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Bi-Univalent Functions Involving Error Function Subordinating to Limacon Domain

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ABSTRACT. By using the analytic structure of the error function, the geometric depth of limacon type domains, the algebraic strength of the Hankel determinant and Fekete- Szegö functional we introduce in this study a new subclass of bi-univalent functions. This subclass is defined by subordination to the normalized error function and limacon mappings. Bounds for the initial Taylor- Maclaurin coefficients of functions in this subclass are determined. Furthermore the Fekete–Szegö functional and Hankel determinant for this subclass is also addressed.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk. A function V belongs to the class A if it is analytic in $\mathbb{D}$ and satisfies the normalization conditions $V(0) = 0 \quad \text{and} \quad V'(0) = 1$. Every function $V \in A$ admits the expansion of the form

$$V(z) = z + \sum_{\nu=2}^{\infty} b_\nu z^\nu, \quad z \in \mathbb{D}. \tag{1.1}$$

The subclass $S \in A$ comprises of analytic functions with one-one property in $\mathbb{D}$. A function V is said to be subordinate to another analytic function W denoted $V \prec W$ if there exists a Schwarz function ω analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $V(z) = W(\omega(z))$.
If W is univalent in $\mathbb{D}$ then subordination relation is equivalent to

$$V(0) = W(0) \quad \text{and} \quad V(\mathbb{D}) \subset W(\mathbb{D}). \tag{1.2}$$

By the principle of subordination [16] if V is univalent and maps a domain $\mathbb{D}_1$ onto another domain $\mathbb{D}_2$ then the inverse function $W = V^{-1}$ defined by

$$W(V(z)) = z, \forall z \in \mathbb{D}_1$$

is analytic and univalent on $\mathbb{D}_2$.
Closely related to analytic subordination is the concept of differential subordination which has been widely used in geometric function theory to investigate properties of analytic functions involving their derivatives. Let $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and let h be univalent in $\mathbb{D}$. If an analytic function p in $\mathbb{D}$ satisfies a second-order differential subordination of the form

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \prec h(z), \quad z \in \mathbb{D},$$

then p is said to be a solution of the corresponding differential subordination.

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The inverse function W can be written as

$$W(w) = w - b_2w^2 + (2b_2^2 - b_3)w^3 - (5b_3^2 - 5b_2b_3 + b_4)w^4 + \cdots . \tag{1.3}$$

A function $V \in S$ is called *bi-univalent* if both V and V^{-1} are univalent in $\mathbb{D}$ and we denote by Σ the class of bi-univalent functions. While the results of univalent functions exists bi-univalent functions present more challenges particularly in determining sharp coefficient bounds. Lewin [12] initiated the study of coefficient problems for functions in Σ focusing on estimating $|b_2|$,$|b_3|$ and $|b_4|$. Despite progress the problem of estimating sharp bounds for higher order coefficients

$$|b_\nu|, \nu \geq 4$$

remain unresolved [3, 17]. Typical examples of bi-univalent functions include

$$V_1(z) = \frac{z}{1-z}, \quad V_2(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

with corresponding inverses

$$V_1^{-1}(w) = \frac{w}{1+w}, \quad V_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}.$$

In this context, the Hankel determinant serves as a powerful analytic tool for probing the structure and growth of analytic functions. Given an analytic function V of the form (1.1) the Hankel determinant defined by Noonan and Thomas [18] is given as

$$H_l(\nu) = \begin{vmatrix} b_\nu & b_{\nu+1} & \cdots & b_{\nu+l-1} \\ b_{\nu+1} & b_{\nu+2} & \cdots & b_{\nu+l} \\ \cdots & \cdots & \cdots & \cdots \\ b_{\nu+l-1} & b_{\nu+l} & \cdots & b_{\nu+2l-2} \end{vmatrix}, [b_1 = 1]. \tag{1.4}$$

For $l = 2$ and $\nu = 1$ (1.4) reduces to

$$H_2(1) = \begin{vmatrix} b_1 & b_2 \\ b_2 & b_3 \end{vmatrix} = b_3 - b_2^2 \, [\because b_1 = 1]. \tag{1.5}$$

This determinant is widely studied due to its connection to Fekete- Szegö functional a significant quantity in geometric function theory. Specifically, for a real paramater μ the Fekete-Szegö functional is given by

$$|b_3 - \mu b_2^2|$$

with the special case as $\mu = 1$.
The second Hankel determinant for $l = 2$ and $\nu = 2$ from (1.4) is reduced as follows

$$H_2(2) = \begin{vmatrix} b_2 & b_3 \\ b_3 & b_4 \end{vmatrix} = b_2b_4 - b_3^2. \tag{1.6}$$

One such approach to defining such subclasses is through analytic subordination to special functions. Ma and Minda [13] introduced a unified technique for defining subclasses like $S^*(\phi)$ and $C(\phi)$ using analytic function ϕ with positive real part:

$$\frac{zV'(z)}{V(z)} \prec \phi(z), \quad \text{or} \quad 1 + \frac{zV''(z)}{V'(z)} \prec \phi(z).$$

Numerous subclasses have emerged by selecting appropriate ϕ functions associated with cassinian ovals[22], cresent shaped regions[20], Bernoulli lemniscates [14, 23], Booth lemniscates [9], rational function [10], exponential maps [15], sigmoid- type curves[5] and limacon type regions.

A notable limacon domain is the bean-shaped region introduced by Yunus et al. [24] bounded by

$$\Omega(\mathbb{D}) = \left\{u = x + iy : (4x^2 + 4y^2 - 8x - 5)^2 + 8(4x^2 + 4y^2 - 12x - 3) = 0\right\}.$$

and mapped by

$$L(z) = 1 + \sqrt{2}z + \frac{1}{2}z^2. \tag{1.7}$$

Such mappings serve as powerful tools in defining function classes through subordination.

Among many special functions the error function and imaginary error function play an important role in various scientific fields including probability, statistics partial differential equations and other engineering applications. The classical error function erf [1] and the imaginary error function erfi [2, 4] are defined as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu z^{2\nu+1}}{(2\nu+1)\nu!},$$

$$\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \sum_{\nu=0}^{\infty} \frac{z^{2\nu+1}}{(2\nu+1)\nu!}$$

respectively. Ramachandran et al. [21] introduced the normalized variant,

$$\operatorname{Erf}(z) = \frac{\sqrt{\pi z}}{2}\operatorname{erf}(\sqrt{z}) = z + \sum_{\nu=2}^{\infty} \frac{(-1)^{\nu-1} z^\nu}{(2\nu-1)(\nu-1)!}.$$

which preserves normalization and analyticity in $\mathbb{D}$ and maps onto domains suitable for subordination.

Closely related is the convolution product (or Hadamard product), which is used to generate new subclasses by combining two power series term-wise. For functions $f(z) = \sum_{\nu=2}^{\infty} a_\nu z^\nu$ and $g(z) = \sum_{\nu=2}^{\infty} b_\nu z^\nu$, the convolution is:

$$(f * g)(z) = \sum_{\nu=2}^{\infty} a_\nu b_\nu z^\nu.$$

Using this operation, one defines function families such as:

$$\operatorname{Erf} * S = (\operatorname{Erf} * V)(z) = \left\{G(z) = z + \sum_{\nu=2}^{\infty} \frac{(-1)^{\nu-1} c_\nu}{(2\nu-1)(\nu-1)!} z^\nu : V \in S\right\}.$$

and using the normalized analytic imaginary error function

$$\operatorname{Erfi}(z) = \frac{\sqrt{\pi z}}{2} erf(\sqrt{z}) = z + \sum_{\nu=2}^{\infty} \frac{z^\nu}{(2\nu-1)(\nu-1)!} \tag{1.8}$$

$$EV(z) = (\operatorname{Erfi} * V)(z) = z + \sum_{\nu=2}^{\infty} \frac{c_\nu z^\nu}{(2\nu--1)(\nu-1)!}. \tag{1.9}$$

The theory of second order differential subordination has been developed extensively by Kanas and Studziński [7] as well as Kanas and Owa [8] who investigated the connection between differential subordination and subordination relations involving expressions $\frac{V(z)}{z}$, $V'(z)$ and $1 + z\frac{V''(z)}{V'(z)}$.

The subclasses defined through linear combinations of functions and its first two derivatives in [8] served as the motivation to extend this framework by incorporating convolution operators associated with imaginary error function. Furthermore, the resulting expressions are subordinated to limacon type regions thereby introducing a new subclass of bi-univalent functions as follows

Definition 1.1. *A function* $V \in \Sigma$ *given by* (1.1) *is said to be in the family* $\hat{\mathcal{T}}_{\Sigma}(t, \mu, \kappa)$ *if it satisfies the two conditions given below:*

$$(1-\kappa)\frac{EV(z)}{z} + \kappa(EV(z))' + \mu z(EV(z))'' \prec L(t, z)$$

and

$$(1-\kappa)\frac{EW(w)}{w} + \kappa(EW(w))' + \mu w(EW(w))'' \prec L(t, w),$$

where $z, w \in \mathbb{D}, \kappa, \mu \geq 0, t \in \left(\frac{1}{2}, 1\right]$, *and the function* $W = V^{-1}$ *is given by* (1.3).

2. Main results

2.1. **Coefficient estimates of the class** $\hat{\mathcal{T}}_{\Sigma}(t, \mu, \kappa)$**.** In this section, we provide bounds for the initial Taylor-Maclaurin coefficients for functions belonging to the class $\hat{\mathcal{T}}_{\Sigma}(t, \mu, \kappa)$ using the following lemma

Lemma 2.1. [19] *Let* P *be the class of all analytic function* $q(z)$ *of the form*

$$q(z) = 1 + \sum_{\nu=1}^{\infty} q_{\nu} z^{\nu} \tag{2.10}$$

with $\Re(q(z)) > 0$ *for all* $z \in \mathbb{D}$. *Then,* $|q_{\nu}| \leq 2$ *for every* $\nu = 1, 2, \ldots$.

Theorem 2.1. *Let the function* $V(z) \in \hat{\mathcal{T}}_{\Sigma}(t, \mu, \kappa)$.*Then*

$$|b_2| \leq \frac{6\sqrt{10}}{\sqrt{|18\sqrt{2}(1+2\kappa+6\mu) + 5\sqrt{2}(1+\kappa+2\mu)^2(2\sqrt{2}+1)|}} \tag{2.11}$$

$$|b_3| \leq \frac{18}{(1+\kappa+2\mu)^2} + \frac{10\sqrt{2}}{(1+2\kappa+6\mu)} \tag{2.12}$$

and

$$|b_4| \leq \frac{126(1+2\sqrt{2})}{(1+3\kappa+12\mu)} + \frac{300}{(1+\kappa+2\mu)(1+2\kappa+6\mu)}. \tag{2.13}$$

Proof. Let $V(z)$ be in the class $\hat{\mathcal{T}}_{\Sigma}(t, \mu, \kappa)$. We have two analytic functions q and r defined in the unit disk $\mathbb{D}$ such that

$$(1-\kappa)\frac{EV(z)}{z} + \kappa\left((EV(z))' + \mu z(EV(z))''\right) \prec L(t, q(z)) \tag{2.14}$$

and

$$(1-\kappa)\frac{EW(w)}{\omega} + \kappa\left((EW(w))' + \mu w(EW(w))''\right) \prec L(t, r(w)), \tag{2.15}$$

where $q(z)=\sum_{n=1}^{\infty} q_\nu z^\nu$ and $r(w)=\sum_{n=1}^{\infty} r_\nu w^\nu$ for all $z,w\in\mathbb{U}$. Then,we have

$$(1-\kappa)\frac{\left(z+\sum_{\nu=2}^{\infty}\frac{b_\nu z^\nu}{(2\nu-1)(\nu-1)!}\right)}{z}+\kappa\left(z+\sum_{\nu=2}^{\infty}\frac{b_\nu z^\nu}{(2\nu-1)(\nu-1)!}\right)'$$
$$+\mu z\left(z+\sum_{\nu=2}^{\infty}\frac{b_\nu z^\nu}{(2\nu-1)(\nu-1)!}\right)''$$
$$=1+\frac{q_1}{\sqrt{2}}z+\left[\frac{1}{\sqrt{2}}(q_2-\frac{q_1^2}{2})+\frac{q_1^2}{8}\right]z^2+\left[\frac{1}{\sqrt{2}}(q_3-q_1q_2+\frac{q_1^3}{4})+\frac{q_1}{4}(q_2-\frac{q_1^2}{2}\right]z^3\ldots$$

and

$$(1-\kappa)\frac{\left(w+\sum_{\nu=2}^{\infty}\frac{b_\nu w^\nu}{(2\nu-1)(\nu-1)!}\right)}{w}+\kappa\left(w+\sum_{\nu=2}^{\infty}\frac{b_\nu w^\nu}{(2\nu-1)(\nu-1)!}\right)'$$
$$+\mu w\left(w+\sum_{\nu=2}^{\infty}\frac{b_\nu w^\nu}{(2\nu-1)(\nu-1)!}\right)''$$
$$=1+\frac{r_1}{\sqrt{2}}w+\left[\frac{1}{\sqrt{2}}(r_2-\frac{r_1^2}{2})+\frac{r_1^2}{8}\right]w^2+\left[\frac{1}{\sqrt{2}}(r_3-r_1r_2+\frac{r_1^3}{4})+\frac{r_1}{4}(r_2-\frac{r_1^2}{2}\right]w^3\ldots.$$

Now, equating the coefficients we get

$$\frac{1+\kappa+2\mu}{3.1!}b_2=\frac{q_1}{\sqrt{2}} \tag{2.16}$$

$$\frac{1+2\kappa+6\mu}{5.2!}b_3=\frac{1}{\sqrt{2}}(q_2-\frac{q_1^2}{2})+\frac{q_1^2}{8} \tag{2.17}$$

$$\frac{(1+3\kappa+12\mu)}{7.3!}b_4=\left[\frac{1}{\sqrt{2}}(q_3-q_1q_2+\frac{q_1^3}{4})+\frac{q_1}{4}(q_2-\frac{q_1^2}{2})\right] \tag{2.18}$$

$$-\frac{(1+\kappa+2\mu)}{3.1!}b_2=\frac{r_1}{\sqrt{2}} \tag{2.19}$$

$$\frac{(1+2\kappa+6\mu)(2b_2^2-b_3)}{5.2!}=\frac{1}{\sqrt{2}}(r_2-\frac{r_1^2}{2})+\frac{r_1^2}{8} \tag{2.20}$$

$$\frac{-(1+3\kappa+12\mu)(b_4+5b_2^3-b_3b_2)}{7.3!}=\left[\frac{1}{\sqrt{2}}(r_3-r_1r_2+\frac{r_1^3}{4})+\frac{r_1}{4}(r_2-\frac{r_1^2}{2})\right]. \tag{2.21}$$

It follows from (2.16) and (2.19)

$$q_1=-r_1 \tag{2.22}$$

$$\frac{2}{9}(1+\kappa+2\mu)^2b_2^2=\frac{1}{2}(q_1^2+r_1^2). \tag{2.23}$$

Subtracting (2.16) and (2.19)

$$\frac{(1+\kappa+2\mu)2b_2}{3.1!}=\frac{q_1-r_1}{\sqrt{2}}$$

$$b_2=\frac{3(q_1-r_1)}{2\sqrt{2}(1+\kappa+2\mu)}. \tag{2.24}$$

Adding (2.17) and (2.20) we get

$$\frac{2{b_2}^2(1+\kappa+2\mu)}{10}=\frac{1}{\sqrt{2}}\left((q_2+r_2)-\frac{(q_1^2+r_1^2)}{2}\right)+\frac{q_1^2+r_1^2}{8} \tag{2.25}$$

Substituting the value of $({q_1}^2 + {r_1}^2)$ from (2.23) in the right- hand side of (2.25), we deduce that

$$\left[\frac{\sqrt{2}(1+2\kappa+6\mu)}{5} + \frac{2(1+\kappa+2\mu)^2}{9} + \frac{\sqrt{2}(1+\kappa+2\mu)^2}{18}\right]{b_2}^2 = (q_2 + r_2) \tag{2.26}$$

Hence,

$${b_2}^2 = \frac{(q_2 + r_2)}{\left[\frac{\sqrt{2}(1+2\kappa+6\mu)}{5} + \frac{\sqrt{2}(1+\kappa+2\mu)^2}{18}[2\sqrt{2}+1].\right]} \tag{2.27}$$

Using lemma 2.1 in (2.26), we get

$$|b_2| \leq \frac{6\sqrt{10}}{\sqrt{|18\sqrt{2}(1+2\kappa+6\mu) + 5\sqrt{2}(1+\kappa+2\mu)^2(2\sqrt{2}+1)}}. \tag{2.28}$$

Subtracting (2.20) and (2.17) yields,

$$\frac{2(1+2\kappa+6\mu)}{10}(b_3 - b_2^2) = \frac{1}{\sqrt{2}}(q_2 - r_2) - \frac{1}{\sqrt{2}}(q_1^2 - r_1^2) + \frac{(q_1^2 - r_1^2)}{8} \tag{2.29}$$

Substituting (2.22) in (2.29) we get

$$b_3 = \frac{5(q_2 - r_2)}{\sqrt{2}(1+2\kappa+6\mu)} + \frac{9({q_1}^2 + {r_1}^2)}{4(1+\kappa+2\mu)^2} \tag{2.30}$$

And in view of (2.29) we obtain

$$|b_3| \leq \frac{10\sqrt{2}}{(1+2\kappa+6\mu)^2} + \frac{18}{(1+\kappa+2\mu)^2}. \tag{2.31}$$

Subtracting equations (2.18) and (2.21)

$$\frac{(1+3\kappa+12\mu)(2b_4 + 5b_2^3 - 5b_3b_2)}{7.3!} =$$

$$\left[\frac{1}{\sqrt{2}}((q_3 - r_3) - (q_1q_2 - r_1r_2) + \frac{(q_1^3 - r_1^3)}{4} + \frac{(q_1q_2 - r_1r_2)}{4} - \frac{(q_1^3 - r_3)}{8})\right]. \tag{2.32}$$

Equating (2.24) and (2.30) gives

$$b_4 = \frac{21\sqrt{2}(q_3 - r_3)}{2(1+3\kappa+12\mu)} + \frac{21(1-2\sqrt{2})(q_1q_2 - r_1r_2)}{2(1+3\kappa+12\mu)} + \frac{21(\sqrt{2}-1)q_1^3}{4(1+3\kappa+12\mu)} + \frac{75q_1(q_2 - r_2)}{2(1+2\kappa+6\mu)(1+\kappa+2\mu)} \tag{2.33}$$

$$|b_4| \leq \frac{126(1+2\sqrt{2})}{(1+3\kappa+12\mu)} + \frac{300}{(1+\kappa+2\mu)(1+2\kappa+6\mu)}. \tag{2.34}$$

□

By taking different values for μ we obtain the following corollaries

Corollary 2.1. *For $\mu = 0$, we have $\hat{\mathcal{T}}_\Sigma(t, 0, \kappa) = \hat{\mathcal{T}}_\Sigma(t, \kappa)$ then*

$$|b_2| \leq \frac{6\sqrt{10}}{\sqrt{|18\sqrt{2}(1+2\kappa) + 5\sqrt{2}(1+\kappa)^2(2\sqrt{2}+1)|}}$$

$$|b_3| \leq \frac{18}{(1+\kappa)^2} + \frac{10\sqrt{2}}{(1+2\kappa)}$$

and

$$|b_4| \leq \frac{126(1+2\sqrt{2})}{(1+3\kappa)} + \frac{300}{(1+2\kappa)(1+\kappa)}.$$

Corollary 2.2. *For $\mu = 0$ and $\kappa = 1$, we have $\hat{\mathcal{T}}_\Sigma(t,0,1) = \hat{\mathcal{T}}_\Sigma(t,1)$ then*

$$|b_2| \leq \frac{6\sqrt{10}}{\sqrt{|54\sqrt{2}+20\sqrt{2}(2\sqrt{2}+1)|}}, \quad |b_3| \leq \frac{27+20\sqrt{2}}{6}$$

and

$$|b_4| \leq \frac{163+126\sqrt{2}}{2}.$$

Corollary 2.3. *For $\mu = 0$ and $\kappa = 0$, we have $\hat{\mathcal{T}}_\Sigma(t,0,0) = \hat{\mathcal{T}}_\Sigma(t)$ then*

$$|b_2| \leq \frac{6\sqrt{5}}{\sqrt{|18\sqrt{2}+5\sqrt{2}(2\sqrt{2}+1)|}}$$
$$|b_3| \leq 18+10\sqrt{2}$$

and

$$|b_4| \leq 426+252\sqrt{2}.$$

2.2. **Fekete-Szegö functional of the class $\hat{\mathcal{T}}_\Sigma(t,\mu,\kappa)$.** The results of this section depend on the following lemma which is well-known and useful in establishing the Fekete-Szegö functional results.

Lemma 2.2. [11] *Let $\nu, l \in \mathbb{R}$ and $p, q \in \mathbb{C}$. If $|p| < r$ and $|q| < r$*

$$|(\nu+l)p + (\nu-l)q| \leq \begin{cases} 2|\nu|r, & \text{if } |\nu| \geq |l| \\ 2|l|r & \text{if } |\nu| \leq |l| \end{cases}$$

We consider the Fekete- Szegö functional for the functions in the class $\hat{\mathcal{T}}_\Sigma(t,\mu,\kappa)$.

Theorem 2.2. *Let the function $V(z) \in \hat{\mathcal{T}}_\Sigma(t,\mu,\kappa)$. Then, for some $\alpha \in \mathbb{R}$,*

$$|b_3 - \alpha {b_2}^2| \leq \begin{cases} \frac{10}{(1+2\kappa+6\mu)}, & 0 \leq H(\alpha) \leq \frac{5}{(1+2\kappa+6\mu)} \\ 2|H(\alpha)|, & H(\alpha) \geq \frac{5}{(1+2\kappa+6\mu)}. \end{cases} \tag{2.35}$$

Proof. For some real number α, using equation (2.30), we have

$$b_3 - \alpha {b_2}^2 = \frac{5}{(1+2\kappa+6\mu)}\left[\frac{(q_2-r_2)}{\sqrt{2}} - \frac{(q_1^2-r_1^2)}{\sqrt{2}(1+2\kappa+6\mu)} + \frac{(q_1^2-r_1^2)}{8(1+2\kappa+6\mu)}\right] + (1-\alpha)b_2^2.$$

Substituting (2.27), we have

$$b_3 - \alpha {b_2}^2 = \frac{5}{(1+2\kappa+6\mu)}\left[\frac{(q_2-r_2)}{\sqrt{2}} - \frac{(q_1^2-r_1^2)}{\sqrt{2}} + \frac{(q_1^2-r_1^2)}{8}\right]$$
$$+ (1-\alpha)\frac{90(q_2+r_2)}{18\sqrt{2}(1+2\kappa+6\mu)+5\sqrt{2}(1+\kappa+2\mu)^2(2\sqrt{2}+1)}$$

(2.36)

$$b_3 - \alpha {b_2}^2 = \Bigg[\Big[\frac{90}{18\sqrt{2}(1+2\kappa+6\mu)+5\sqrt{2}(1+\kappa+2\mu)^2(2\sqrt{2}+1)} + \frac{5}{(1+2\kappa+6\mu)}\Big]q_2 + \Big[\frac{90}{18\sqrt{2}(1+2\kappa+6\mu)+5\sqrt{2}(1+\kappa+2\mu)^2(2\sqrt{2}+1)} - \frac{5}{(1+2\kappa+6\mu)}\Big]r_2\Bigg]$$

$$= \Big[[H(\alpha) + \frac{5}{(1+2\kappa+6\mu)}]q_2 + [H(\alpha) - \frac{5}{(1+2\kappa+6\mu)}]r_2\Big]$$

where,

$$H(\alpha) = \frac{90}{18\sqrt{2}(1+2\kappa+6\mu)+5\sqrt{2}(1+\kappa+2\mu)^2(2\sqrt{2}+1)}.$$

Thus, the desired inequality is obtained by applying Lemma 2.2 . □

Corollary 2.4. *Let the function $V(z) \in \hat{\mathcal{T}}_\Sigma(t,0,\kappa) = \hat{\mathcal{T}}_\Sigma(t,\kappa)$. Then, for some $\alpha \in \mathbb{R}$*

$$|b_3 - \alpha {b_2}^2| \le \begin{cases} \frac{10}{(1+2\kappa)} & , 0 \le H(\alpha) \le \frac{5}{(1+2\kappa)} \\ 2|H(\alpha)| & , H(\alpha) \ge \frac{5}{(1+2\kappa)}. \end{cases} \tag{2.37}$$

Corollary 2.5. *Let the function $V(z) \in \hat{\mathcal{T}}_\Sigma(t,0,1) = \hat{\mathcal{T}}_\Sigma(t,1)$. Then, for some $\alpha \in \mathbb{R}$*

$$|b_3 - \alpha {b_2}^2| \le \begin{cases} \frac{10}{3} & , 0 \le H(\alpha) \le \frac{5}{3} \\ 2|H(\alpha)| & , H(\alpha) \ge \frac{5}{3}. \end{cases} \tag{2.38}$$

Corollary 2.6. *Let the function $V(z) \in \hat{\mathcal{T}}_\Sigma(t,0,0) = \hat{\mathcal{T}}_\Sigma(t)$. Then, for some $\alpha \in \mathbb{R}$*

$$|b_3 - \alpha {b_2}^2| \le \begin{cases} 10 & , 0 \le H(\alpha) \le 5 \\ 2|H(\alpha)| & , H(\alpha) \ge 5. \end{cases} \tag{2.39}$$

2.3. **Hankel determinant.** We use the following lemma to find the second Hankel determinant for the set of all bi-univalent functions in the class $\hat{\mathcal{T}}_\Sigma(t,\mu,\kappa)$

Lemma 2.3. [6] *If a function $q \in P$, then*

$$2q_2 = q_1^2 + (4-q_1^2)k \tag{2.40}$$

$$4q_3 = q_1^3 + 2q_1(4-q_1^2)k - q_1(4-q_1^2)k^2 + 2(4-q_1^2)(1-|k|^2)m, \tag{2.41}$$

for some k, m with $|k| \le 1$ and $|m| \le 1$.

Theorem 2.3. *Let $V(z)$ given by (1.1) be in the class $\hat{\mathcal{T}}_\Sigma(t,\mu,\kappa)$. Then we have*

$$|b_2b_4 - b_3^2| \le \frac{1008+252\sqrt{2}}{(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{324}{(1+\kappa+2\mu)^4}. \tag{2.42}$$

Proof. Using (2.24), (2.30) and (2.33) we get

$$b_2b_4 - b_3^2 = \frac{63q_1(q_3-r_3)}{(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{63(1-2\sqrt{2})q_1^2(q_2-r_2)}{\sqrt{2}(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{63(\sqrt{2}-1)q_1^4}{2\sqrt{2}(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{135q_1^2(q_2-r_2)}{2\sqrt{2}(1+\kappa+2\mu)^2(1+3\kappa+12\mu)} - \frac{81q_1^4}{4(1+\kappa+2\mu)^4} - \frac{25(q_2-r_2)^2}{2(1+2\kappa+6\mu)^2}.$$

According to Lemma 2.3 and (2.22) we obtain

$$q_2 - r_2 = \frac{4-q_1^2}{2}(k-l), \quad q_2 + r_2 = q_1^2 + \frac{4-q_1^2}{2}(k+l), \tag{2.43}$$

and

$$\begin{aligned} q_3 - r_3 = {}& \frac{q_1^3}{2} + \frac{(4-q_1^2)q_1}{2}(k+l) - \frac{(4-q_1^2)q_1}{4}(k^2+l^2) \\ & + \frac{4-q_1^2}{2}\left[(1-|k|^2)m - (1-|l|^2)n\right], \end{aligned} \tag{2.44}$$

for some k, l, m, and n with $|k| \le 1$, $|l| \le 1$, $|m| \le 1$ and $|n| \le 1$. Since $q \in \mathcal{P}$ we have $|q_1| \le 2$. Letting $q_1 = q$ we may assume without loss of generality that $q \in [0,2]$.

$$\begin{aligned} |b_2b_4 - b_3^2| \le {}& \frac{63q_1^4}{2(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{63q_1^2(k+l)(4-q_1^2)}{2(1+\kappa+2\mu)(1+3\kappa+12\mu)} \\ & + \frac{63q_1^2(4-q_1^2)(k^2+l^2)}{4(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{63(1-2\sqrt{2})q_1^2(4-q_1^2)(k-l)}{2\sqrt{2}(1+\kappa+2\mu)(1+3\kappa+12\mu)} \\ & + \frac{63(\sqrt{2}-1)q_1^4}{2\sqrt{2}(1+\kappa+2\mu)(1+3\kappa+6\mu)} + \frac{135q_1^2(4-q_1^2)(k-l)}{4\sqrt{2}(1+\kappa+2\mu)(1+3\kappa+12\mu)} \\ & - \frac{81q_1^4}{4(1+\kappa+2\mu)^4} - \frac{25(4-q_1^2)^2(k-l)}{8(1+2\kappa+6\mu)^2}. \end{aligned}$$

Substituting (2.43) and (2.44) and setting $\eta = |k|$, $\rho = |l|$ we apply the triangle inequality and standard bounds for analytic functions in $\mathbb{D}$ to estimate $|b_2b_4 - b_3^2|$ yielding the required inequality.

$$\begin{aligned} |b_2b_4 - b_3^2| \le {}& \Bigg(\frac{63}{2(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{63(\sqrt{2}+1)}{2\sqrt{2}(1+\kappa+2\mu)(1+3\kappa+6\mu)} \\ & + \frac{81}{4(1+\kappa+2\mu)^4}\Bigg)q^4 + \Bigg(\frac{63q^2(4-q^2)}{2(1+\kappa+2\mu)(1+3\kappa+12\mu)} \\ & + \frac{63(1-2\sqrt{2}q^2(4-q^2)}{2\sqrt{2}(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{135q^2(4-q^2)}{4\sqrt{2}(1+\kappa+2\mu)^2(1+3\kappa+12\mu)}\Bigg)(\rho+\eta) \\ & + \frac{63q^2(4-q^2)(\rho^2+\eta^2)}{4(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{25(4-q^2)^2(\rho+\eta)^2}{8(1+2\kappa+6\mu)^2} \\ \le {}& H_1 + H_2(\eta+\rho) + H_3(\eta^2+\rho^2) + H_4(\eta+\rho)^2 = H(\eta,\rho) \end{aligned}$$

where

$$\begin{aligned} H_1 &= \left(\frac{126\sqrt{2}+63}{2\sqrt{2}(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{81}{4(1+\kappa+2\mu)^4}\right)q^4 \ge 0 \\ H_2 &= \left(\frac{63(1+3\sqrt{2})q^2(4-q^2)}{2\sqrt{2}(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{135q^2(4-q^2)}{4\sqrt{2}(1+\kappa+2\mu)^2(1+3\kappa+12\mu)}\right) \ge 0 \\ H_3 &= \frac{63q^2(4-q^2)}{4(1+\kappa+2\mu)(1+3\kappa+12\mu)} \ge 0 \\ H_4 &= \frac{24(4-q^2)}{8(1+2\kappa+6\mu)^2} \ge 0. \end{aligned}$$

(1) Let $q = 0$. Since $H_1(0,\mu,\kappa) = H_2(0,\mu,\kappa) = H_3(0,\mu,\kappa) = 0$ and

$$H_4(0,\mu,\kappa) = \frac{50}{(1+2\kappa+6\mu)^2}$$

the function $\Psi(\rho,\eta)$ is written as follows:

$$\Psi(\rho,\eta) = \frac{50}{(1+2\kappa+6\mu)^2}(\rho+\eta)^2, \quad (\rho,\eta) \in \Delta.$$

It is obvious that the function Ψ reaches its maximum near the closed-square boundary Δ. Now, differentiating the function $\Psi(\rho,\eta)$ we have

$$\Psi_\rho(\rho,\eta) = \frac{50}{(1+2\kappa+6\mu)^2}(\rho+\eta)$$

for each $\eta \in [0,1]$.
The function $\Psi(\rho,\eta)$ is an increasing function and reaches its maximum at $\rho = 1$ since $\Psi_\rho(\rho,\eta) \geq 0$. Therefore

$$\max\{\Psi(\rho,\eta) : \eta \in [0,1]\} = \Psi(1,\eta) = \frac{50}{(1+2\kappa+6\mu)^2}(1+\eta)^2, \quad \eta \in [0,1].$$

Differentiating $\Psi(1,\eta)$ we obtain

$$\Psi'(1,\eta) = \frac{100}{(1+2\kappa+6\mu)^2}(1+\eta), \quad \eta \in [0,1].$$

Since $\Psi'(1,\eta) > 0$ the function $\Psi(1,\eta)$ is an increasing function and the maximum occurs at $\eta = 1$. Hence

$$\max\{\Psi(1,\eta) : \eta \in [0,1]\} = \Psi(1,1) = \frac{200}{(1+2\kappa+6\mu)^2} = \left(\frac{10\sqrt{2}}{1+2\kappa+6\mu)}\right)^2.$$

Thus by taking $q = 0$ we obtain

$$\Psi(\rho,\eta) \leq \max\{\Psi(\rho,\eta) : (\rho,\eta) \in [0,1]^2\} = \Psi(1,1) = \left(\frac{10\sqrt{2}}{1+2\kappa+6\mu}\right)^2.$$

We know that $|b_2b_4 - b_3^2| \leq \Psi(\rho,\eta)$ so we can have

$$|b_2b_4 - b_3^2| \leq \left(\frac{10\sqrt{2}}{1+2\kappa+6\mu}\right)^2.$$

(2) Now taking $q = 2$. Since $H_2(2,\mu,\kappa) = H_3(2,\mu,\kappa) = H_4(2,\mu,\kappa) = 0$ and

$$\begin{aligned} H_1(2,\mu,\kappa) &= \left(\frac{126\sqrt{2}+63}{(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{81}{\sqrt{2}(1+\kappa+2\mu)^4}\right)4\sqrt{2} \\ &= \left(\frac{1008+252\sqrt{2}}{(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{324}{(1+\kappa+2\mu)^4}\right) \end{aligned}$$

the function $\Psi(\rho,\eta)$ is a constant as follows:

$$\Psi(\rho,\eta) = H_1(2) = \frac{1008+252\sqrt{2}}{(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{324}{(1+\kappa+2\mu)^4}.$$

Thus, we obtain

$$|b_2b_4 - b_3^2| \leq \frac{1008+252\sqrt{2}}{(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{324}{(1+\kappa+2\mu)^4},$$

in the case of $q = 2$.

Consequently, in light of these two instances we write

$$|b_2b_4 - b_3^2| \leq \max\left\{\left(\frac{10\sqrt{2}}{1+2\kappa+6\mu}\right)^2, \frac{1008+252\sqrt{2}}{(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{324}{(1+\kappa+2\mu)^4}\right\}.$$

Therefore,

$$|b_2b_4 - b_3^2| \leq \frac{1008+252\sqrt{2}}{(1+\kappa+2\mu)(1+3\kappa+12\mu)} + \frac{324}{(1+\kappa+2\mu)^4} \tag{2.45}$$

□

The following remark is provided to demonstrate that the subclass $\hat{\mathcal{T}}_\Sigma(t,\mu,\kappa)$ is non-empty and that the function belonging to this class satisfy all the coefficient estimates, the Fekete - Szegö inequality and the second Hankel determinant bounds obtained.

Remark 2.1. *Let $V(z) = z + \varepsilon z^2$ where $0 < \varepsilon \leq \frac{1}{2}$. Then $V \in \Sigma$ satisfies the defining subclass conditions of $\hat{\mathcal{T}}_\Sigma(t,\mu,\kappa)$. Moreover this function satisfies the coefficient estimates, the Fekete-Szegö inequality and the second Hankel determinant bounds obtained in Theorem 2.1 -2.3 and their corresponding corollaries.*

Verification. For the function $V(z) = z + \varepsilon z^2$ with $0 < \varepsilon \leq \frac{1}{2}$ we have $b_2 = \varepsilon$ and $b_3 = b_4 = 0$.
Since all the right- hand side in the coefficient estimates obtained in Theorem 2.1 and Corollaries 2.1-2.3 are positive for $\kappa \geq 0$ and $\mu \geq 0$, it follows immediately that the inequalities for b_2, b_3 and b_4 are satisfied.
Further for any $\alpha \in \mathbb{R}$

$$|b_3 - \alpha b_2^2| = |\alpha|\varepsilon^2,$$

which satisfies the Fekete - Szegö inequality given in Theorem 2.2 and Corollaries 2.4 -2.6 for admissible parameter values.
Finally the second Hankel determinant satisfies

$$|b_2b_4 - b_3^2| = 0,$$

which trivially satisfies the bound obtained in Theorem 2.3. Hence the chosen example satisfies the inequalities derived. □

3. Conclusion

We introduced a subclass of bi-univalent functions defined using the error function subordinated to a limaçon domain. Coefficient estimates for the initial Taylor coefficients were obtained, along with bounds for the Fekete- Szegö functional and the second Hankel determinant. These results generalize several existing results. As a future work, one may consider to extend these methods to higher order Hankel determinant and corresponding q- analogues.

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