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Conformable Bilateral Laplace Transform on Time Scales

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ABSTRACT. In this paper, we define the conformable bilateral Laplace transform on arbitrary time scales. We proved the decay property of the generalized exponential function as $t \in \mathbb{T}$ asymptotically approaches minus infinity. Then, the conditions for the absolute and uniform convergence of the conformable bilateral Laplace transform are provided. We specify the class of functions for which the transform exists and provide an inversion formula to reconstruct the original function on a time scale. Finally, the uniqueness theorem is proved for the proposed transform.

1. INTRODUCTION

In 1763, Euler introduced the unilateral Laplace transform to solve differential equations. However, between 1779 and 1812, Laplace used this transform in extensive studies, particularly in the field of probability theory. The bilateral Laplace transform is an integral transform that generalizes the standard unilateral Laplace transform to functions defined across the entire real line and also shows a strong connection with the Fourier transform. The p-multiplied version of the bilateral Laplace transform developed by Van der Pol and Bremmer [24]. This transform unifies the unilateral Laplace and Fourier transforms, providing a framework for Heaviside's operational calculus. This version of the bilateral Laplace transform was modified by Paley, Wiener, and Widder [16, 25], removing the multiplicative p-factor and the application of Stieltjes integral. Their contribution has solidified its significance beyond the unilateral Laplace transform.

Implemented by Hilger [15] in 1988, timescale theory provides a unified and extended framework for discrete and continuous dynamic systems. A time scale, denoted as $\mathbb{T}$, is any non-empty closed subset of the real numbers that inherits the topological properties of the real numbers. As dynamic equations are the core of time scale theory, integral transform methods are crucial for solving them. Consequently, various integral transforms on time scales have been generalized [5, 7, 8, 9, 12, 13, 17, 18, 19, 20, 21, 22].

The concept of derivatives of non-integer order, known as fractional derivatives, first appeared in the letter between L'Hopital and Leibniz in which the question of a halforder derivative was posed. Since then, many formulations of fractional derivatives have appeared. Recently, a new definition of fractional derivative, named "conformable fractional derivative", is introduced. This new fractional derivative is compatible with the classical derivative and time scales derivative and it has attracted the attention in domains such as mechanics, electronics and anomalous diffusion. If $\mathbb{T}$ is a time scale and $f : \mathbb{T} \to \mathbb{R}$ is a given function for which the conformable fractional derivative $f^{(\alpha)}(t)$, $\alpha \in (0, 1]$, $t \in \mathbb{T}$, exists, then when $\alpha = 1$ we get the time scale derivative $f^{\Delta}(t)$, $t \in \mathbb{T}$, of the function f, and if $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$, we get the classical derivative $f'(t)$, $t \in \mathbb{R}$, of the function f. Thus, the results obtained in the framework of the conformable fractional calculus unify

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the results obtained in the framework of the classical calculus and time scales calculus. Researchers have generalized the conformable fractional calculus on arbitrary time scales using the principles of classical fractional calculus [1, 2, 3]. The main aim of this paper is to define the bilateral Laplace transform in the framework of the conformable fractional calculus, called conformable bilateral Laplace transform, and to develop some of its properties. Firstly, the decay property of the generalized exponential function is proved as $t \in \mathbb{T}$ asymptotically tends to minus infinity. Some conditions for absolute and uniform convergence of the conformable bilateral Laplace transform are given. We determine the function class for which the conformable bilateral Laplace transform exists and we provide an inversion formula to recover the original function. To the best of our knowledge no such investigation in the existing references.

This paper is organized as follows. Section 2, we recall some basic concepts and notions of the conformable fractional calculus. Section 3, we define the conformable bilateral Laplace transform on time scales along with its absolute and uniform convergence. Finally, the inversion formula for the proposed transform is formulated.

2. Preliminaries

We assume that the reader is attentive to the basics of time scale calculus, details can be found in [1, 4, 6, 10, 11, 14, 26, 27]. Here we compile definitions and theorems that are be most relevant to our discussion. Here, we assume that the time scale $\mathbb{T}$ under consideration is unbounded above and below and contain the origin as its component.

Definition 2.1. *For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.*

Definition 2.2. *The forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined as $\mu(t) = \sigma(t) - t$.*

Definition 2.3. *Let $t \in \mathbb{T}$. If $\sigma(t) = t$ and $t < \sup \mathbb{T}$, then t is said to be right-dense. If $\sigma(t) > t$, then t is said to be right-scattered. Similarly, if $\rho(t) = t$ and $t > \inf \mathbb{T}$, then t is said to be left-dense. If $\rho(t) < t$, then t is said to be left-scattered.*

Definition 2.4. *If $\sup \mathbb{T} = m$ is left-scattered, then define $\mathbb{T}^\kappa = \mathbb{T} \backslash \{m\}$. Otherwise, define $\mathbb{T}^\kappa = \mathbb{T}$.*

Definition 2.5. *Assume that $g : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in (0, 1]$. If $g^{(\alpha)}(t)$ is the number, provided it exists, with the property that, given any $\epsilon > 0$, there exists a neighbourhood $U \subset \mathbb{T}$ of t, with $\delta > 0$, such that*

$$|(g(\sigma(t)) - g(s))|t|^{1-\alpha} - g^{(\alpha)}(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$. We call $g^{(\alpha)}(t)$ the conformable fractional derivative of g of order α at t.

Definition 2.6. *Assume that $g : \mathbb{T} \to \mathbb{R}$ is a regulated function. Then, the conformable α-fractional integral of g, for $\alpha \in (0, 1]$, is given by*

$$\int g(t) \Delta^\alpha t = \int g(t)|t|^{\alpha-1} \Delta t.$$

Definition 2.7. *If a function $g : \mathbb{T} \to \mathbb{R}$ is continuous at the right-dense points in $\mathbb{T}$, and has a finite limit at the left-dense points in $\mathbb{T}$, then g is said to be rd-continuous. We denote the set of all rd-continuous functions by $\mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.*

Definition 2.8. *A function $g : \mathbb{T} \to \mathbb{R}$ is said to be 'α-regressive' provided*

$$1 + \mu(t) g(t) |t|^{\alpha-1} \neq 0, \text{ for all } t \in \mathbb{T}^\kappa,$$

holds. Similarly, $g:\mathbb{T}\to\mathbb{R}$ is said to be 'α-positively regressive' provided

$$1+\mu(t)g(t)|t|^{\alpha-1}>0,\ \textit{for all } t\in\mathbb{T}^{\kappa}$$

holds. We denote the set of all α-regressive and rd-continuous functions $g:\mathbb{T}\to\mathbb{R}$ by $\mathcal{R}^{\alpha}$. The set $\mathcal{R}^{\alpha+}$ is the subset of $\mathcal{R}^{\alpha}$ containing all α-positively regressive functions.

Definition 2.9. *In $\mathcal{R}^{\alpha}$ 'α-circle plus' addition $\oplus_{\alpha}$ is defined as below*

$$(g_1\oplus_{\alpha}g_2)(t)=g_1(t)+g_2(t)+\mu(t)g_1(t)g_2(t)|t|^{\alpha-1},\ \textit{for all } t\in\mathbb{T}^{\kappa}.$$

The set $\mathcal{R}^{\alpha}$ forms an abelian group under $\oplus_{\alpha}$.
For $g\in\mathcal{R}^{\alpha}$, the inverse of g is given as

$$\ominus_{\alpha}g=\frac{-g(t)}{1+\mu(t)g(t)|t|^{\alpha-1}},\ \textit{for all } t\in\mathbb{T}^{\kappa}.$$

Definition 2.10. *For $\mathcal{R}^{\alpha}$, 'α-circle minus' substraction $\ominus_{\alpha}$ is defined as*

$$(g_1\ominus_{\alpha}g_2)(t)=\frac{g_1(t)-g_2(t)}{1+\mu(t)g_2(t)|t|^{\alpha-1}},\ \textit{for all } t\in\mathbb{T}^{\kappa}.$$

Definition 2.11. *Let $g\in\mathfrak{R}^{\alpha}$, then the generalized exponential function is defined by*

$$\mathbb{E}_g(t,s)=exp\Big(\int_s^t\xi_{\mu(\tau)}\big(g(\tau)|\tau|^{\alpha-1}\big)\Delta\tau\Big),\ \textit{for all } t,s\in\mathbb{T}.\tag{2.1}$$

Following the concept of the cylindrical transformation, [4, Definition 2.21] *Equation 2.1 can be rewritten as*

$$\mathbb{E}_g(t,s)=exp\Big(\int_s^t\frac{1}{\mu(\tau)}Log(1+\mu(\tau)g(\tau)|\tau|^{\alpha-1})\Delta\tau\Big),\ \textit{for all } t,s\in\mathbb{T}.\tag{2.2}$$

Next theorem collects some important properties of the generalized exponential function.

Theorem 2.1. *If $g_1,g_2\in\mathcal{R}^{\alpha}$, then for all $s,t\in\mathbb{T}$,*

1. $\mathbb{E}_0(t,s)=1$ *and* $\mathbb{E}_{g_1}(t,t)=1$.
2. $\mathbb{E}_{g_1}(t,s)\mathbb{E}_{g_1}(s,r)=\mathbb{E}_{g_1}(t,r)$.
3. $\mathbb{E}_{g_1}(t,s)=\frac{1}{\mathbb{E}_{g_1}(s,t)}=\mathbb{E}_{\ominus_{\alpha}g_1}(s,t)$.
4. $\mathbb{E}_{g_1}(t,s)\mathbb{E}_{g_2}(t,s)=\mathbb{E}_{g_1\oplus_{\alpha}g_2}(t,s)$.
5. $\frac{\mathbb{E}_{g_1}(t,s)}{\mathbb{E}_{g_2}(t,s)}=\mathbb{E}_{g_1\ominus_{\alpha}g_2}(t,s)$.
6. $\mathbb{E}_{g_1}(\sigma(t),s)=(1+\mu(t)g_1(t)|t|^{\alpha-1})\mathbb{E}_{g_1}(t,s)$.
7. $\mathbb{E}_{\ominus_{\alpha}g_1}(\sigma(t),s)=\frac{\mathbb{E}_{\ominus_{\alpha}g_1}(t,s)}{1+\mu(t)g_1(t)|t|^{\alpha-1}}$.
8. $\mathbb{E}_{g_1}^{\Delta}(t,s)=g_1(t)\mathbb{E}_{g_1}(t,s)|t|^{\alpha-1}$.
9. $\mathbb{E}_{g_1}^{(\alpha)}(t,s)=g_1(t)\mathbb{E}_{g_1}(t,s)$.

Definition 2.12. *For $h>0$, the Hilger complex numbers, and the Hilger real axis are defined as $\mathbb{C}_h=\{\,z\in\mathbb{C}:z\neq-\frac{1}{h}\}$ and, $\mathbb{R}_h=\{z\in\mathbb{C}_h:z\in\mathbb{R}$ and, $z>-\frac{1}{h}\}$ respectively. For $h=0$, we have $\mathbb{C}_0:=\mathbb{C}$ and $\mathbb{R}_0:=\mathbb{R}$.*

Definition 2.13. *Let $h>0$, and $z\in\mathbb{C}_h$. The Hilger real part of z is defined as*

$$Re_h(z):=\frac{|hz+1|-1}{h}.$$

3. The Conformable Bilateral Laplace Transform

Suppose that $\mathbb{T}$ is a time scale with forward jump operator and delta differentiation operator σ and Δ, respectively, and $\inf \mathbb{T} = -\infty$, $\sup \mathbb{T} = \infty$. For $s \in \mathbb{T}$, define

$$\begin{aligned} \mu_*(s) &= \inf_{t\in[s,\infty)} \mu(t), \\ \mu^*(s) &= \sup_{t\in(-\infty,s]} \mu(t), \\ \overline{\mu}(s) &= \inf_{t\in(-\infty,s]} \mu(t). \end{aligned}$$

For $s, t \in \mathbb{T}$ and $\lambda \in \mathcal{R}^\alpha(\mathbb{T})$, set

$$M_\lambda(t,s) = \int_t^s \frac{|\tau|^{\alpha-1}}{1+\lambda\mu(\tau)}\Delta\tau.$$

Lemma 3.1. *Let $s \in \mathbb{T}$, $\lambda \in \mathcal{R}^{\alpha+}((-\infty, s])$. Then*

(1) $M_\lambda^\Delta(t,s) < 0$ for all $t \in (-\infty, s)$, where the differentiation is with respect to t.
(2) $\lim_{t\to-\infty} M_\lambda(t,s) = \infty$.

Proof. (1) By the definition of the function M_λ, it follows

$$\begin{aligned} M_\lambda^\Delta(t,s) &= -\frac{|t|^{\alpha-1}}{1+\lambda\mu(t)} \\ &< 0, \quad t \in (-\infty, s]. \end{aligned}$$

(2) Because we will investigate the behaviour when $t \to -\infty$, without loss of generality, assume that $t < 0$. We will consider two cases.

(a) Let $\sup_{t\in(-\infty,s]} \mu(t) < \infty$. Then

$$\begin{aligned} 1+\lambda\mu(t) &\le 1+|\lambda|\mu(t) \\ &\le 1+|\lambda| \sup_{t\in(-\infty,s]} \mu(t), \quad t \in (-\infty, s]. \end{aligned}$$

Hence,

$$\begin{aligned} M_\lambda(t,s) &= \int_t^s \frac{|\tau|^{\alpha-1}}{1+\lambda\mu(\tau)}\Delta\tau \\ &\ge \int_t^s \frac{|\tau|^{\alpha-1}}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \end{aligned}$$

(i) Let $s > 0$. Then

$$\int_t^s \frac{|\tau|^{\alpha-1}}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau$$

$$\begin{aligned}
&= \int_t^0 \frac{|\tau|^{\alpha-1}}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau + \int_0^s \frac{|\tau|^{\alpha-1}}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \\
&\geq \int_t^0 \frac{|t|^{\alpha-1}}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau + \int_0^s \frac{|\tau|^{\alpha-1}}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \\
&= \frac{|t|^{\alpha-1}(-t)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)} + \int_0^s \frac{|\tau|^{\alpha-1}}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \\
&\to \infty, \quad \text{as} \quad t\to-\infty.
\end{aligned}$$

(ii) Let $s \leq 0$. Then

$$\begin{aligned}
\int_t^s \frac{|\tau|^{\alpha-1}}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau &\geq \int_t^s \frac{|t|^{\alpha-1}}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \\
&= \frac{|t|^{\alpha-1}(s-t)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)} \\
&\to \infty, \quad \text{as} \quad t\to-\infty.
\end{aligned}$$

(b) Suppose that $\sup_{t\in(-\infty,s]}\mu(t) = \infty$. Because $\lambda \in \mathcal{R}^{\alpha+}((-\infty,s])$, we have that $\lambda \geq 0$. If $\lambda = 0$, then

$$M_\lambda(t,s) = \int_t^s |\tau|^{\alpha-1}\Delta\tau.$$

(i) Let $s > 0$. Then

$$\begin{aligned}
M_\lambda(t,s) &= \int_t^0 |\tau|^{\alpha-1}\Delta\tau + \int_0^s |\tau|^{\alpha-1}\Delta\tau \\
&\geq |t|^{\alpha-1}(-t) + \int_0^s |\tau|^{\alpha-1}\Delta\tau \\
&\to \infty, \quad \text{as} \quad t\to-\infty.
\end{aligned}$$

(ii) Let $s \leq 0$. Then

$$\begin{aligned}
M_\lambda(t,s) &\geq \int_t^s |t|^{\alpha-1}\Delta\tau \\
&= |t|^{\alpha-1}(s-t) \\
&\to \infty, \quad \text{as} \quad t\to-\infty.
\end{aligned}$$

Assume that $\lambda > 0$. Since $\sup_{t\in(-\infty,s]}\mu(t) = \infty$, there exists a decreasing and divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset (-\infty,s]$ of right-scattered points such that

$\{\mu(\xi_k)|\xi_k|^{\alpha-1}\}_{k\in\mathbb{N}}$ is an increasing and divergent sequence. Hence, we get

$$\begin{aligned} M_\lambda(t,s) &= \int_t^s \frac{|\tau|^{\alpha-1}}{1+\lambda\mu(\tau)}\Delta\tau \\ &\geq \sum_{\sigma(\xi_k)\geq t, \xi_k\leq s} \int_{\xi_k}^{\sigma(\xi_k)} |\tau|^{\alpha-1}\Delta\tau \\ &= \sum_{\sigma(\xi_k)\geq t, \xi_k\leq s} \mu(\xi_k)|\xi_k|^{\alpha-1} \\ &\to \infty, \quad \text{as} \quad t\to-\infty. \end{aligned}$$

This completes the proof.

□

Theorem 3.2 (Decay of the Generalized Exponential Function)**.** *Let $s\in\mathbb{T}$, $\lambda\in\mathcal{R}^{\alpha+}((-\infty,s])$ and*

$$\overline{\mathbb{C}}_{\mu^*(s)}(\lambda)=\left\{z\in\mathbb{C}: Re_{\mu^*(s)}(z)<\lambda\right\}.$$

Then, for any $z\in\overline{\mathbb{C}}_{\mu^*(s)}(\lambda)$, we have the following properties.

(1)

$$|\mathbb{E}_{\lambda\ominus_\alpha z}(t,s)|\leq \mathbb{E}_{\lambda\ominus_\alpha Re_{\mu^*(s)}(z)}(t,s), \quad t\in(-\infty,s],$$

(2)

$$\lim_{t\to-\infty}\mathbb{E}_{\lambda\ominus_\alpha Re_{\mu^*(s)}(z)}(t,s)=0,$$

(3)

$$\lim_{t\to-\infty}\mathbb{E}_{\lambda\ominus_\alpha z}(t,s)=0.$$

Proof. Let $\Psi_h(z,\lambda)$ be the following function

$$\Psi_h(z,\lambda)=\begin{cases}\frac{1}{h}\log\left|\frac{1+h\lambda}{1+hz}\right| & \text{if} \quad h>0 \\ \lambda-\text{Re}(z) \quad \text{if} \quad h=0. \end{cases}$$

(1) As in [23], we have

$$\Psi_h(z,\lambda)=\Psi_h(\text{Re}_h(z),\lambda)$$

and

$$|\mathbb{E}_{\lambda\ominus_\alpha z}(t,s)| \quad \leq \quad \mathbb{E}_{\lambda\ominus_\alpha \text{Re}_{\mu^*(s)}(z)}(t,s), \quad t\in(-\infty,s].$$

(2) For $z\in\overline{\mathbb{C}}_{\mu^*(s)}(\lambda)$, we have

$$\begin{aligned} \lambda\ominus_\alpha \text{Re}_{\mu^*(s)}(z) &= \frac{\lambda-\text{Re}_{\mu^*(s)}(z)}{1+\mu(t)\text{Re}_{\mu^*(s)}(z)|t|^{\alpha-1}} \\ &> \frac{\lambda-\text{Re}_{\mu^*(s)}(z)}{1+\lambda\mu(t)|t|^{\alpha-1}} \\ &> 0, \quad t\in(-\infty,s], \end{aligned}$$

and

$$\begin{aligned}
1+\mu(t)\left(\lambda \ominus_\alpha \operatorname{Re}_{\mu^*(s)}(z)\right)|t|^{\alpha-1} &= \frac{1+\lambda\mu(t)|t|^{\alpha-1}}{1+\mu(t)\operatorname{Re}_{\mu^*(s)}(z)|t|^{\alpha-1}} \\
&> \frac{1+\lambda\mu(t)|t|^{\alpha-1}}{1+\lambda\mu(t)|t|^{\alpha-1}} \\
&= 1, \quad t \in (-\infty, s].
\end{aligned}$$

Thus,

$$\lambda \ominus_\alpha \operatorname{Re}_{\mu^*(s)}(z) \in \mathcal{R}^{\alpha+}((-\infty, s])$$

and

$$\begin{aligned}
\mathbb{E}_{\lambda\ominus_\alpha \operatorname{Re}_{\mu^*(s)}(z)}(t,s) &= e^{\left(\lambda-\operatorname{Re}_{\mu^*(s)}(z)\right)\int_s^t \frac{|\tau|^{\alpha-1}}{1+\mu(\tau)\operatorname{Re}_{\mu^*(s)}(z)|\tau|^{\alpha-1}}\Delta\tau} \\
&= e^{\left(\operatorname{Re}_{\mu^*(s)}(z)-\lambda\right)M_{\operatorname{Re}_{\mu^*(s)}(z)|\tau|^{\alpha-1}}(t,s)} \\
&\to 0, \quad \text{as} \quad t\to-\infty.
\end{aligned}$$

(3) By 1 and 2, we obtain

$$\begin{aligned}
|\mathbb{E}_{\lambda\ominus_\alpha z}(t,s)| &\le \mathbb{E}_{\lambda\ominus_\alpha \operatorname{Re}_{\mu^*(s)}(z)}(t,s) \\
&\to 0, \quad \text{as} \quad t\to-\infty.
\end{aligned}$$

This completes the proof.

□

Definition 3.14. *Suppose that $s \in \mathbb{T}$, $f : \mathbb{T} \to \mathbb{R}$ is regulated. Then the conformable bilateral Laplace transform of f is defined by*

$$\mathcal{L}^b(f)(z,s) = \int_{-\infty}^{\infty} f(t)\mathbb{E}^\sigma_{\ominus_\alpha z}(t,s)\Delta^\alpha t$$

for $z \in \mathbb{C}$ for which $1+\mu(t)z|t|^{\alpha-1} \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the improper integral exists.

Definition 3.15 (Conformable Double Exponential Order)**.** *Let $s \in \mathbb{T}$. A function $f \in \mathcal{C}_{rd}(\mathbb{T})$ has conformable double exponential order (β,γ) on $\mathbb{T}$ if*

(1) $\beta \in \mathcal{R}^{\alpha+}([s,\infty))$, $\gamma \in \mathcal{R}^{\alpha+}((-\infty,s])$,

(2) there exist positive constants K_β, K_γ such that

$$\begin{aligned}
|f(t)| &\le K_\beta \mathbb{E}_\beta(t,s), \quad t\in[s,\infty), \\
|f(t)| &\le K_\gamma \mathbb{E}_\gamma(t,s), \quad t\in(-\infty,s].
\end{aligned}$$

Example 3.1. *Let $\gamma > 0$. Consider the function*

$$f(t) = \mathbb{E}_\gamma(t,s), \quad t\in\mathbb{T}.$$

If $s \le t$, then $\beta = \gamma$. If $s > t$, then $0 \le f(t) \le 1$ and $\mathbb{E}_{\ominus_\alpha\gamma}(t,s) > 1$. Thus,

$$f(t) \le \mathbb{E}_{\ominus_\alpha\gamma}(t,s).$$

Therefore f is of double exponential order $(\gamma, \ominus_\alpha\gamma)$.

Lemma 3.2. *Let $s \in \mathbb{T}$ and $f \in \mathcal{C}_{rd}(\mathbb{T})$ be a function of conformable double exponential order (β, γ). Then*

(1) $\lim_{t\to\infty} f(t)\mathbb{E}_{\ominus_\alpha z}(t,s) = 0$ *for any* $z \in \mathbb{C}_{\mu_*(s)}(\beta)$.

(2) $\lim_{t\to-\infty} f(t)\mathbb{E}_{\ominus_\alpha z}(t,s) = 0$ *for any* $z \in \overline{\mathbb{C}}_{\mu^*(s)}(\gamma)$.

Proof. Since f is of double exponential order (β, γ), there exist positive constants K_β, K_γ such that

$$|f(t)| \leq K_\beta \mathbb{E}_\beta(t,s), \quad t \in [s,\infty),$$

$$|f(t)| \leq K_\gamma \mathbb{E}_\gamma(t,s), \quad t \in (-\infty,s].$$

(1) The first statement follows by Lemma 3.4 in [23].

(2) By Theorem 3.2, we have

$$\begin{aligned} |f(t)\mathbb{E}_{\ominus_\alpha z}(t,s)| &\leq K_\gamma \left|\mathbb{E}_{\gamma\ominus_\alpha z}(t,s)\right| \\ &\leq K_\gamma \mathbb{E}_{\gamma\ominus_\alpha \mathrm{Re}_{\mu^*(s)}(z)}(t,s) \\ &\to 0, \quad \text{as} \quad t \to -\infty. \end{aligned}$$

This completes the proof. □

For $z \in \mathbb{C}$, denote

$$\overline{\overline{\mu}}(s) = \mu^*(s) \quad \text{if} \quad \mathrm{Re}_{\overline{\mu}(s)}(z) \leq 0$$

and

$$\overline{\overline{\mu}}(s) = \overline{\mu}(s) \quad \text{if} \quad \mathrm{Re}_{\overline{\mu}(s)}(z) > 0.$$

Definition 3.16. *Let $s \in \mathbb{T}$, $\beta \in \mathcal{R}^{\alpha+}([s,\infty))$, $\gamma \in \mathcal{R}^{\alpha+}((-\infty,s])$. We say that (s,β,γ) is an admissible triple if*

$$\begin{aligned} \mathbb{C}_{s,\beta,\gamma} &= \Big\{ z \in \mathbb{C} : Re_{\mu^*(s)}(z) < \gamma, \quad Re_{\mu_*(s)}(z) > \beta, \\ & \qquad 1 + \overline{\overline{\mu}}(s) Re_{\overline{\mu}(s)}(z) \neq 0 \Big\}. \end{aligned}$$

Theorem 3.3 (Absolute Convergence of the Conformable Bilateral Laplace Transform)**.** *Let (s,β,γ) be an admissible triple, $f \in \mathcal{C}_{rd}(\mathbb{T})$ be of double exponential order (β,γ). Then $\mathcal{L}^b(f)(\cdot,s)$ exists on $\mathbb{C}_{s,\beta,\gamma}$ and converges absolutely.*

Proof. Since f is of double exponential order (β, γ), there exist positive constants K_β, K_γ such that

$$|f(t)| \leq K_\beta \mathbb{E}_\beta(t,s), \quad t \in [s,\infty),$$

$$|f(t)| \leq K_\gamma \mathbb{E}_\gamma(t,s), \quad t \in (-\infty,s].$$

Next, for $t \leq s$, we have

$$\begin{aligned} |1 + \mu(t) z |t|^{\alpha-1}| &= 1 + \mu(t)\mathrm{Re}_{\mu(t)}(z)|t|^{\alpha-1} \\ &\geq 1 + \overline{\overline{\mu}}(s)\mathrm{Re}_{\overline{\mu}(s)}(z)|t|^{\alpha-1}. \end{aligned}$$

For $z \in \mathbb{C}_{s,\beta,\gamma}$, we find

$$\begin{aligned}\left|\mathcal{L}^b(f)(z,s)\right| &= \left|\int_s^\infty f(\tau)\mathbb{E}_{\ominus_\alpha z}(\sigma(\tau),s)\Delta^\alpha\tau + \int_{-\infty}^s f(\tau)\mathbb{E}_{\ominus_\alpha z}(\sigma(\tau),s)\Delta^\alpha\tau\right| \\ &\leq \int_s^\infty \left|f(\tau)\mathbb{E}_{\ominus_\alpha z}(\sigma(\tau),s)\right|\Delta^\alpha\tau + \int_{-\infty}^s \left|f(\tau)\mathbb{E}_{\ominus_\alpha z}(\sigma(\tau),s)\right|\Delta^\alpha\tau \\ &= J_1 + J_2.\end{aligned}$$

By the proof of Theorem 3.2 in [23], we have

$$J_1 \leq \frac{K_\beta}{\mathrm{Re}_{\mu_*(s)}(z) - \beta}, \quad z \in \mathbb{C}_{s,\alpha,\gamma}.$$

Consider the integral

$$J = \int_t^s \left|f(\tau)\mathbb{E}_{\ominus_\alpha z}(\sigma(\tau),s)\right|\Delta^\alpha\tau, \quad t \leq s.$$

Using Theorem 3.2, 1, 2, we obtain

$$\begin{aligned}J &\leq K_\gamma \int_t^s \frac{|\mathbb{E}_{\gamma\ominus_\alpha z}(\tau,s)|}{|1+\mu(\tau)z|\tau|^{\alpha-1}|}\Delta^\alpha\tau \\ &= K_\gamma \int_t^s \frac{|\mathbb{E}_{\gamma\ominus_\alpha z}(\tau,s)|}{1+\mu(\tau)\mathrm{Re}_{\mu(\tau)}(z)|\tau|^{\alpha-1}}\Delta^\alpha\tau \\ &\leq K_\gamma \int_t^s \frac{\mathbb{E}_{\gamma\ominus_\alpha \mathrm{Re}_{\mu(\tau)}(z)}(\tau,s)}{1+\mu(\tau)\mathrm{Re}_{\mu(\tau)}(z)|\tau|^{\alpha-1}}\Delta^\alpha\tau \\ &= K_\gamma \int_t^s \frac{1}{\gamma - \mathrm{Re}_{\mu(\tau)}(z)}\mathbb{E}^{(\alpha)}_{\gamma\ominus_\alpha \mathrm{Re}_{\mu(\tau)}(z)}(\tau,s)\Delta^\alpha\tau \\ &\leq \frac{K_\gamma}{\gamma - \mathrm{Re}_{\mu(\tau)}(z)}\left(1 - \mathbb{E}_{\gamma\ominus_\alpha \mathrm{Re}_{\mu(\tau)}(z)}(t,s)\right)\end{aligned}$$

Therefore

$$J_2 \leq \frac{K_\gamma}{\gamma - \mathrm{Re}_{\mu^*(s)}(z)}$$

and

$$\left|\mathcal{L}^b(f)(z,s)\right| \leq \frac{K_\beta}{\mathrm{Re}_{\mu_*(s)}(z) - \beta} + \frac{K_\gamma}{\gamma - \mathrm{Re}_{\mu^*(s)}(z)}.$$

Consequently the conformable bilateral Laplace transform of the function f converges absolutely on $\mathbb{C}_{s,\beta,\gamma}$. This completes the proof. □

Corollary 3.1. *Let (s,β,γ) be an admissible triple, $f \in \mathcal{C}_{rd}(\mathbb{T})$ be of conformable double exponential order (β,γ). Then*

$$\lim_{|z|\to\infty} \mathcal{L}^b(f)(z,s) = 0.$$

Proof. Note that $|z| \to \infty$ implies $\mathrm{Re}_{\mu_*(s)}(z) \to \infty$, $\mathrm{Re}_{\mu^*(s)}(z) \to \infty$ and $\mathrm{Re}_{\overline{\mu}(s)}(z) \to \infty$. This completes the proof. □

Theorem 3.4 (Uniform Convergence of the Bilateral Laplace Transform)**.** *Let (s,β,γ) be an admissible triple, $f \in \mathcal{C}_{rd}(\mathbb{T})$ be of conformable double exponential order (α,γ). Then $\mathcal{L}^b(f)(\cdot,s)$ converges uniformly in $\mathbb{C}_{s,\beta,\gamma}$.*

Proof. Let $\epsilon > 0$ be arbitrarily chosen. By [23], we have that there exists an $r \in [s, \infty)$ such that

$$\left| \int_t^\infty f(\tau)\mathbb{E}_{\ominus_\alpha z}(\sigma(\tau), s)\Delta^\alpha \tau \right| \leq \epsilon, \quad t \in [r, \infty), \quad z \in \mathbb{C}_{s,\alpha,\gamma}.$$

Let $t \in (-\infty, s]$ and $a \leq t$. By [23], we get

$$\begin{aligned} \left| \int_a^t f(\tau)\mathbb{E}_{\ominus_\alpha z}(\sigma(\tau), s)\Delta^\alpha \tau \right| &\leq \frac{K_\gamma}{\gamma - \mathrm{Re}_{\mu^*(s)}(z)} \left(\mathbb{E}_{\gamma \ominus_\alpha \mathrm{Re}_{\mu(\tau)}(z)}(t, s) - \mathbb{E}_{\gamma \ominus_\alpha \mathrm{Re}_{\mu(\tau)}(z)}(a, s) \right) \\ &\leq \frac{K_\gamma}{\gamma - \mathrm{Re}_{\mu^*(s)}(z)} \mathbb{E}_{\gamma \ominus_\alpha \mathrm{Re}_{\mu(\tau)}(z)}(t, s) \\ &\leq \frac{K_\gamma}{\gamma - \mathrm{Re}_{\mu^*(s)}(z)} \mathbb{E}_{\gamma \ominus_\alpha \mathrm{Re}_{\mu^*(s)}(z)}(t, s). \end{aligned}$$

Thus, there exists an $r_1 \in (-\infty, s]$, such that

$$\left| \int_{-\infty}^t f(\tau)\mathbb{E}_{\ominus_\alpha z}(\sigma(\tau), s)\Delta^\alpha \tau \right| \leq \epsilon, \quad t \in (-\infty, r_1], \quad z \in \mathbb{C}_{s,\beta,\gamma}.$$

This completes the proof. □

Theorem 3.5 (Inverse of the transform). *Let (s, α_1, γ_1) be an admissible triple and $f \in \mathcal{C}_{rd}([s, \infty))$ be of conformable double exponential order (α_1, γ_1). Consider $\mathcal{L}^b(f)(\cdot, s)$ on $\mathbb{C}_{s,\alpha_1,\gamma_1}$ and suppose that it has finitely many regressive poles of finite order $\{z_1, z_2, \ldots, z_n\}$ and $\widetilde{F}^b_{\mathbb{R}}(z)$ is the bilateral Laplace transform of the function $\widetilde{f}$ on $\mathbb{R}$ that corresponds to the transform $\mathcal{L}^b(f)(z, s)$ of f on $\mathbb{T}$. If*

$$\int_{c-i\infty}^{c+i\infty} \left| \widetilde{F}^b_{\mathbb{R}}(z) \right| |dz| < \infty,$$

then

$$f(t) = \sum_{i=1}^n res_{z=z_i} \mathbb{E}_z(t, s)\mathcal{L}^b(f)(z, s)$$

has conformable bilateral Laplace transform $\mathcal{L}^b(f)(z, s)$ for all $z \in \mathbb{C}_{s,\alpha_1,\gamma_1} \bigcap \mathbb{C}$.

Proof. Without loss of generality, suppose that $s = 0$. We have that $\mathcal{L}^b(f)(\cdot)$ converges uniformly on $\mathbb{C}_{s,\alpha_1,\gamma_1} \bigcap \mathbb{C}$ and hence, it is analytic in this region. Next, we have that

$$\lim_{|z| \to \infty} \mathcal{L}^b(f)(z) = 0.$$

Let C be the collection of the bilateral Laplace transforms over $\mathbb{R}$, D be the collection of the conformable bilateral Laplace transforms over $\mathbb{T}$, i.e., $C = \left\{ \widetilde{F}^b_{\mathbb{R}}(z) \right\}$, $D = \left\{ \mathcal{L}^b(f)(z) \right\}$, where

$$\begin{aligned} \widetilde{F}^b_{\mathbb{R}}(z) &= G(z)e^{-z\tau}, \\ \mathcal{L}^b(f)(z) &= G(z)\mathbb{E}_{\ominus_\alpha z}(\tau, 0) \end{aligned}$$

for G a rational function and for $\tau \in \mathbb{T}$ a constant. Let $C_{p-c0}(\mathbb{R}, \mathbb{R})$ denotes the space of piecewise continuous functions of exponential order, $C_{prd-ez}(\mathbb{T}, \mathbb{R})$ denotes the space of piecewise right-dense continuous functions of conformable double exponential type in which the exponential function coincides with a conformable Hilger exponential function.

Each of $\theta, \gamma, \theta^{-1}, \gamma^{-1}$ map functions involving the continuous exponential to the time scale conformable exponential and vice versa. For example, γ maps the function

$$\widetilde{F}^b_{\mathbb{R}}(z) = \frac{e^{-za}}{z}$$

to the function

$$\mathcal{L}^b(f)(z) = \frac{\mathbb{E}_{\ominus_\alpha z}(a,0)}{z},$$

while γ^{-1} maps $\mathcal{L}^b(f)(z)$ back to $\widetilde{F}^b_{\mathbb{R}}(z)$ in the obvious manner. If the representation of $\mathcal{L}^b(f)(z)$ is independent of the exponential, that is $\tau = 0$, then γ and its inverse γ^{-1} will act as the identity. For example,

$$\gamma\left(\frac{1}{1+z^2}\right) = \gamma^{-1}\left(\frac{1}{1+z^2}\right) = \frac{1}{1+z^2}.$$

The map θ will set the continuous exponential function to the time scale conformable exponential function in the following manner: if we write $\widetilde{f} \in C_{p-c0}(\mathbb{R},\mathbb{R})$ as

$$\widetilde{f}(t) = \sum_{i=1}^{n} \text{Res}_{z=z_i} e^{zt} \widetilde{F}^b_{\mathbb{R}}(z),$$

then

$$\theta\left(\widetilde{f}(t)\right) = \sum_{i=1}^{n} \text{Res}_{z=z_i} \mathbb{E}_z(t,0)\mathcal{L}^b(f)(z).$$

To go from $\widetilde{F}^b_{\mathbb{R}}(z)$ to $\mathcal{L}^b(f)(z)$, we simply switch expressions involving the continuous exponential in $\widetilde{F}^b_{\mathbb{R}}$ with the time scale conformable exponential giving $\mathcal{L}^b(f)(z)$ as was done for γ and its inverse θ^{-1} will then act on $g \in C_{prd-ez}(\mathbb{T},\mathbb{R})$,

$$g(t) = \sum_{i=1}^{n} \text{Res}_{z=z_i} \mathbb{E}_z(t,0) G_{\mathbb{T}}(z)$$

as

$$\theta^{-1}(g(t)) = \sum_{i=1}^{n} \text{Res}_{z=z_i} e^{zt} \widetilde{G}_{\mathbb{R}}(z).$$

For example, for the unit step function $\widetilde{u}_a(t)$ on $\mathbb{R}$, we know from the continuous result that we may write the step function as

$$\widetilde{u}_a(t) = \text{Res}_{z=0} e^{zt} \frac{e^{-az}}{z},$$

so that if $a \in \mathbb{T}$, then

$$\theta\left(\widetilde{u}_a(t)\right) = \text{Res}_{z=0} \mathbb{E}_z(t,0) \frac{\mathbb{E}_{\ominus_\alpha z}(a,0)}{z}.$$

Now, for a given time scale conformable bilateral Laplace transform $\mathcal{L}^b(f)(z)$, we begin by mapping to $\widetilde{F}^b_{\mathbb{R}}(z)$ via γ^{-1}. We have that the inverse of $\widetilde{F}^b_{\mathbb{R}}(z)$ exists for all z with $z \in \mathbb{C}_{s,\alpha_1,\gamma_1} \bigcap \mathbb{C}$ and it is given by

$$\widetilde{f}(t) = \sum_{i=1}^{n} \text{Res}_{z=z_i} e^{zt} \widetilde{F}^b_{\mathbb{R}}(z).$$

Applying θ to $\widetilde{f}(t)$ to retrieve the time scale function, we get

$$f(t) = \sum_{i=1}^{n} \text{Res}_{z=z_i} \mathbb{E}_z(t,0)\mathcal{L}^b(f)(z),$$

whereby

$$\left(\gamma \circ \widetilde{F}^{b}_{\mathbb{R}} \circ \theta^{-1}\right)(f(t)) = \mathcal{L}^{b}(f)(z).$$

This completes the proof. □

Theorem 3.6 (Uniqueness of the inverse)**.** *If the functions $f, g : \mathbb{T} \to \mathbb{R}$ satisfy Theorem 3.5 and have the same conformable bilateral Laplace transform, then $f = g$ a.e.*

Proof. Suppose that

$$\int_{-\infty}^{\infty} \mathbb{E}^{\sigma}_{\ominus_\alpha z}(t,s) f(t) \Delta^\alpha t = \int_{-\infty}^{\infty} \mathbb{E}^{\sigma}_{\ominus_\alpha z}(t,s) g(t) \Delta^\alpha t.$$

Hence, $h = f - g$ has conformable bilateral Laplace transform zero and $h \in \ker\mathcal{L}^b$. If we denote by $\mathcal{L}^{b-1}$ the inversion of $\mathcal{L}^b$, then

$$\mathcal{L}^{b-1} \circ \mathcal{L}^{b}(h) = \mathcal{L}^{b-1}(0) = 0 = h.$$

Therefore $f = g$ a.e. This completes the proof. □

4. Conclusions

We generalized the conformable bilateral Laplace transform on time scales and established several foundational results. We study the decay behaviour of the generalized exponential function as $t \in \mathbb{T}$ tends to negative infinity. Additionally, the conditions necessary for the absolute and uniform convergence of the proposed transform are provided and the class of functions for which the transform is well-defined is identified. An explicit inversion formula was derived to recover the original function on given time scale. Finally, we proved uniqueness theorem, ensuring the distinctiveness of the transform representation.

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