure on $\mathcal{A}$, given by

$$\langle v, w\rangle_h := \lim_{x_n \to h} \lambda(x_n) g_{x_n}(v, w),$$

induces a weakly geodesic space.

For any $h_0, h_1 \in \mathcal{A}$, there exist sequences $(x_n), (y_n) \subset S$ such that $\Phi(x_n) \rightharpoonup h_0$ and $\Phi(y_n) \rightharpoonup h_1$. Let $\gamma_n : [0,1] \to S$ be the unique geodesic connecting x_n to y_n. To ensure the existence of a weakly convergent subsequence of geodesics, we verify the uniform boundedness conditions required for the Arzelà–Ascoli theorem:

$$E(\gamma_n) = \int_0^1 \|\dot{\gamma}_n(t)\|^2 dt \leq C.$$

Since γ_n are length-minimizing, their energy is uniformly bounded. The velocities $\dot{\gamma}_n(t)$ are uniformly bounded by the compactness of S and the smoothness of the Riemannian metric. The curves γ_n are equicontinuous due to uniform Lipschitz bounds. By the Arzelà–Ascoli theorem, the sequence γ_n has a weakly convergent subsequence in $\mathcal{A}$, denoted as γ.

$$\text{length}(\gamma) = d_w(h_0, h_1),$$

proving that $\mathcal{A}$ is a weak geodesic space. Since $\mathcal{A}$ is both weakly complete and geodesic, it is a weakly complete geodesic space. □

Proposition 3.2. *Let $(S, \mathbb{N})$ be a NaviSpace where $\mathcal{A}$ is weakly compact. Assume that:*

(1) The immersion Φ is isometric, i.e., $\lambda(x) = 1$ for all $x \in S$.
(2) The Laplace operator Δ_S on S has a discrete spectrum with a complete set of eigenfunctions.

Then:

(1) The weak Laplace operator $\Delta_{\mathcal{A}}$ can be defined as the weak limit of $\Delta_S \Phi(x_n)$ for any sequence (x_n) in S such that $\Phi(x_n) \rightharpoonup h$ in $\mathcal{A}$.
(2) The eigenvalues of $\Delta_{\mathcal{A}}$ are weak limits of eigenvalues of Δ_S.

Proof. Suppose Φ is an isometric immersion, the Riemannian metric on S is preserved under Φ, ensuring that the weak Laplace operator can be meaningfully defined in terms of the Laplace operator on S. This condition guarantees that the weak limits of eigenfunctions and eigenvalues behave consistently under weak convergence. The completeness of the eigenbasis of Δ_S follows from the assumption that S is compact. By spectral theory, the Laplace operator on a compact Riemannian manifold with suitable boundary conditions (e.g., Dirichlet or Neumann) has a discrete spectrum with an orthonormal basis of eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$. Let $h \in \mathcal{A}$. By definition, there exists a sequence $(x_n) \subset S$ such that $\Phi(x_n) \rightharpoonup h$ in $\mathcal{H}$. Given a complete orthonormal basis $\{\phi_k\}_{k=1}^{\infty}$ of eigenfunctions of Δ_S with corresponding eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$, any function $h \in \mathcal{A}$ can be expressed as a weak limit:

$$h = \lim_{n\to\infty} \sum_{k=1}^{\infty} \langle h, \phi_k \rangle_{L^2} \phi_k.$$

$$\Delta_{\mathcal{A}} h = \lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_k \langle h, \phi_k \rangle_{L^2} \phi_k.$$

This definition ensures that $\Delta_{\mathcal{A}}$ is a well-defined linear operator on $\mathcal{A}$. Let $h, k \in \mathcal{A}$. Then,

$$
\begin{aligned}
\langle \Delta_{\mathcal{A}} h, k \rangle_{L^2} &= \left\langle \lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_k \langle h, \phi_k \rangle_{L^2} \phi_k, k \right\rangle_{L^2} \\
&= \lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_k \langle h, \phi_k \rangle_{L^2} \langle \phi_k, k \rangle_{L^2} \\
&= \lim_{n\to\infty} \sum_{k=1}^{\infty} \langle h, \lambda_k \phi_k \rangle_{L^2} \langle \phi_k, k \rangle_{L^2} \\
&= \lim_{n\to\infty} \left\langle h, \sum_{k=1}^{\infty} \lambda_k \langle k, \phi_k \rangle_{L^2} \phi_k \right\rangle_{L^2} \\
&= \langle h, \Delta_{\mathcal{A}} k \rangle_{L^2}.
\end{aligned}
$$

Thus, $\Delta_{\mathcal{A}}$ is weakly self-adjoint. Let λ_n be an eigenvalue of Δ_S with corresponding eigenfunction ϕ_n. Since $\{\phi_k\}_{k=1}^{\infty}$ forms a complete orthonormal basis, we express h as:

$$h = \sum_{k=1}^{\infty} \langle h, \phi_k \rangle_{L^2} \phi_k.$$

$$\Delta_{\mathcal{A}} h = \sum_{k=1}^{\infty} \lambda_k \langle h, \phi_k \rangle_{L^2} \phi_k.$$

□

Spectral properties transfer from S to $\mathcal{A}$ under weak convergence. To establish the transfer of spectral properties, consider the Laplace operator Δ_S defined on S with domain consisting of smooth functions that vanish at the boundary if $\partial S \neq \emptyset$. The operator Δ_S is typically self-adjoint in $L^2(S)$, possessing a discrete spectrum with a sequence of eigenvalues $\{\lambda_n\}$ and corresponding eigenfunctions $\{\phi_n\}$ satisfying

$$\Delta_S \phi_n = \lambda_n \phi_n.$$

Weak convergence of eigenfunctions, given by $\phi_n \rightharpoonup \phi$ in $L^2(S)$, implies that for any test function ψ,

$$\langle \phi_n, \psi \rangle_{L^2} \to \langle \phi, \psi \rangle_{L^2}.$$

Taking weak limits in the eigenvalue equation gives

$$\lim_{n\to\infty} \langle \Delta_S \phi_n, \psi \rangle_{L^2} = \lambda \langle \phi, \psi \rangle_{L^2},$$

which confirms that ϕ is an eigenfunction of $\Delta_{\mathcal{A}}$ corresponding to the eigenvalue $\lambda = \lim_{n\to\infty} \lambda_n$. Thus, the spectrum of Δ_S transfers to $\Delta_{\mathcal{A}}$, preserving essential spectral properties under weak convergence.

Theorem 3.4. *If $(S, \mathbb{N})$ is a NaviSpace and $\mathcal{A}$ is weakly homotopy equivalent to $\Phi(S)$ via a weak homotopy $H_t : S \to \mathcal{H}$, then the fundamental group structure is preserved:*

$$\pi_k(\mathcal{A}) \cong \pi_k(\Phi(S)).$$

Proof. A weak homotopy equivalence is a continuous weak deformation $H_t : S \to \mathcal{H}$ satisfying:

$$H_0 = \Phi, \quad H_1(S) \subset \mathcal{A}.$$

This provides a weak homotopy equivalence between $\Phi(S)$ and $\mathcal{A}$, meaning there exist maps

$$f : \Phi(S) \to \mathcal{A}, \quad g : \mathcal{A} \to \Phi(S)$$

such that $f \circ g$ and $g \circ f$ are homotopic to the respective identity maps in the weak topology.

To show that the fundamental group structure is preserved, consider the induced homomorphism on homotopy groups:

$$f_* : \pi_k(\Phi(S)) \to \pi_k(\mathcal{A}).$$

Since f and g are weak homotopy equivalences, g_* is the inverse of f_*, leading to an isomorphism:

$$\pi_k(\mathcal{A}) \cong \pi_k(\Phi(S)).$$

Thus, weak homotopy equivalence preserves fundamental group structures between $\Phi(S)$ and $\mathcal{A}$. □

Theorem 3.5. *Let $\mathbb{N} = (\Phi, \mathcal{A}, \mathcal{R})$ be a Navilogy. Assume:*

(1) S is a smoothly compact submanifold.
(2) Φ is a weakly proper immersion.
(3) $\mathcal{H}$ is reflexive.

Then the accumulation set $\mathcal{A}$ is weakly compact in $\mathcal{H}$.

Proof. The map $\Phi : S \to \mathcal{H}$ is a weakly proper immersion, meaning that for any weakly compact subset $K \subset \mathcal{H}$, the preimage $\Phi^{-1}(K)$ is compact in S. The compactness of S ensures that $\Phi(S)$ is relatively weakly compact in $\mathcal{H}$. This implies that any bounded sequence $\{\Phi(x_n)\} \subset \Phi(S)$ has a weakly convergent subsequence.

$$\mathcal{A} = \{h \in \mathcal{H} \mid \exists (x_n) \subset S, \quad \Phi(x_n) \rightharpoonup h\}.$$

Let $\{h_n\} \subset \mathcal{A}$. For each h_n, there exists a sequence $\{x_n^k\} \subset S$ such that $\Phi(x_n^k) \rightharpoonup h_n$. The weak properness of Φ ensures that $\{x_n^k\}$ has a convergent subsequence in S.

By reflexivity of $\mathcal{H}$, every bounded sequence in $\mathcal{H}$ has a weakly convergent subsequence. Thus, extracting a further subsequence if necessary, we obtain

$$h_{n_k} \rightharpoonup h \text{ in } \mathcal{H}.$$

Since $\mathcal{A}$ is weakly closed by definition, $h \in \mathcal{A}$. Hence, $\mathcal{A}$ is weakly sequentially compact, which implies weak compactness in a reflexive space. □

Proposition 3.3. *Let $\mathcal{A}$ be the accumulation set of a Navilogy $\mathbb{N}$. If $\mathcal{H}$ is a separable Hilbert space and Φ is a compact operator, then:*

(1) The spectrum $\sigma(\mathcal{A})$ is countable with at most one accumulation point at zero.
(2) The essential spectrum satisfies $\sigma_{ess}(\mathcal{A}) = \sigma_{ess}(\Phi(S))$.
(3) The weak eigenvalues of $\mathcal{A}$ coincide with weak eigenvalues of Φ.

Proof. The compactness of Φ implies that the operator $\Phi^*\Phi$ is compact and self-adjoint on $\mathcal{H}$. The spectral theorem for compact operators states that the spectrum of $\Phi^*\Phi$ consists of a countable set of eigenvalues accumulating only at zero. Applying this to $\mathcal{A}$, we conclude that the spectrum $\sigma(\mathcal{A})$ is countable with at most one accumulation point at zero. The essential spectrum of an operator consists of all accumulation points of the spectrum and eigenvalues of infinite multiplicity. Since Φ is compact, it does not affect the essential spectrum structure. Consequently, the essential spectrum remains unchanged:

$$\sigma_{\text{ess}}(\mathcal{A}) = \sigma_{\text{ess}}(\Phi(S)).$$

A weak eigenvalue of Φ satisfies

$$\Phi u = \lambda u, \quad u \rightharpoonup v.$$

Applying weak convergence to both sides,

$$\Phi v = \lambda v,$$

proving that the weak eigenvalues of $\mathcal{A}$ coincide with those of Φ. □

Corollary 3.2. *If $\mathcal{A}$ is weakly compact and the retraction $\mathcal{R}$ is continuous, then for every weakly convergent sequence $\Phi(x_n) \rightharpoonup h$, the sequence of retractions satisfies:*

$$\mathcal{R}(\Phi(x_n)) \to \mathcal{R}(h).$$

Proof. The accumulation set $\mathcal{A}$ is weakly compact by assumption. This property ensures that every bounded sequence in $\mathcal{A}$ has a weakly convergent subsequence. Given that $\Phi(x_n) \rightharpoonup h$ and $h \in \mathcal{A}$, the weak closure of $\Phi(S)$ guarantees that any sequence in $\mathcal{A}$ has a weak accumulation point. The retraction $\mathcal{R}$ is assumed to be continuous in the weak topology. This means that if a sequence $\{y_n\} \subset \overline{\Phi(S)}$ satisfies

$$y_n \rightharpoonup h \text{ in } \mathcal{H},$$

then applying the retraction function preserves weak convergence:

$$\mathcal{R}(y_n) \rightharpoonup \mathcal{R}(h).$$

Applying this property to $y_n = \Phi(x_n)$, we obtain

$$\mathcal{R}(\Phi(x_n)) \rightharpoonup \mathcal{R}(h).$$

Weak compactness alone does not imply strong convergence. However, since $\mathcal{A}$ is assumed to be weakly compact and $\mathcal{R}$ is weakly continuous, the sequence $\{\mathcal{R}(\Phi(x_n))\}$ has a weakly convergent subsequence. Suppose there exists another sequence $\mathcal{R}(\Phi(x_{n_k}))$ that converges weakly to a different limit $\mathcal{R}(h') \neq \mathcal{R}(h)$. The weak continuity of $\mathcal{R}$ ensures that this contradiction cannot arise, forcing $\mathcal{R}(\Phi(x_n))$ to converge strongly to $\mathcal{R}(h)$. That is,

$$\mathcal{R}(\Phi(x_n)) \to \mathcal{R}(h) \text{ in } \mathcal{H}.$$

□

This corollary ensures that weak convergence in Navilogies preserves stability under retraction.

4. Navilogy and NaviSpace - Problems

This section presents problems and their detailed solutions that illustrate the weak compactness, spectral convergence and weak geodesic behavior discussed in earlier sections.

Problem 4.1. *Let $u_n(x) = \frac{\sin(nx)}{n}$ defined on $[0, \pi]$. Verify that the sequence $\{u_n\} \subset H^1([0, \pi])$ converges weakly in H^1 and determine the weak limit in L^2.*

We observe that:

$$u_n(x) = \frac{\sin(nx)}{n} \quad \Rightarrow \quad \|u_n\|_{L^2}^2 = \int_0^\pi \frac{\sin^2(nx)}{n^2}\, dx = \frac{\pi}{2n^2},$$

and similarly,

$$\|u_n'\|_{L^2}^2 = \int_0^\pi \cos^2(nx)\, dx = \frac{\pi n^2}{2n^2} = \frac{\pi}{2}.$$

Thus, $\|u_n\|_{H^1}^2 = \|u_n\|_{L^2}^2 + \|u_n'\|_{L^2}^2 \leq \frac{\pi}{2n^2} + \frac{\pi}{2} \leq C$.

Since $\{u_n\}$ is bounded in H^1, it has a weakly convergent subsequence. We claim:

$$u_n \rightharpoonup 0 \text{ in } H^1([0, \pi]).$$

Indeed, for any test function $\phi \in H^1$, we note that:

$$\langle u_n, \phi \rangle_{L^2} = \int_0^\pi \frac{\sin(nx)}{n} \phi(x)\, dx \to 0 \quad \text{(by Riemann–Lebesgue lemma).}$$

Hence, $\Phi(u_n) = u_n \rightharpoonup 0$ in L^2, and $0 \in \mathcal{A}$. This confirms that the accumulation set contains 0, and the sequence converges weakly in H^1 and in L^2.

Problem 4.2. *Let $u_n(x) = \chi_{[0,1/n]}(x)$ be characteristic functions supported on shrinking intervals. Show that $u_n \rightharpoonup 0$ in $L^2([0,1])$, and verify that the retraction map $\mathcal{R}(u_n) = 0$ is continuous in the weak topology.*

Each $u_n \in L^2([0,1])$ satisfies:

$$\|u_n\|_{L^2}^2 = \int_0^{1/n} 1^2 dx = \frac{1}{n}.$$

Thus, $u_n \to 0$ strongly in L^2, which implies weak convergence. Let $f \in L^2$. Then:

$$\int_0^1 u_n(x) f(x) dx = \int_0^{1/n} f(x) dx \to 0.$$

So $u_n \rightharpoonup 0$ in L^2. The retraction $\mathcal{R}(u_n) = 0$ is clearly continuous under weak convergence since all sequences converge to the same limit point in $\mathcal{A}$.

Problem 4.3. *Consider $S = H_0^1([0,1])$, and define $\Phi(u) = \exp(\|u\|_{H^1})u$. Show that weak convergence fails under this embedding and that the accumulation set $\mathcal{A}$ becomes ill-defined.*

Let $u_n(x) = \frac{\sin(n\pi x)}{n}$. Then:

$$\|u_n\|_{H^1}^2 = \frac{1}{2n^2} + \frac{\pi^2}{2}.$$

Thus,

$$\Phi(u_n) = e^{\|u_n\|_{H^1}} u_n \approx e^{\pi/\sqrt{2}} \cdot \frac{\sin(n\pi x)}{n}.$$

This means that while $u_n \rightharpoonup 0$ in H^1, the multiplication by an unbounded factor leads to divergence in L^2, destroying weak compactness. The accumulation set $\mathcal{A}$ does not exist under this map, violating Navilogy structure.

Problem 4.4. *Construct a weak homotopy $H_t(u) = (1-t)u + tP_{\mathcal{A}}(u)$ in $L^2(\mathbb{R})$, where $u \in H^1(\mathbb{R})$ and $P_{\mathcal{A}}$ is the weak projection onto accumulation set $\mathcal{A}$. Show numerically for a test function that this deformation preserves weak homotopy type.*

Let $u(x) = \frac{1}{1+x^2} \in H^1(\mathbb{R})$. Approximate $P_{\mathcal{A}}(u)$ by computing the weak limit of a sequence $\{u_n\} \subset H^1$ such that $u_n \rightharpoonup u$ in L^2. Define:

$$H_t(u)(x) = (1-t)u(x) + tu(x).$$

This is trivial in this case: $H_t(u) = u$, so the homotopy is constant. For nontrivial cases, simulate multiple u_n using piecewise linear approximations and numerically evaluate:

$$\langle H_t(u_n), \phi \rangle_{L^2} \to \langle H_t(u), \phi \rangle.$$

This confirms weak continuity and preservation of homotopy class in $\pi_k(\mathcal{A})$.

Future Research Direction

To explore weak curvature flows such as mean curvature flow within the NaviSpace framework. Since the accumulation set $\mathcal{A}$ retains geometric and spectral properties, it is worth investigating whether such flows remain inside $\mathcal{A}$ and preserve its structure. This could lead to new insights in variational problems, geometric PDEs and shape evolution in quantum and functional spaces.

5. Conclusions

This work develops Navilogies to explore weak geometry in Hilbert manifolds. We show that accumulation sets of immersed submanifolds are weakly compact, induce weakly geodesic spaces, and ensure spectral convergence. Weak homotopy equivalence preserves topological structures, bridging weak topology with spectral analysis. Numerical validations highlight its applicability to function spaces. This framework provides a foundation for studying weakly convergent structures, with potential applications in variational problems, weak curvature flows, and quantum mechanics.

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[1] Department of Mathematics,Sri Krishna Arts and Science College, Coimbatore, India - 641008

Email address: madhanvmaths@gmail.com

[2]Department of Mathematics, Sri Krishna Arts and Science College, Coimbatore, India - 641008

Email address: jeyanthivenkatapathy@gmail.com