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Essential Conditions for Subclasses of Spiral-Like Functions and Their Univalent Derivatives associated with Poisson distribution series

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ABSTRACT. In this article, we investigate the necessary and sufficient conditions for specific functions that involve Poisson distribution series to be in few subclasses of analytic functions, where both h and h' are univalent in the open unit disc $\mathbb{E}$. We also consider an integral operator associated with Poisson distribution series and discuss several mapping properties of integral operator. Furthermore, we point out certain corollaries and consequences of the main results. In addition, we determine the necessary conditions for specific subclasses of analytic functions linked to Poisson distribution series to belong to subclasses of spiral-like univalent functions.

1. OVERVIEW AND KEY CONCEPTS

We define $\mathcal{A}$ as the class of analytic functions $h(z)$ that are defined on the open unit disk, $\mathbb{E} := \{z : |z| < 1\}$, which takes the normalized form:

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n. \tag{1.1}$$

Additionally, we use $\mathcal{S}$ to refer to the family of univalent functions (i.e., analytic and injective) within $\mathcal{A}$. Geometric Function Theory focuses on the geometric characteristics of functions that belong to $\mathcal{S}$ or a specific subset of it. Let $\mathcal{S}_1$ illustrate the subfamily of $\mathcal{S}$ that encompasses functions h for which both h and its derivative h' are univalent in $\mathbb{E}$. A function $h(z)$ expressed in the form (1.1) belongs to $\mathcal{S}_u$ if both h and its first u derivatives are univalent in $\mathbb{E}$. For any function $h \in \mathcal{A}$ presented in the form (1.1), and if $f \in \mathcal{A}$ is given by

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n,$$

we define the Hadamard product (or Convolution) of h and f as

$$(h * f)(z) := z + \sum_{n=2}^{\infty} h_n f_n z^n, \quad z \in \mathbb{E}.$$

Let $\mathcal{S}^*(\eta)$ and $\mathcal{K}(\eta)$ be defined as the subclasses of $\mathcal{S}$ that contain functions which are starlike of order η and convex of order η, under the condition that $0 \leq \eta < 1$. The analytic characterizations of these two classes are given by:

$$\mathcal{S}^*(\eta) := \left\{ h \in \mathcal{A} : \Re\left(\frac{zh'(z)}{h(z)}\right) > \eta \right\}$$

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and

$$\mathcal{K}(\eta) := \left\{ h \in \mathcal{A} : \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > \eta \right\}.$$

Additionally, $\mathcal{K} \equiv \mathcal{K}(0)$ and $\mathcal{S}^* \equiv \mathcal{S}^*(0)$, the well-recognized standard class of convex and starlike functions. It is a confirmed fact that

$$zh'(z) \in \mathcal{S}^*(\eta) \iff h \in \mathcal{K}(\eta).$$

Let $\mathcal{T}$ signify the class of all functions characterized by

$$h(z) = z - \sum_{n=2}^{\infty} h_n z^n, \quad h_n \geq 0, \tag{1.2}$$

normalized according to the conditions $h(0) = h'(0) - 1 = 0$, which are analytic in $\mathbb{E}$. The subclasses of $\mathcal{T}$ are represented by $\mathcal{T}^*(\eta)$ and $\mathcal{C}(\eta)$, which correspond to starlike of order η and convex of order η, respectively. Silverman [23] explored functions in the classes $\mathcal{T}^*(\eta) = \mathcal{T} \cap \mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta) = \mathcal{T} \cap \mathcal{K}(\eta)$. Moreover, let $\mathcal{T}_1$ denote the subfamily of $\mathcal{T}$ comprising functions h such that both h and its derivative h' are univalent in $\mathbb{E}$. It is obvious that the second coefficient of a function in $\mathcal{T}_1$ cannot be zero. Thus, the class $\mathcal{T}_1$ is non-empty since the function $z - \frac{z^2}{2}$ is included in $\mathcal{T}_1$. A function $h(z)$ presented in the form (1.1) is considered to be in $\mathcal{T}_n$ if h and its first n derivatives are univalent in $\mathbb{E}$. If h belongs to $\mathcal{T}_n$, then it is said to be in the class $\mathcal{T}_\infty$, which represents the limiting or extremal class obtained as the parameter n tends to infinity. In this sense, $\mathcal{T}_\infty$ collects all functions of the class $\mathcal{T}$ that satisfy the defining condition of $\mathcal{T}_n$, for every admissible value of n, or equivalently, that remain invariant under the strongest form of the imposed constraint. Thus, membership of h in $\mathcal{T}_n$, implies that h also belongs to the broader class $\mathcal{T}_\infty$.

A function $h \in \mathcal{A}$ belongs to the $\mathcal{UCV}$ class of uniformly convex functions within the unit disk $\mathbb{E}$ if it meets the criteria of being a normalized convex function in $\mathbb{E}$. Additionally, it has the property that for any circular arc Ω located in $\mathbb{E}$, with its center ζ also within $\mathbb{E}$, the resulting image curve $h(\Omega)$ is a convex arc. The idea of uniformly convex functions, referred to as $\mathcal{UCV}$, was first presented by Goodman [8]. Rønning [21] demonstrated that a function $h(z)$ expressed in the format (1.1) belongs to $\mathcal{UCV}$ if and only if

$$\Re\left(1 + (z - \zeta)\frac{h''(z)}{h'(z)}\right) \geq 0, \quad (z, \zeta) \in \mathbb{E} \times \mathbb{E}.$$

He further introduced the concept of uniformly starlike functions, and the analytic criteria is expressed as follows: $h \in \mathcal{UST}$ if and only if

$$\Re\left(\frac{h(\zeta) - h(z)}{(\zeta - z)h'(z)}\right) \geq 0, \quad (z, \zeta) \in \mathbb{E} \times \mathbb{E}.$$

As a further point, we identify two essential subclasses of $\mathcal{S}$, known as $\varkappa - \mathcal{UCV}$ and $\varkappa - \mathcal{ST}$ which comprise functions that are $\varkappa-$uniformly convex and $\varkappa - starlike$ in $\mathbb{E}$, respectively. The analytic representations of these two classes are given as

$$\varkappa - \mathcal{UCV} := \left\{ h \in \mathcal{S} : \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > \varkappa \left|\frac{zh'(z)}{h(z)}\right|, \ (z \in \mathbb{E} \ and \ 0 \leq \varkappa < \infty) \right\}$$

and

$$\varkappa - \mathcal{ST} := \left\{ h \in \mathcal{S} : \Re\left(\frac{zh'(z)}{h(z)}\right) > \varkappa \left|\frac{zh'(z)}{h(z)} - 1\right|, \ (z \in \mathbb{E} \ and \ 0 \leq \varkappa < \infty) \right\}.$$

The $\varkappa - \mathcal{UCV}$ class was introduced by Kanas and Wiśniowska [12], where its geometric definition and its connections to conic domains were examined. The $\varkappa - \mathcal{ST}$ class was explored in [11]. Indeed, it is associated with the $\varkappa - \mathcal{UCV}$ class through the well-known Alexander equivalence that links the standard classes of convex and starlike functions; for additional developments concerning both the $\varkappa - \mathcal{UCV}$ and $\varkappa - \mathcal{ST}$ classes, refer to the work of Kanas and Srivastava [10]. Specifically, when $\varkappa = 1$, we derive

$$1 - \mathcal{UCV} \equiv \mathcal{UCV} \quad and \quad 1 - \mathcal{ST} \equiv \mathcal{SP},$$

where $\mathcal{UCV}$ and $\mathcal{SP}$ denote the well-known classes of uniformly convex functions and parabolic starlike functions in $\mathbb{E}$, respectively (see details in [8, 9, 21, 25, 26]). Furthermore, utilizing a specific fractional calculus operator, Srivastava and Mishra [28] conducted a comprehensive and unified analysis of the $\mathcal{UCV}$ and $\mathcal{SP}$ classes. A function h belonging to the class $\mathcal{A}$ is said to be spiral-like if

$$\Re\left(e^{-i\alpha}\frac{zh'(z)}{h(z)}\right) > 0, \quad z \in \mathbb{E},$$

for some α in $\mathbb{C}$ such that $|\alpha| < \dfrac{\pi}{2}$. The concept of spiral-like functions was first introduced in [27]. Furthermore, a function h is defined as convex spiral-like if $zh'(z)$ exhibits spiral-like properties. According to Murugusundramoorthy [15, 16], we examine the following subclasses of spiral-like functions as outlined below:

For $0 \le \delta < 1$, $0 \le \eta < 1$, and $|\alpha| < \dfrac{\pi}{2}$, we define the class $\mathcal{S}(\alpha, \eta, \delta)$ as follows:

$$\mathcal{S}(\alpha, \eta, \delta) := \left\{ h \in \mathcal{A} : \Re\left(e^{i\alpha}\frac{zh'(z)}{(1-\delta)h(z) + \delta zh'(z)}\right) > \eta\cos\alpha, \quad z \in \mathbb{E} \right\}.$$

Based on Alexander's relation (refer to [7]), we introduce the following subclass $\mathcal{K}(\alpha, \eta, \delta)$. For $0 \le \delta < 1$, $0 \le \eta < 1$, and $|\alpha| < \dfrac{\pi}{2}$, we define the class $\mathcal{K}(\alpha, \eta, \delta)$ as follows:

$$\mathcal{K}(\alpha, \eta, \delta) := \left\{ h \in \mathcal{A} : \Re\left(e^{i\alpha}\frac{h'(z) + zh''(z)}{\delta zh''(z) + h'(z)}\right) > \eta\cos\alpha, \quad z \in \mathbb{E} \right\}.$$

The class $\mathcal{M}(\beta)$ consists of starlike functions where $1 < \beta \le \dfrac{4}{3}$,

$$\mathcal{M}(\beta) := \left\{ h \in \mathcal{A} : \Re\left(\frac{zh'(z)}{h(z)}\right) < \beta, \quad z \in \mathbb{E} \right\}$$

and the class $\mathcal{N}(\beta)$ consists of convex functions where $1 < \beta \le \dfrac{4}{3}$,

$$\mathcal{N}(\beta) := \left\{ h \in \mathcal{A} : \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) < \beta, \quad z \in \mathbb{E} \right\},$$

was introduced by Uralegaddi et al. [29] (refer to [5, 6]). Additionally, let $\mathcal{M}^*(\beta)$ be defined as $\mathcal{M}(\beta) \cap \mathcal{T}$ and $\mathcal{N}^*(\beta)$ defined as $\mathcal{N}(\beta) \cap \mathcal{T}$. We present two new subclasses of $\mathcal{S}$, specifically $\mathcal{M}(\xi, \beta)$ and $\mathcal{N}(\xi, \beta)$, to explore certain inclusion properties.

For specific values of β $\left(1 < \beta \le \dfrac{4}{3}\right)$ and ξ $(0 \le \xi < 1)$, we consider $\mathcal{M}(\xi, \beta)$ and $\mathcal{N}(\xi, \beta)$, as two new subclasses of $\mathcal{S}$ that include functions of the form (1.1) which meet the analytic criteria

$$\mathcal{M}(\xi, \beta) := \left\{ h \in \mathcal{A} : \Re\left(\frac{zh'(z)}{(1-\xi)h(z) + \xi zh'(z)}\right) < \beta, \quad z \in \mathbb{E} \right\}.$$

and

$$\mathcal{N}(\xi, \beta) := \left\{ h \in \mathcal{A} : \Re\left(\frac{h'(z) + zh''(z)}{\xi zh''(z) + h'(z)}\right) < \beta, \quad z \in \mathbb{E} \right\}.$$

Note that $\mathcal{M}^*(\xi,\beta) = \mathcal{M}(\xi,\beta) \cap \mathcal{T}$ and $\mathcal{N}^*(\xi,\beta) = \mathcal{N}(\xi,\beta) \cap \mathcal{T}$. It is important to observe that $\mathcal{M}^*(0,\beta) \equiv \mathcal{M}^*(\beta)$ and $\mathcal{N}^*(0,\beta) \equiv \mathcal{N}^*(\beta)$; $\mathcal{M}^*(\beta)$ and $\mathcal{N}^*(\beta)$ represent the subclasses examined by Uralegaddi et al. [29].

1.1. **Poisson distribution series.** It is widely recognized that special functions (series) are crucial in the field of geometric function theory, particularly in de Branges' proof of the renowned Bieberbach conjecture. The unexpected application of special functions (hypergeometric functions) has sparked a resurgence of interest in function theory over the past few decades. There exists a vast body of literature addressing the geometric characteristics of various types of special functions, notably the generalized Gaussian hypergeometric functions [4, 13, 14, 24], as well as the Bessel functions [1, 2, 3, 20].

A variable y is considered to follow a Poisson distribution if it can assume the values $0, 1, 2, 3, \dots$ with probabilities

$$e^{-k},\ k\frac{e^{-k}}{1!},\ k^2\frac{e^{-k}}{2!}, k^3\frac{e^{-k}}{3!}, \dots$$

respectively, where k is referred to as the parameter. Consequently

$$\mathcal{P}(y=m) = \frac{k^m e^{-k}}{m!}, \quad m = 0, 1, 2, 3, \dots.$$

Recently, Porwal [17, 19] introduced a power series whose coefficients represent probabilities of the Poisson distribution

$$\Psi(k,z) = z + \sum_{n=2}^{\infty} \frac{k^{n-1}}{(n-1)!} e^{-k} z^n, \quad z \in \mathbb{E}.$$

The ratio test indicates that the radius of convergence for the series is infinite. In [19], Porwal also defined the series

$$\Phi(k,z) := 2z - \Psi(k,z) = z - \sum_{n=2}^{\infty} \frac{k^{n-1}}{(n-1)!} e^{-k} z^n, \quad z \in \mathbb{E}.$$

Presently, we are looking to the linear operator

$$\mathcal{J}(k,z) : \mathcal{A} \longrightarrow \mathcal{A}$$

defined through the convolution or Hadamard product

$$\mathcal{J}(k,z)h = \Psi(k,z) * h(z) = z + \sum_{n=2}^{\infty} \frac{k^{n-1}}{(n-1)!} e^{-k} a_n z^n, \quad z \in \mathbb{E}.$$

For the purpose of convenience in the sections that follow, we will refer to the following identities

$$\sum_{n=2}^{\infty} \frac{k^{n-1}}{(n-1)!} = e^k - 1$$

$$\sum_{n=2}^{\infty} \frac{k^{n-1}}{(n-2)!} = ke^k$$

$$\sum_{n=2}^{\infty} \frac{k^{n-1}}{(n-3)!} = k^2 e^k$$

$$\sum_{n=2}^{\infty} \frac{k^{n-1}}{(n-4)!} = k^3 e^k.$$

We are now going to specify the sufficient conditions for the function h to be included in the classes discussed above.

Lemma 1.1. [22] *Assume a function* $h(z)$ *is expressed as in* (1.1) *where* $h_2 \neq 0$ *for* $n \geq 3$. *Then,* $h \in \mathcal{S}_1$ *if*

$$\sum_{n=3}^{\infty} n(n-1)|h_n| \leq 2|h_2|.$$

Lemma 1.2. [22] *Assume a function* $h(z)$ *is expressed as in* (1.1) *where* $h_2 > 0$. *Then,* $h \in \mathcal{T}_1$ *if*

$$\sum_{n=3}^{\infty} n(n-1)h_n \leq 2h_2.$$

Additionally, the condition mentioned is sufficient for $0 < h_2 \leq \frac{1}{3}$.

Lemma 1.3. [22] *Assume a function* $h(z)$ *is expressed as in* (1.1) *where* $\prod_{n=2}^{m+1} h_n \neq 0$. *Then,* $h \in \mathcal{T}_m$ *if*

$$\sum_{n=2+r}^{\infty} (n-r)(n-r+1)\dots nh_n \leq (n+1)!h_{n+1},$$

for $r = 1, 2, \dots, m$.

Lemma 1.4. [15, 16] *Assume a function* $h(z)$ *is expressed as in* (1.1). *Then,* $h \in \mathcal{S}(\alpha, \eta, \delta)$ *if*

$$\sum_{n=2}^{\infty} \left[(1-\delta)(n-1)\sec\alpha + (1-\eta)(1+n\delta-\delta)\right]|h_n| \leq (1-\eta),$$

where $0 \leq \delta < 1$, $0 \leq \eta < 1$, *and* $|\alpha| < \frac{\pi}{2}$.

Lemma 1.5. [15, 16] *Assume a function* $h(z)$ *is expressed as in* (1.1). *Then,* $h \in \mathcal{K}(\alpha, \eta, \delta)$ *if*

$$\sum_{n=2}^{\infty} n\left[(1-\delta)(n-1)\sec\alpha + (1-\eta)(1+n\delta-\delta)\right]|h_n| \leq (1-\eta),$$

where $0 \leq \delta < 1$, $0 \leq \eta < 1$, *and* $|\alpha| < \frac{\pi}{2}$.

Lemma 1.6. [18] *Assume a function* $h(z)$ *is expressed as in* (1.1). *Then,* $h \in \mathcal{M}^*(\xi, \beta)$ *if*

$$\sum_{n=2}^{\infty} \left[n - (1+n\xi-\xi)\beta\right]|h_n| \leq \beta - 1,$$

where $1 < \beta \leq \frac{4}{3}$ *and* $0 \leq \xi < 1$.

Lemma 1.7. [18] *Assume a function* $h(z)$ *is expressed as in* (1.1). *Then,* $h \in \mathcal{N}^*(\xi, \beta)$ *if*

$$\sum_{n=2}^{\infty} n\left[n - (1+n\xi-\xi)\beta\right]|h_n| \leq \beta - 1,$$

where $1 < \beta \leq \frac{4}{3}$ *and* $0 \leq \xi < 1$.

In this article, we investigate the necessary and sufficient conditions together with inclusion relations for certain functions involving Poisson distribution series to determine their membership in specific subclasses of analytic functions, where both h and h' are univalent in the open unit disk $\mathbb{E}$. We also consider an integral operator associated with Poisson distribution series and discuss several mapping properties. Furthermore, we point out certain corollaries and consequences of the main results. In addition, we determine the necessary conditions for specific subclasses of analytic functions linked to Poisson distribution series to belong to subclasses of spiral-like univalent functions.

2. Sufficient conditions for Poisson distribution series

Theorem 2.1. *A necessary condition for $\Psi(k,z)$ to belong to $\mathcal{S}_1$ is that inequality*

$$e^k(k+2) \le 4, \tag{2.3}$$

holds. If $0 < k \le \dfrac{e^k}{3}$, then condition (2.3) *is necessary and also sufficient for $\Phi(k,z)$ to be in $\mathcal{T}_1$.*

Proof. To establish that $\Psi(k,z)$ belongs to $\mathcal{S}_1$ as mentioned in Lemma 1.1, it is enough to prove that the inequality is fulfilled:

$$\sum_{n=3}^{\infty} n(n-1)\frac{k^{n-1}}{(n-1)!}e^{-k} \le 2ke^{-k}. \tag{2.4}$$

The left side of the inequality (2.4) could be denoted as

$$\begin{aligned}\mathcal{L}(k) :=& \sum_{n=3}^{\infty} n(n-1)\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& \sum_{n=3}^{\infty}[n^2-n]\frac{k^{n-1}}{(n-1)!}e^{-k}.\end{aligned}$$

When we express $n^2-n=(n-2)(n-1)+2(n-1)$, we find

$$\begin{aligned}\mathcal{L}(k) :=& \sum_{n=3}^{\infty}[n^2-n]\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& \sum_{n=3}^{\infty}[(n-2)(n-1)+2(n-1)]\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& \sum_{n=3}^{\infty}(n-2)(n-1)\frac{k^{n-1}}{(n-1)!}e^{-k} + 2\sum_{n=3}^{\infty}(n-1)\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& e^{-k}\sum_{n=3}^{\infty}\frac{k^{n-1}}{(n-3)!} + 2e^{-k}\sum_{n=3}^{\infty}\frac{k^{n-1}}{(n-2)!}\\ =& e^{-k}\left[\sum_{n=2}^{\infty}\frac{k^{n}}{(n-2)!} + 2\sum_{n=2}^{\infty}\frac{k^{n}}{(n-1)!}\right]\\ =& ke^{-k}\left[\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-2)!} + 2\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}\right]\\ =& ke^{-k}[ke^k+2e^k-2].\end{aligned}$$

Thus, we have

$$\mathcal{L}(k) = ke^{-k}[ke^k+2e^k-2]. \tag{2.5}$$

Note that, the value of expression (2.5) is bounded above by $2ke^{-k}$ provided that (2.3) is true. As stated in Lemma 1.2, condition (2.3) is enough for $\Phi(k,z)$ to be part of $\mathcal{T}_1$. This essentially completes the proof of Theorem 2.1. □

Theorem 2.2. *A necessary condition for $\Psi(k,z)$ to belong to $\mathcal{S}_2$ is that inequality*

$$e^k(k+3) \le 6, \tag{2.6}$$

holds. The necessity of condition (2.6) *is evident for $\Phi(k,z)$ to exist in $\mathcal{T}_2$.*

Proof. To establish that $\Psi(k,z)$ belongs to $\mathcal{T}_2$ as mentioned in Lemma 1.3, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=4}^{\infty} n(n-1)(n-2)\frac{k^{n-1}}{(n-1)!}e^{-k} \leq k^2 e^{-k}. \tag{2.7}$$

The left side of the inequality (2.7) could be denoted as

$$\begin{aligned}\mathcal{L}(k) :=& \sum_{n=4}^{\infty} n(n-1)(n-2)\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& \sum_{n=4}^{\infty}[n^3-3n^2+2n]\frac{k^{n-1}}{(n-1)!}e^{-k}.\end{aligned}$$

When we express $n^3-3n^2+2n = (n-3)(n-2)(n-1)+3(n-2)(n-1)$, we find

$$\begin{aligned}\mathcal{L}(k) :=& \sum_{n=4}^{\infty}[n^3-3n^2+2n]\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& \sum_{n=4}^{\infty}[(n-3)(n-2)(n-1)+3(n-2)(n-1)]\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& \sum_{n=4}^{\infty}(n-3)(n-2)(n-1)\frac{k^{n-1}}{(n-1)!}e^{-k} + 3\sum_{n=4}^{\infty}(n-2)(n-1)\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& e^{-k}\sum_{n=4}^{\infty}\frac{k^{n-1}}{(n-4)!} + 3e^{-k}\sum_{n=4}^{\infty}\frac{k^{n-1}}{(n-3)!}\\ =& e^{-k}\left[\sum_{n=2}^{\infty}\frac{k^{n+1}}{(n-2)!} + 3\sum_{n=2}^{\infty}\frac{k^{n+1}}{(n-1)!}\right]\\ =& k^2e^{-k}\left[\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-2)!} + 3k^2\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}\right]\\ =& k^2e^{-k}[ke^k+3e^k-3].\end{aligned}$$

Thus, we have

$$\mathcal{L}(k) = k^2e^{-k}[ke^k+3e^k-3]. \tag{2.8}$$

Note that, the value of expression (2.8) is bounded above by k^2e^{-k} provided that (2.6) is true. As stated in Lemma 1.3, condition (2.6) is enough for $\Phi(k,z)$ to be part of $\mathcal{T}_2$. This essentially completes the proof of Theorem 2.2. □

Theorem 2.3. *If $0 \leq \delta < 1$, $0 \leq \eta < 1$, and $|\alpha| < \frac{\pi}{2}$, then a sufficient condition for $\Psi(k,z)$ to belong to $\mathcal{S}(\alpha,\eta,\delta)$ is that inequality*

$$ke^k[(1-\delta)\sec\alpha + \delta(1-\eta)] + (1-\eta)(e^k-1) \leq e^k(1-\eta), \tag{2.9}$$

holds.

Proof. To establish that $\Psi(k,z)$ belongs to $\mathcal{S}(\alpha,\eta,\delta)$ as mentioned in Lemma 1.4, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=2}^{\infty}[(1-\delta)(n-1)\sec\alpha + (1-\eta)(1+n\delta-\delta)]\frac{k^{n-1}}{(n-1)!}e^{-k} \leq (1-\eta). \tag{2.10}$$

The left side of the inequality (2.10) could be denoted as

$$\begin{aligned}
\mathcal{L}(k,\alpha,\delta,\eta) :=& \sum_{n=2}^{\infty} \left[(1-\delta)(n-1)\sec\alpha + (1-\eta)(1+n\delta-\delta)\right] \frac{k^{n-1}}{(n-1)!}e^{-k} \\
=& \sum_{n=2}^{\infty} \left[((1-\delta)\sec\alpha + \delta(1-\eta))(n-1) + (1-\eta)\right] \frac{k^{n-1}}{(n-1)!}e^{-k} \\
=& [(1-\delta)\sec\alpha + \delta(1-\eta)] \sum_{n=2}^{\infty}(n-1)\frac{k^{n-1}}{(n-1)!}e^{-k} + (1-\eta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}e^{-k} \\
=& [(1-\delta)\sec\alpha + \delta(1-\eta)]e^{-k}\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-2)!} + (1-\eta)e^{-k}\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!} \\
=& [(1-\delta)\sec\alpha + \delta(1-\eta)]e^{-k} \times ke^{k} + (1-\eta)e^{-k}(e^{k}-1).
\end{aligned}$$

Thus, we have

$$\mathcal{L}(k,\alpha,\delta,\eta) = k[(1-\delta)\sec\alpha + \delta(1-\eta)] + (1-\eta)e^{-k}(e^{k}-1). \tag{2.11}$$

Note that, the value of expression (2.11) is bounded above by $1-\eta$ provided that (2.9) is true. As stated in Lemma 1.4, condition (2.9) is enough for $\Psi(k,z)$ to be part of $\mathcal{S}(\alpha,\eta,\delta)$. This essentially completes the proof of Theorem 2.3. □

By establishing $\delta = 0$, we can refine the assertion of Theorem 2.3 as detailed below.

Corollary 2.1. *If $0 \le \eta < 1$, and $|\alpha| < \frac{\pi}{2}$, then a sufficient condition for $\Psi(k,z)$ to belong to $\mathcal{S}(\alpha,\eta)$ is that inequality*

$$ke^{k}\sec\alpha + (1-\eta)(e^{k}-1) \le e^{k}(1-\eta),$$

holds.

Theorem 2.4. *If $0 \le \delta < 1$, $0 \le \eta < 1$, and $|\alpha| < \frac{\pi}{2}$, then a sufficient condition for $\Psi(k,z)$ to belong to $\mathcal{K}(\alpha,\eta,\delta)$ is that inequality*

$$k^2e^{k}[(1-\delta)\sec\alpha+\delta(1-\eta)]+[2(1-\eta)+(1-\eta)(2\delta+1)]ke^{k}+(1-\eta)(e^{k}-1) \le e^{k}(1-\eta), \tag{2.12}$$

holds.

Proof. To establish that $\Psi(k,z)$ belongs to $\mathcal{K}(\alpha,\eta,\delta)$ as mentioned in Lemma 1.5, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=2}^{\infty} n\left[(1-\delta)(n-1)\sec\alpha + (1-\eta)(1+n\delta-\delta)\right]\frac{k^{n-1}}{(n-1)!}e^{-k} \le (1-\eta). \tag{2.13}$$

The left side of the inequality (2.13) could be denoted as

$$\begin{aligned}
\mathcal{L}(k,\alpha,\delta,\eta) :=& \sum_{n=2}^{\infty} n\left[(1-\delta)(n-1)\sec\alpha + (1-\eta)(1+n\delta-\delta)\right]\frac{k^{n-1}}{(n-1)!}e^{-k} \\
=& \sum_{n=2}^{\infty}[(1-\delta)\sec\alpha + \delta(1-\eta)](n^2-n)\frac{k^{n-1}}{(n-1)!}e^{-k} + \sum_{n=2}^{\infty}(1-\eta)n\frac{k^{n-1}}{(n-1)!}e^{-k}.
\end{aligned}$$

When we express $n^2 - n = (n-2)(n-1) + 2(n-1)$ and $n = (n-1) + 1$, we find

$$\mathcal{L}(k,\alpha,\delta,\eta) := \sum_{n=2}^{\infty}[(1-\delta)\sec\alpha + \delta(1-\eta)](n^2-n)\frac{k^{n-1}}{(n-1)!}e^{-k} + \sum_{n=2}^{\infty}(1-\eta)n\frac{k^{n-1}}{(n-1)!}e^{-k}$$

$$= [(1-\delta)\sec\alpha + \delta(1-\eta)]\sum_{n=2}^{\infty}[(n-2)(n-1) + 2(n-1)]\frac{k^{n-1}}{(n-1)!}e^{-k}$$

$$+ (1-\eta)\sum_{n=2}^{\infty}[(n-1)+1]\frac{k^{n-1}}{(n-1)!}e^{-k}$$

$$= [(1-\delta)\sec\alpha + \delta(1-\eta)]\sum_{n=2}^{\infty}(n-2)(n-1)\frac{k^{n-1}}{(n-1)!}e^{-k}$$

$$+ 2[(1-\delta)\sec\alpha + \delta(1-\eta)]\sum_{n=2}^{\infty}(n-1)\frac{k^{n-1}}{(n-1)!}e^{-k}$$

$$+ (1-\eta)\sum_{n=2}^{\infty}[(n-1)+1]\frac{k^{n-1}}{(n-1)!}e^{-k}$$

$$= e^{-k}[(1-\delta)\sec\alpha + \delta(1-\eta)]\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-3)!}$$

$$+ e^{-k}[2(1-\delta)\sec\alpha + (2\delta+1)(1-\eta)]\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-2)!} + e^{-k}(1-\eta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}$$

$$= k^2[(1-\delta)\sec\alpha + \delta(1-\eta)] + k[2(1-\delta)\sec\alpha + (2\delta+1)(1-\eta)] + (1-\eta)(1-e^k).$$

Thus, we have

$$\mathcal{L}(k,\alpha,\delta,\eta) = k^2[(1-\delta)\sec\alpha + \delta(1-\eta)] + k[2(1-\delta)\sec\alpha + (2\delta+1)(1-\eta)] + (1-\eta)(1-e^k). \tag{2.14}$$

Note that, the value of expression (2.14) is bounded above by $1-\eta$ provided that (2.12) is true. As stated in Lemma 1.5, condition (2.12) is enough for $\Psi(k,z)$ to be part of $\mathcal{K}(\alpha,\eta,\delta)$. This essentially completes the proof of Theorem 2.4. □

By establishing $\delta = 0$, we can refine the assertion of Theorem 2.4 as detailed below.

Corollary 2.2. *If $0 \le \eta < 1$, and $|\alpha| < \frac{\pi}{2}$, then a sufficient condition for $\Psi(k,z)$ to belong to $\mathcal{K}(\alpha,\eta)$ is that inequality*

$$k^2e^k\sec\alpha + 3(1-\eta)ke^k + (1-\eta)(e^k-1) \le e^k(1-\eta),$$

holds.

Theorem 2.5. *If $1 < \beta \le \frac{4}{3}$ and $0 \le \xi < 1$, then a sufficient condition for $\Psi(k,z)$ to belong to $\mathcal{M}^*(\xi,\beta)$ is that inequality*

$$k(1-\xi\beta) + (1-\beta)(1-e^{-k}) \le \beta - 1, \tag{2.15}$$

holds.

Proof. To establish that $\Psi(k,z)$ belongs to $\mathcal{M}^*(\xi,\beta)$ as mentioned in Lemma 1.6, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=2}^{\infty}[n - (1 + n\xi - \xi)\beta]\frac{k^{n-1}}{(n-1)!}e^{-k} \le \beta - 1. \tag{2.16}$$

The left side of the inequality (2.16) could be denoted as

$$\begin{aligned}\mathcal{L}(k,\xi,\beta) :=& \sum_{n=2}^{\infty} [n-(1+n\xi-\xi)\beta]\,\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& \sum_{n=2}^{\infty} [(n-1)(1-\xi\beta)+(1-\beta)]\,\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& \sum_{n=2}^{\infty}(n-1)(1-\xi\beta)\frac{k^{n-1}}{(n-1)!}e^{-k}+\sum_{n=2}^{\infty}(1-\beta)\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& e^{-k}(1-\xi\beta)\sum_{n=2}^{\infty}(n-1)\frac{k^{n-1}}{(n-1)!}+e^{-k}(1-\beta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}\\ =& e^{-k}(1-\xi\beta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-2)!}+e^{-k}(1-\beta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}\\ =& e^{-k}(1-\xi\beta)k^2e^k+e^{-k}(1-\beta)(e^k-1).\end{aligned}$$

Thus, we have

$$\mathcal{L}(k,\xi,\beta)=k(1-\xi\beta)+(1-\beta)(1-e^{-k}). \tag{2.17}$$

Note that, the value of expression (2.17) is bounded above by $\beta-1$ provided that (2.15) is true. As stated in Lemma 1.6, condition (2.15) is enough for $\Psi(k,z)$ to be part of $\mathcal{M}^*(\xi,\beta)$. This essentially completes the proof of Theorem 2.5. □

By establishing $\xi=0$, we can refine the assertion of Theorem 2.5 as detailed below.

Corollary 2.3. *If $1<\beta\leq\frac{4}{3}$, then a sufficient condition for $\Psi(k,z)$ to belong to $\mathcal{M}^*(\beta)$ is that inequality*

$$k+(1-\beta)(1-e^{-k})\leq\beta-1,$$

holds.

Theorem 2.6. *If $1<\beta\leq\frac{4}{3}$ and $0\leq\xi<1$, then a sufficient condition for $\Psi(k,z)$ to belong to $\mathcal{N}^*(\xi,\beta)$ is that inequality*

$$k^2(1-\xi\beta)+k(3-2\xi\beta-\beta)+(1-\beta)(1-e^{-k})\leq\beta-1, \tag{2.18}$$

holds.

Proof. To establish that $\Psi(k,z)$ belongs to $\mathcal{N}^*(\xi,\beta)$ as mentioned in Lemma 1.6, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=2}^{\infty} n\,[n-(1+n\xi-\xi)\beta]\,\frac{k^{n-1}}{(n-1)!}e^{-k}\leq\beta-1. \tag{2.19}$$

The left side of the inequality (2.19) could be denoted as

$$\begin{aligned}\mathcal{L}(k,\xi,\beta) :=& \sum_{n=2}^{\infty} [n-(1+n\xi-\xi)\beta]\,\frac{k^{n-1}}{(n-1)!}e^{-k}\\ =& \sum_{n=2}^{\infty}\left[(n^2-n)(1-\xi\beta)+n(1-\beta)\right]\frac{k^{n-1}}{(n-1)!}e^{-k}.\end{aligned}$$

When we express $n^2-n=(n-2)(n-1)+2(n-1)$ and $n=(n-1)+1$, we find

$$\begin{aligned}
\mathcal{L}(k,\xi,\beta) :=& \sum_{n=2}^{\infty}\left[(n^2-n)(1-\xi\beta)+n(1-\beta)\right]\frac{k^{n-1}}{(n-1)!}e^{-k}\\
=& \sum_{n=2}^{\infty}\left[(n-2)(n-1)(1-\xi\beta)+(n-1)(3-2\xi\beta-\beta)+(1-\beta)\right]\frac{k^{n-1}}{(n-1)!}e^{-k}\\
=& \sum_{n=2}^{\infty}(n-2)(n-1)(1-\xi\beta)\frac{k^{n-1}}{(n-1)!}e^{-k}+\sum_{n=2}^{\infty}(n-1)(3-2\xi\beta-\beta)\frac{k^{n-1}}{(n-1)!}e^{-k}\\
&+\sum_{n=2}^{\infty}(1-\beta)\frac{k^{n-1}}{(n-1)!}e^{-k}\\
=& e^{-k}(1-\xi\beta)\sum_{n=2}^{\infty}(n-2)(n-1)\frac{k^{n-1}}{(n-1)!}+e^{-k}(3-2\xi\beta-\beta)\sum_{n=2}^{\infty}(n-1)\frac{k^{n-1}}{(n-1)!}\\
&+e^{-k}(1-\beta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}\\
=& e^{-k}(1-\xi\beta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-3)!}+e^{-k}(3-2\xi\beta-\beta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-2)!}+e^{-k}(1-\beta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}\\
=& e^{-k}(1-\xi\beta)k^2e^k+e^{-k}(3-2\xi\beta-\beta)ke^k+e^{-k}(1-\beta)(e^k-1).
\end{aligned}$$

Thus, we have

$$\mathcal{L}(k,\xi,\beta)=k^2(1-\xi\beta)+k(3-2\xi\beta-\beta)+(1-\beta)(1-e^{-k}). \tag{2.20}$$

Note that, the value of expression (2.20) is bounded above by $\beta-1$ provided that (2.18) is true. As stated in Lemma 1.7, condition (2.18) is enough for $\Psi(k,z)$ to be part of $\mathcal{N}^*(\xi,\beta)$. This essentially completes the proof of Theorem 2.6. □

By establishing $\xi=0$, we can refine the assertion of Theorem 2.6 as detailed below.

Corollary 2.4. *If* $1<\beta\leq\frac{4}{3}$, *then a sufficient condition for* $\Psi(k,z)$ *to belong to* $\mathcal{N}^*(\beta)$ *is that inequality*

$$k^2+k(3-\beta)+(1-\beta)(1-e^{-k})\leq\beta-1,$$

holds.

3. Sufficient conditions for Integral Form of Poisson distribution series

Let us examine the Integral operators $\mathcal{G}(k,z)$ and $\mathcal{G}_1(k,z)$ defined as follows:

$$\mathcal{G}(k,z):=\int_0^z\frac{\Psi(k,z)}{z}dz=z+\sum_{n=2}^{\infty}\frac{k^{n-1}}{n!}e^{-k}z^n,\quad z\in\mathbb{E}$$

and

$$\mathcal{G}_1(k,z):=\int_0^z\frac{\Phi(k,z)}{z}dz=z-\sum_{n=2}^{\infty}\frac{k^{n-1}}{n!}e^{-k}z^n,\quad z\in\mathbb{E}.$$

Theorem 3.7. *A necessary condition for* $\mathcal{G}(k,z)$ *to belong to* $\mathcal{S}_1$ *is that inequality*

$$e^k\leq 4, \tag{3.21}$$

holds. If $0<k\leq\frac{2e^k}{3}$, *then condition* (3.21) *is necessary and also sufficient for* $\mathcal{G}_1(k,z)$ *to be in* $\mathcal{T}_1$.

Proof. To establish that $\mathcal{G}(k, z)$ belongs to $\mathcal{S}_1$ as mentioned in Lemma 1.1, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=3}^{\infty} n(n-1)\frac{k^{n-1}}{n!}e^{-k} \leq ke^{-k}. \tag{3.22}$$

The left side of the inequality (3.22) could be denoted as

$$\begin{aligned}
\mathcal{L}(k) :=& \sum_{n=3}^{\infty} n(n-1)\frac{k^{n-1}}{n!}e^{-k} \\
=& \sum_{n=3}^{\infty} \frac{k^{n-1}}{(n-2)!}e^{-k} \\
=& \sum_{n=2}^{\infty} \frac{k^{n}}{(n-1)!}e^{-k} \\
=& ke^{-k}\sum_{n=2}^{\infty} \frac{k^{n-1}}{(n-1)!} \\
=& ke^{-k}(e^k - 1).
\end{aligned}$$

Thus, we have

$$\mathcal{L}(k) = ke^{-k}(e^k - 1). \tag{3.23}$$

Note that, the value of expression (3.23) is bounded above by $2ke^{-k}$ provided that (3.21) is true. As stated in Lemma 1.2, condition (3.21) is enough for $\mathcal{G}_1(k, z)$ to be part of $\mathcal{T}_1$. This essentially completes the proof of Theorem 3.7. □

Theorem 3.8. *A necessary condition for $\mathcal{G}(k, z)$ to belong to $\mathcal{S}_2$ is that inequality*

$$e^k \leq 2, \tag{3.24}$$

holds. The necessity of condition (3.24) *is evident for $\mathcal{G}_1(k, z)$ to exist in $\mathcal{T}_2$.*

Proof. To establish that $\mathcal{G}(k, z)$ belongs to $\mathcal{T}_2$ as mentioned in Lemma 1.3, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=4}^{\infty} n(n-1)(n-2)\frac{k^{n-1}}{n!}e^{-k} \leq k^2e^{-k}. \tag{3.25}$$

The left side of the inequality (3.25) could be denoted as

$$\begin{aligned}\mathcal{L}(k) :=& \sum_{n=4}^{\infty} n(n-1)(n-2)\frac{k^{n-1}}{n!}e^{-k} \\ =& \sum_{n=4}^{\infty} \frac{k^{n-1}}{(n-3)!}e^{-k} \\ =& e^{-k}\sum_{n=2}^{\infty} \frac{k^{n+1}}{(n-1)!} \\ =& k^2e^{-k}\sum_{n=2}^{\infty} \frac{k^{n-1}}{(n-1)!} \\ =& k^2e^{-k}(e^k-1).\end{aligned}$$

Thus, we have

$$\mathcal{L}(k) = k^2e^{-k}(e^k-1). \tag{3.26}$$

Note that, the value of expression (3.26) is bounded above by k^2e^{-k} provided that (3.24) is true. As stated in Lemma 1.3, condition (3.24) is enough for $\mathcal{G}_1(k,z)$ to be part of $\mathcal{T}_2$. This essentially completes the proof of Theorem 3.8. $\square$

Theorem 3.9. *If $0 \le \delta < 1$, $0 \le \eta < 1$, and $|\alpha| < \frac{\pi}{2}$, then a sufficient condition for $\mathcal{G}(k,z)$ to belong to $\mathcal{S}(\alpha,\eta,\delta)$ is that inequality*

$$[(1-\delta)\sec\alpha+\delta(1-\eta)](1-e^{-k})+(1-\delta)(1-\eta-\sec\alpha)\int_0^1\left(\frac{\Psi(k,z)}{z}-1\right)dz \le (1-\eta), \tag{3.27}$$

holds.

Proof. To establish that $\mathcal{G}(k,z)$ belongs to $\mathcal{S}(\alpha,\eta,\delta)$ as mentioned in Lemma 1.4, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=2}^{\infty}[(1-\delta)(n-1)\sec\alpha+(1-\eta)(1+n\delta-\delta)]\frac{k^{n-1}}{n!}e^{-k} \le (1-\eta). \tag{3.28}$$

The left side of the inequality (3.28) could be denoted as

$$\begin{aligned}\mathcal{L}(k,\alpha,\delta,\eta) :=& \sum_{n=2}^{\infty} \left[(1-\delta)(n-1)\sec\alpha + (1-\eta)(1+n\delta-\delta)\right] \frac{k^{n-1}}{n!}e^{-k} \\ =& \sum_{n=2}^{\infty} n\left[(1-\delta)\sec\alpha + \delta(1-\eta)\right] \frac{k^{n-1}}{n!}e^{-k} \\ &+(1-\delta)(1-\eta-\sec\alpha)\sum_{n=2}^{\infty}\frac{k^{n-1}}{n!}e^{-k} \\ =& e^{-k}\left[(1-\delta)\sec\alpha + \delta(1-\eta)\right]\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!} \\ &+e^{-k}(1-\delta)(1-\eta-\sec\alpha)\sum_{n=2}^{\infty}\frac{k^{n-1}}{n!} \\ =& [(1-\delta)\sec\alpha+\delta(1-\eta)](1-e^{-k}) + (1-\eta) \\ &+(1-\delta)(1-\eta-\sec\alpha)\int_0^1\left(\frac{\Psi(k,z)}{z}-1\right)dz.\end{aligned}$$

Thus, we have

(3.29)

$$\mathcal{L}(k,\alpha,\delta,\eta) = [(1-\delta)\sec\alpha+\delta(1-\eta)](1-e^{-k})+(1-\delta)(1-\eta-\sec\alpha)\int_0^1\left(\frac{\Psi(k,z)}{z}-1\right)dz.$$

Note that, the value of expression (3.29) is bounded above by $1-\eta$ provided that (3.27) is true. As stated in Lemma 1.4, condition (3.27) is enough for $\mathcal{G}(k,z)$ to be part of $\mathcal{S}(\alpha,\eta,\delta)$. This essentially completes the proof of Theorem 3.9. □

By establishing $\delta = 0$, we can refine the assertion of Theorem 3.9 as detailed below.

Corollary 3.5. *If* $0 \le \eta < 1$, *and* $|\alpha| < \frac{\pi}{2}$, *then a sufficient condition for* $\mathcal{G}(k,z)$ *to belong to* $\mathcal{S}(\alpha,\eta)$ *is that inequality*

$$\sec\alpha(1-e^{-k}) + (1-\eta) + (1-\eta-\sec\alpha)\int_0^1\left(\frac{\Psi(k,z)}{z}-1\right)dz \le (1-\eta),$$

holds.

Theorem 3.10. *If* $0 \le \delta < 1$, $0 \le \eta < 1$, *and* $|\alpha| < \frac{\pi}{2}$, *then a sufficient condition for* $\mathcal{G}(k,z)$ *to belong to* $\mathcal{K}(\alpha,\eta,\delta)$ *is that inequality*

$$k(1-\delta)\sec\alpha + k\delta(1-\eta) + (1-\eta)(1-e^{-k}) \le (1-\eta), \tag{3.30}$$

holds.

Proof. To establish that $\mathcal{G}(k,z)$ belongs to $\mathcal{K}(\alpha,\eta,\delta)$ as mentioned in Lemma 1.4, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=2}^{\infty} n\left[(1-\delta)(n-1)\sec\alpha + (1-\eta)(1+n\delta-\delta)\right]\frac{k^{n-1}}{n!}e^{-k} \le (1-\eta). \tag{3.31}$$

The left side of the inequality (2.13) could be denoted as

$$\begin{aligned}\mathcal{L}(k,\alpha,\delta,\eta) :=& \sum_{n=2}^{\infty} n\left[(1-\delta)(n-1)\sec\alpha + (1-\eta)(1+n\delta-\delta)\right]\frac{k^{n-1}}{n!}e^{-k}\\ =&\sum_{n=2}^{\infty}[(1-\delta)\sec\alpha+\delta(1-\eta)]n(n-1)\frac{k^{n-1}}{n!}e^{-k}+\sum_{n=2}^{\infty}(1-\eta)n\frac{k^{n-1}}{n!}e^{-k}\\ =&[(1-\delta)\sec\alpha+\delta(1-\eta)]\sum_{n=2}^{\infty}n(n-1)\frac{k^{n-1}}{n!}+(1-\eta)e^{-k}\sum_{n=2}^{\infty}n\frac{k^{n-1}}{n!}\\ =&[(1-\delta)\sec\alpha+\delta(1-\eta)]\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-2)!}+(1-\eta)e^{-k}\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}\\ =&k(1-\delta)\sec\alpha+k\delta(1-\eta)+(1-\eta)(1-e^{-k}).\end{aligned}$$

Thus, we have

$$\mathcal{L}(k,\alpha,\delta,\eta) = k(1-\delta)\sec\alpha + k\delta(1-\eta) + (1-\eta)(1-e^{-k}). \tag{3.32}$$

Note that, the value of expression (3.32) is bounded above by $1-\eta$ provided that (3.30) is true. As stated in Lemma 1.5, condition (3.30) is enough for $\mathcal{G}(k,z)$ to be part of $\mathcal{K}(\alpha,\eta,\delta)$. This essentially completes the proof of Theorem 3.10. □

By establishing $\delta = 0$, we can refine the assertion of Theorem 3.10 as detailed below.

Corollary 3.6. *If* $0 \le \eta < 1$, *and* $|\alpha| < \frac{\pi}{2}$, *then a sufficient condition for* $\mathcal{G}(k,z)$ *to belong to* $\mathcal{K}(\alpha,\eta)$ *is that inequality*

$$k\sec\alpha + (1-\eta)(1-e^{-k}) \le (1-\eta),$$

holds.

Theorem 3.11. *If* $1 < \beta \le \frac{4}{3}$ *and* $0 \le \xi < 1$, *then a sufficient condition for* $\mathcal{G}(k,z)$ *to belong to* $\mathcal{M}^*(\xi,\beta)$ *is that inequality*

$$(1-\xi\beta)(e^k-1) - \beta(1-\xi)\int_0^1 \left(\frac{\Psi(k,z)}{z} - 1\right) dz \le e^k(\beta-1), \tag{3.33}$$

holds.

Proof. To establish that $\mathcal{G}(k,z)$ belongs to $\mathcal{M}^*(\xi,\beta)$ as mentioned in Lemma 1.6, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=2}^{\infty}\left[n-(1+n\xi-\xi)\beta\right]\frac{k^{n-1}}{n!}e^{-k} \le \beta - 1. \tag{3.34}$$

The left side of the inequality (3.34) could be denoted as

$$\begin{aligned}\mathcal{L}(k,\xi,\beta) :=& \sum_{n=2}^{\infty}\left[n-(1+n\xi-\xi)\beta\right]\frac{k^{n-1}}{n!}e^{-k}\\ =& \sum_{n=2}^{\infty}\left[n(1-\xi\beta)-\beta(1-\xi)\right]\frac{k^{n-1}}{n!}e^{-k}\\ =& \sum_{n=2}^{\infty}n(1-\xi\beta)\frac{k^{n-1}}{n!}e^{-k}-\sum_{n=2}^{\infty}\beta(1-\xi)\frac{k^{n-1}}{n!}e^{-k}\\ =& e^{-k}(1-\xi\beta)\sum_{n=2}^{\infty}n\frac{k^{n-1}}{n!}-e^{-k}\beta(1-\xi)\sum_{n=2}^{\infty}\frac{k^{n-1}}{n!}\\ =& e^{-k}(1-\xi\beta)\sum_{n=2}^{\infty}\frac{k^{n-1}}{(n-1)!}-e^{-k}\beta(1-\xi)\sum_{n=2}^{\infty}\frac{k^{n-1}}{n!}\\ =& e^{-k}(1-\xi\beta)(e^{k}-1)e^{-k}\beta(1-\xi)\int_0^1\left(\frac{\Psi(k,z)}{z}-1\right)dz.\end{aligned}$$

Thus, we have

$$\mathcal{L}(k,\xi,\beta)=e^{-k}(1-\xi\beta)(e^{k}-1)e^{-k}\beta(1-\xi)\int_0^1\left(\frac{\Psi(k,z)}{z}-1\right)dz. \tag{3.35}$$

Note that, the value of expression (3.35) is bounded above by $\beta-1$ provided that (3.33) is true. As stated in Lemma 1.6, condition (3.33) is enough for $\mathcal{G}(k,z)$ to be part of $\mathcal{M}^*(\xi,\beta)$. This essentially completes the proof of Theorem 3.11. □

By establishing $\xi=0$, we can refine the assertion of Theorem 3.11 as detailed below.

Corollary 3.7. *If $1<\beta\le\frac{4}{3}$, then a sufficient condition for $\mathcal{G}(k,z)$ to belong to $\mathcal{M}^*(\beta)$ is that inequality*

$$(e^{k}-1)-\beta\int_0^1\left(\frac{\Psi(k,z)}{z}-1\right)dz\le e^{k}(\beta-1),$$

holds.

Theorem 3.12. *If $1<\beta\le\frac{4}{3}$ and $0\le\xi<1$, then a sufficient condition for $\mathcal{G}(k,z)$ to belong to $\mathcal{N}^*(\xi,\beta)$ if and only if inequality* (2.15) *is satisfied.*

Proof. To establish that $\mathcal{G}(k,z)$ belongs to $\mathcal{N}^*(\xi,\beta)$ as mentioned in Lemma 1.6, it is enough to prove that the inequality below is fulfilled:

$$\sum_{n=2}^{\infty}n\left[n-(1+n\xi-\xi)\beta\right]\frac{k^{n-1}}{n!}e^{-k}\le\beta-1. \tag{3.36}$$

The left side of the inequality (3.36) could be denoted as

$$\begin{aligned}\mathcal{L}(k,\xi,\beta) :=& \sum_{n=2}^{\infty}n\left[n-(1+n\xi-\xi)\beta\right]\frac{k^{n-1}}{n!}e^{-k}\\ =& \sum_{n=2}^{\infty}\left[n-(1+n\xi-\xi)\beta\right]\frac{k^{n-1}}{(n-1)!}e^{-k}.\end{aligned}$$

Utilizing the procedure from Theorem 2.5, we derive the required result. This essentially completes the proof of Theorem 3.12. □

4. Concluding remarks and observations

In this research, we have identified the sufficient conditions for the functions $\Psi(k,z)$ to be classified within the classes $\mathcal{S}_1$, $\mathcal{T}_1$, $\mathcal{S}_2$ and $\mathcal{T}_2$. Furthermore, we have determined the mapping properties of the integral operator $\mathcal{G}(k,z)$ as well as a finding related to the particular integral operator acting on $\Psi(k,z)$.

We established sufficient conditions and inclusion results for functions $h \in \mathcal{A}$ to be classified within the classes $\mathcal{S}(\alpha,\eta,\delta)$, $\mathcal{K}(\alpha,\eta,\delta)$, $\mathcal{M}^*(\xi,\beta)$ and $\mathcal{N}^*(\xi,\beta)$, as well as information concerning the images of functions when the convolution operator is applied with Poisson distribution series.

The study also suggests that by employing q-calculus for values of $0 < q < 1$, along with orthogonal polynomials and different kinds of operators, one may get interesting results.

References

[1] Baricz, A. Geometric properties of generalized Bessel functions. *Publ. Math. Debrecen* **73** (2008), no. 1-2, 155–178.

[2] Baricz, A. Geometric properties of generalized Bessel functions of complex order. *Mathematica* **48(71)** (2006), no. 1, 13–18.

[3] Baricz, A. *Generalized Bessel functions of the first kind.* Lecture Notes in Mathematics, 1994, Springer, Berlin, 2010.

[4] Cho, N.E.; Woo, S.Y.; Owa, S. Uniform convexity properties for hypergeometric functions. *Fract. Calc. Appl. Anal.* **5** (2002), no. 3, 303–313.

[5] Dixit, K.K.; Chandra, V. On subclass of univalent functions with positive coefficients. *Aligarh Bull. Math.* **27** (2008), no. 2, 87–93.

[6] Dixit, K.K.; Pathak, A.L. A new class of analytic functions with positive coefficients. *Indian J. Pure Appl. Math.* **34** (2003), no. 2, 209–218.

[7] Duren, P.L. *Univalent functions.* Grundlehren der mathematischen Wissenschaften, 259, Springer, New York, 1983.

[8] Goodman, A.W. On uniformly convex functions. *Ann. Polon. Math.* **56** (1991), no. 1, 87–92.

[9] Goodman, A.W. *Univalent functions.* Vol. II, Mariner Publishing Co., Inc., Tampa, FL, 1983.

[10] Kanas, S.R.; Srivastava, H.M. Linear operators associated with k-uniformly convex functions. *Integral Transform. Spec. Funct.* **9** (2000), no. 2, 121–132.

[11] Kanas, S.R.; Wiśniowska-Wajnryb, A. Conic domains and starlike functions. *Rev. Roumaine Math. Pures Appl.* **45** (2000), no. 4, 647–657.

[12] Kanas, S.R.; Wiśniowska-Wajnryb, A. Conic regions and k-uniform convexity. *J. Comput. Appl. Math.* **105** (1999), no. 1-2, 327–336.

[13] Merkes, E.P.; Scott, W.T. Starlike hypergeometric functions. *Proc. Amer. Math. Soc.* **12** (1961), 885–888.

[14] Mostafa, A.O. A study on starlike and convex properties for hypergeometric functions. *JIPAM. J. Inequal. Pure Appl. Math.* **10** (2009), no. 3, Article 87, 8 pp.

[15] Murugusundaramoorthy, G. Subordination results for spiral-like functions associated with the Srivastava-Attiya operator. *Integral Transforms Spec. Funct.* **23** (2012), no. 2, 97–103.

[16] Murugusundaramoorthy, G.; Răducanu, D.; Vijaya, K. A class of spirallike functions defined by Ruscheweyh-type q-difference operator. *Novi Sad J. Math.* **49** (2019), no. 2, 59–71.

[17] Murugusundaramoorthy, G.; Vijaya, K; Porwal, S. Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series. *Hacet. J. Math. Stat.* **45** (2016), no. 4, 1101–1107.

[18] Murugusundaramoorthy, G.; Porwal, S. Univalent functions with positive coefficients involving Touchard polynomials. *arXiv preprint arXiv:2007.05439* (2020).

[19] Porwal, S. An application of a Poisson distribution series on certain analytic functions. *J. Complex Anal.* **2014**, Art. ID 984135, 3 pp.

[20] Prakash, V.; Breaz, D.; Sivasubramanian, S.; El-Deeb, S.M. Results of certain subclasses of univalent function related to bessel functions. *Mathematics* **13** (2025), no. 4, 569.

[21] Rønning, F. Uniformly convex functions and a corresponding class of starlike functions. *Proc. Amer. Math. Soc.* **118** (1993), no. 1, 189–196.

[22] Silverman, H. Univalent functions having univalent derivatives. *Rocky Mountain J. Math.* **16** (1986), no. 1, 55–61.

[23] Silverman, H. Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51** (1975), 109–116.

[24] Silverman, H. Starlike and convexity properties for hypergeometric functions. *J. Math. Anal. Appl.* **172** (1993), no. 2, 574–581.

[25] Sivasubramanian, S.; Rosy, T.; Muthunagai, K. Certain sufficient conditions for a subclass of analytic functions involving Hohlov operator. *Comput. Math. Appl.* **62** (2011), no. 12, 4479–4485.

[26] Sivasubramanian, S.; Sokół, J. Hypergeometric transforms in certain classes of analytic functions. *Math. Comput. Model.* **54** (2011), no. 11-12, 3076–3082.

[27] Spacek, L. Contribution a la theorie fonctions univalentes. *Casopis Pest. Mat.* **62** (1933), 12–19.

[28] Srivastava, H.M.; Mishra, A.K. Applications of fractional calculus to parabolic starlike and uniformly convex functions. *Comput. Math. Appl.* **39** (2000), no. 3-4, 57–69

[29] Uralegaddi, B.A.; Ganigi, M.D.; Sarangi, S.M. Univalent functions with positive coefficients. *Tamkang J. Math.* **25** (1994), no. 3, 225–230.

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