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Quantitative estimates by linear and non-linear Bernstein-Chlodowsky-Kantorovich Operators

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ABSTRACT. Bernstein polynomials were introduced by Bernstein in 1912 and possess many interesting properties. These properties have led to the discovery of new applications and developments. These generalizations aim to provide adequate and powerful tools for applications such as numerical analysis, computer-aided geometric design, and differential equation solutions. With these applications, the importance of Bernstein polynomials increased and became an important topic in approximation theory. Classical approximation theory is concerned with the representation of continuous functions by simpler functions such as polynomials and trigonometric functions. In the last century, significant attention has been given to the realization that linearity is not a necessary condition for approximation operators. The positive nonlinear operators with maximum and product were presented by Bede. The Choquet integral has a wide range of applications in finance, the study of cooperative games, statistical mechanics, and potential theory. Approximation of max-product operators and Choquet integral operators, which can generate better approximation estimates than their classical counterparts, has been developed in recent years. In this study, firstly we introduce the Choquet integral in relation to Bernstein-Chlodowsky-Kantorovich operators and obtain quantitative estimates in uniform and pointwise approximation using these operators. Then the max-product type of these operators is denoted, and their approximation properties are investigated.

1. INTRODUCTION

In recent years, two significant study lines in function approximation have emerged; one of them is approximation with max-product operators and the other one is approximation using Choquet integral operators. Both research directions develop nonlinear approximation operators that may provide preferable approximation estimates compared to their linear (classical) equivalents. The Korovkin type theorems are formulated based on linear positive approximating operators or functionals in the field of approximation theory. In [1] and [2], B. Bede et al. introduced the concept of constructing nonlinear positive operators through the use of discrete linear approximating operators. In [11], S.G. Gal presented an open problem and introduced the max-product type Bernstein operators (Open Problem 5.5.4, pp. 324-326). The order of approximation of nonlinear approximating operators was examined in [4]-[12] as a result of this open problem.

The open problem formulated by S.G. Gal is directly relevance to the present work. In their work, the authors construct nonlinear, non-polynomial operators by replacing the conventional pair of operations (addition and multiplication) with alternative pairs inspired by fuzzy set theory and techniques from image processing. While the original problem addressed the approximation properties of nonlinear max-product type Bernstein operators, this study proposes a natural extension by introducing a new class of nonlinear operators, namely the Bernstein–Chlodowsky–Kantorovich operators of Choquet type. These operators are constructed by incorporating the Bernstein–Chlodowsky

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basis with Kantorovich-type modifications, and utilizing the Choquet integral with respect to monotone and submodular set functions. In this framework, our approach contributes to the present study of Gal's open problem by expanding the class of nonlinear approximation operators and improving the theory with Choquet-type structures.

The generalization of the Bernstein polynomials, known as the Bernstein-Chlodowsky polynomials, are denoted by

$$B_h(f)(x)=\sum_{k=0}^{h} s_{h,k}(x) f\left(\frac{kb_h}{h}\right),\ x\in[0,b_h],$$

where $b_h\to\infty$ as $h\to\infty$, $\lim_{h\to\infty}\frac{b_h}{h}=0$ and $s_{h,k}(x)=\binom{h}{k}\left(\frac{x}{b_h}\right)^k\left(1-\frac{x}{b_h}\right)^{h-k}$. Univariate and bivariate continuous functions were studied for their approximation properties in [8], [13], [14]. Operators of max-product–Choquet and Kantorovich–Choquet type have attracted significant attention due to their applicability in various real-world problems where classical linear approximation methods are inadequate. In particular, they have been effectively employed in signal and image reconstruction, especially in scenarios involving uncertainty or incomplete information. Furthermore, their integration with Choquet integrals allows for modeling non-additive phenomena, which is particularly relevant in fields such as fuzzy decision-making, multi-criteria analysis, and neural network approximation. The combination of these operator types enhances both the theoretical approximation framework and its practical versatility.

In this work, we define linear and nonlinear Bernstein-Chlodowsky-Kantorovich operators based on the Choquet integral and provide quantitative estimates for their uniform and pointwise approximation properties. This paper is structured as follows. Section 2 provides preliminary information regarding the Choquet integral. In Section 3, we first present the linear and non-linear Bernstein-Chlodowsky-Kantorovich operators of Choquet types and provide quantitative estimates for uniform and pointwise approximation.

2. Preliminaries

This section introduces fundamental principles and findings about the Choquet integral, which will be utilized throughout the principle part of the paper.

Definition 2.1. *Suppose that Ω is a nonempty set and that $\mathcal{C}$ is a σ-algebra of subsets in Ω.*

i. *Assume that $\mu:\mathcal{C}\to[0,+\infty]$. μ is a monotone set function (or capacity) when $\mu(\emptyset)=0$ and $X,Z\in\mathcal{C}$, where $X\subset Z$, meaning $\mu(X)\le\mu(Z)$. If the value of $\mu(X\cup Z)+(X\cap Z)$ is less than or equal to the value of $\mu(X)+\mu(Z)$, for every X and $Z\in\mathcal{C}$, then μ is referred to be submodular. When the value of $\mu(\mathcal{C})$ is equal to 1, μ is said to as normalized.*

ii. *Let μ be a monotone, normalized set function on $\mathcal{C}$. The Choquet integral is characterized as*

$$(C)\int_X g d\mu=\int_0^{+\infty}\mu\left(F_\beta(g)\bigcap X\right)d\beta+\int_{-\infty}^{0}\left[\mu\left(F_\beta(g)\bigcap X\right)-\mu(X)\right]d\beta,$$

where $F_\beta(g)=\{\omega\in\Omega; g(\omega)\ge\beta\}$. Let $(C)\int_X g d\mu$ be a real number, then g is said to be Choquet integrable on set A. Observe that if $g\ge 0$ on X, then in the above formula we obtain $\int_{-\infty}^{0}=0$.

The Choquet integral $(C)\int_X g d\mu$ simplifies to the Lebesgue integral if μ is the Lebesgue measure.

Here we present a few well-known characteristics of the Choquet integral.

Remark 2.1 (Properties of the Choquet Integral). *Let $\mu : \mathcal{C} \to [0, +\infty]$ be a monotone set function. Then, the Choquet integral satisfies the following properties:*

i. *For any positive value of a, we have:*

$$(C) \int_X ag \, d\mu = a \cdot (C) \int_X g \, d\mu.$$

ii. *For any $c \in \mathbb{R}$ and arbitrary measurable functions g, l, we have:*

$$(C) \int_X (g + l) \, d\mu \leq (C) \int_X g \, d\mu + (C) \int_X l \, d\mu,$$

implying that the Choquet integral is sublinear.

iii. *For each $g \leq l$ on X then*

$$(C) \int_X g \, d\mu \leq (C) \int_X l \, d\mu.$$

iv. *Let $g \geq 0$ and $X \subset Z$. Then*

$$(C) \int_X g \, d\mu \leq (C) \int_Z g \, d\mu.$$

Moreover, if μ is finitely subadditive,

$$(C) \int_{X \cup Z} g \, d\mu \leq (C) \int_X g \, d\mu + (C) \int_Z g \, d\mu.$$

v. *It is an immediate consequence of the Choquet integral that*

$$(C) \int_X 1 \, d\mu = \mu(X).$$

vi. *If $\mu(X) = \gamma(M(X))$, where $\gamma : [0, 1] \to [0, 1]$ is an increasing concave function with $\gamma(0) = 0$, $\gamma(1) = 1$, and M is a probability measure on a σ-algebra over Ω, then μ is a monotone and submodular set function.*

Example 2.1. *Let $X = \{x_1, x_2, x_3\}$ and consider a function $f : X \to \mathbb{R}_+$ given by*

$$f(x_1) = 2, \quad f(x_2) = 5, \quad f(x_3) = 3.$$

Let $\mu : 2^X \to [0, 1]$ be a set function defined as:

$$\mu(\emptyset) = 0, \quad \mu(\{x_1\}) = 0.2, \quad \mu(\{x_2\}) = 0.4, \quad \mu(\{x_3\}) = 0.3,$$

$$\mu(\{x_2, x_3\}) = 0.7, \quad \mu(X) = 1.$$

It is easy to verify that μ is both monotone and submodular.

Ordering the values of f in decreasing order:

$$f(x_{(1)}) = 5, \quad f(x_{(2)}) = 3, \quad f(x_{(3)}) = 2,$$

with corresponding sets $A_1 = \{x_2\}$, $A_2 = \{x_2, x_3\}$, and $A_3 = X$.

Then, the discrete Choquet integral is given by:

$$(C) \int f \, d\mu = (5 - 3) \cdot \mu(A_1) + (3 - 2) \cdot \mu(A_2) + (2 - 0) \cdot \mu(A_3),$$

$$= 2 \cdot 0.4 + 1 \cdot 0.7 + 2 \cdot 1 = 3.5.$$

This example illustrates the application of the Choquet integral using a monotone, submodular set function over a finite set.

3. Construction of the Operators

The sigma algebra of all Borel measurable subsets in $P(I)$ which represents the class of all subsets of I is denoted by $\mathcal{B}_I$. Here, $(\Gamma_{h,x})_{h\in\mathbb{N},x\in I}$ denotes a collection of families $\Gamma_{h,x} = \{\mu_{h,k,x}\}_{k=0}^{h}$, comprising monotone, submodular, and strictly positive set functions on $\mathcal{B}_I$, where $I = [0, b_h]$ for Bernstein-Chlodowsky-Kantorovich polynomials. A set function on $\mathcal{B}_I$ is considered strictly positive if, for any open subset $X \subset \mathbb{R}$ with $X \cap I = \emptyset$, it holds that $\mu(X \cap I) > 0$. In this section, we introduce Bernstein-Chlodowsky-Kantorovich operators of Choquet types and investigate some approximation properties of our new operator.

Definition 3.2 (Bernstein–Chlodowsky–Kantorovich Operators of Choquet Type)**.** *Let $\Gamma_{h,x} = \{\mu_{h,k,x}\}_{k=0}^{h}$ be a sequence of monotone and submodular set functions defined on the Borel σ-algebra $\mathcal{B}_I$, with $I = [0, b_h]$.*

Then, the Bernstein–Chlodowsky–Kantorovich operators of Choquet type (Choquet BC-Kantorovich operator)are defined as:

$$CK_{h,\Gamma_{h,x}}(g)(x) = \sum_{k=0}^{h} s_{h,k}(x) \cdot \frac{(C)\int_{b_h k/(h+1)}^{b_h(k+1)/(h+1)} g(t)\, d\mu_{h,k,x}(t)}{\mu_{h,k,x}\left(\left[\frac{b_h k}{h+1}, \frac{b_h(k+1)}{h+1}\right]\right)},$$

where $s_{h,k}(x)$ are given by:

$$s_{h,k}(x) = \binom{h}{k}\left(\frac{x}{b_h}\right)^{k}\left(1 - \frac{x}{b_h}\right)^{h-k}.$$

Here, the function $g : I \to \mathbb{R}_+$ is assumed to be $\mathcal{B}_I$-measurable and bounded on I, ensuring that the Choquet integral is well-defined on each subinterval.

Remark 3.2. *For all h, k, and x, if $\mu_{h,k,x} = M$, which M is the Lebesque measure, then the operators mentioned above are considered classical. Additionally, given that $\mu_{h,k,x} = \delta_{b_h k/h}$ (the Dirac measures), and considering that $\frac{b_h k}{h} \in \left[\frac{b_h k}{h+1}, \frac{b_h(k+1)}{h+1}\right]$, it follows immediately that $CK_{h,\Gamma_{h,x}}(g)(x)$ are the classical Bernstein-Chlodowsky polynomials.*

The following represents, for simplicity, any of the operators $CK_{h,\Gamma_{h,x}}(g)$ by $L_h(g)$.

Theorem 3.1. *Let $I = [0, b_h]$ and let $C_+^b(I)$ denote the space of all continuous, bounded, and positive-valued functions on I. Consider the operator $CK_{h,\Gamma_{h,x}}$ acting on $g \in C_+^b(I)$, and define the auxiliary function $\varphi_x(t) := |t - x|$.*

Then, for every $g \in C_+^b(I)$, $x \in I$, and $h \in \mathbb{N}$, the following estimate holds:

$$|L_h(g)(x) - g(x)| \le 2\,\omega_1\left(g; L_h(\varphi_x)(x)\right)_I, \tag{3.1}$$

where the modulus of continuity is defined by

$$\omega_1(g;\delta)_I := \sup\left\{|g(x) - g(y)| : x, y \in I,\ |x - y| \le \delta\right\}.$$

Proof. For $x \in I$, $h, k \in \mathbb{N}$, let us take $T_{h,k,x} : C_+^b(I) \to \mathbb{R}_+$ describe by

$$T_{h,k,x}(f) = (C)\int_{I_{k,h}} g(t) d\mu_{h,k,x}(t),\ f \in C_+^b(I),$$

where $I_{k,h} = \left[\frac{b_h k}{h+1}, \frac{b_h(k+1)}{h+1}\right]$ for $CK_{h,\Gamma_{h,x}}(g)(x)$. Lemma 3.1 in [5] and its proof show that since $T_{h,k,x}$ is positively homogeneous, sublinear, and monotonically increasing, we obtain

$$|\, T_{h,k,x}(g) - T_{h,k,x}(l)\, | \le T_{h,k,x}\left(|\, g - l\,|\right).$$

This implies that $L_h(\lambda g) = \lambda L_h(g)$, $L_h(g+l) \leq L_h(g) + L_h(l)$, $g \leq l$ on I implies $L_h(g) \leq L_h(l)$ on I, $\forall\, \lambda \geq 0$, $g, l \in C_+^b(I)$, $h \in \mathbb{N}, x \in I$ and that

$$\mid L_h(g)(x) - L_h(l)(x) \mid \leq L_h\left(\mid g - l \mid\right)(x). \tag{3.2}$$

By denoting $e_0(t) = 1$ for all $t \in I$, and by considering the characteristic in Remark 2.1, (i) and Remark 3.2 we obtain the following for any fixed x,

$$\mid L_h(g)(x) - g(x) \mid = \mid L_h\left(g(t)\right)(x) - L_h\left(g(x)\right)(x) \mid \leq L_h\left(\mid g(t) - g(x) \mid\right)(x). \tag{3.3}$$

However, by considering the characteristics of the modulus of continuity, we can conclude that for all $t, x \in I$ and $\delta > 0$, the following is hold

$$\mid g(t) - g(x) \mid \leq \omega_1\left(g; \parallel t - x \parallel\right)_I \leq \left[\frac{1}{\delta} \parallel t - x \parallel + 1\right] \omega_1\left(g; \delta\right)_I. \tag{3.4}$$

From (3.3) and applying L_n to (3.4), utilizing the features of L_n referenced following inequality (3.2), we obtain

$$\mid L_h(g)(x) - g(x) \mid \leq \left[\frac{1}{\delta} L_h(\varphi_x)(x) + 1\right] \omega_1(g; \delta)_I.$$

We are able to acquire the appropriate estimate by selecting the equation $\delta = L_h(\varphi_x)(x)$). □

Remark 3.3. *The proof of Theorem 3.1 relies on the positive homogeneity property of the Choquet integral. Consequently, the function g is required to be non-negative, i.e., $g \in C_+^b(I)$. Theorem 3.1 can be restated with the slightly modified operator described by*

$$L_h^*(g)(x) = L_h(g - m)(x) + m$$

, where $m \in \mathbb{R}$ represents a lower constraint for g, namely $g(x) \geq m$, for all $x \in I$, if g is of arbitrary sign on I. Indeed, this may be deduced directly from the relation $\omega_1\left(g - m; \delta\right)_I = \omega_1\left(g; \delta\right)_I$ and

$$L_h^*(g)(x) - g(x) = L_h(g - m)(x) - (g(x) - m).$$

Remark 3.4. *It is important to emphasize that, owing to the nonlinearity of the Choquet integral as discussed in Remarks 2.1, (i) and 3.2, the L_h operators in Theorem 3.1 are nonlinear, in contrast to classical situations. Particularly, when $\Gamma_{h,x}$ decreases to one element, we will denote $CK_{h,\Gamma_{h,x}}(g) := C_{h,\mu}(g)$. Subsequently, the estimate in (3.1) allows us to obtain concrete quantitative results for certain specific choices of $\Gamma_{h,x}$.*

Corollary 3.1. *Assume that the set function $\mu_{h,k,x}$ is given by $\mu := \sqrt{M}$ for all $h, k \in \mathbb{N}$ and $x \in I$, where M denotes the Lebesgue measure on I.*

Then, for every $g \in C_+^b(I)$, $x \in I$, and $h \in \mathbb{N}$, we have:

$$|C_{h,\mu}(g)(x) - g(x)| \leq 2\,\omega_1\left(g; \frac{\sqrt{x(b_h - x)}}{\sqrt{h}} + \frac{b_h}{h}\right)_I.$$

Proof. Based on Remark 2.1(vi) $\mu = \sqrt{M}$ is a monotone and submodular set function. Furthermore, it is obvious that μ is strictly positive. To evaluate $CK_{h,\mu}(\varphi_x)(x)$, let us indicate

$$\begin{aligned} C_{h,k}(x) =& \frac{(C)\int_{b_h k/(h+1)}^{b_h(k+1)/(h+1)} \mid t - x \mid d\mu_{h,k,x}(t)}{\mu_{h,k,x}\left(\left[\frac{b_h k}{h+1}, \frac{b_h(k+1)}{h+1}\right]\right)} \\ =& \frac{\sqrt{h+1}}{\sqrt{b_h}} (C) \int_{b_h k/(h+1)}^{b_h(k+1)/(h+1)} \mid t - x \mid d\mu_{h,k,x}(t). \end{aligned}$$

Three possibilities: 1. $x \in \left[\frac{b_h k}{h+1}, \frac{b_h(k+1)}{h+1}\right]$, 2. $0 \leq x < \frac{b_h k}{h+1}$, 3. $\frac{b_h(k+1)}{h+1} < x$

Case 1. Since $\mid t-x \mid \leq \frac{b_h}{h+1}$, for all $t, x \in I_{h,k} = \left[\frac{b_h k}{h+1}, \frac{b_h(k+1)}{h+1}\right]$ from Remark 2.1 (ii), we get

$$C_{h,k} \leq \frac{\sqrt{h+1}}{\sqrt{b_h}} \frac{b_h}{h+1} (C) \int_{b_h k/(h+1)}^{b_h(k+1)/(h+1)} 1.d\mu < \frac{b_h}{h}.$$

Case 2. We have $\mid t-x \mid = t-x$ and denoting

$$M(h,k,x,\beta) := \mu\left(\{t \in I_{h,k} : t \geq x+\beta\}\right),$$

we obtain

$$\begin{aligned}
C_{h,k}(x) =& \frac{\sqrt{h+1}}{\sqrt{b_h}} \int_0^\infty M(h,k,x,\beta)d\beta \\
=& \frac{\sqrt{h+1}}{\sqrt{b_h}} \int_0^{\frac{b_h(k+1)}{h+1}-x} M(h,k,x,\beta)d\beta \\
=& \frac{\sqrt{h+1}}{\sqrt{b_h}} \left(\int_0^{\frac{b_h k}{h+1}-x} M(h,k,x,\beta)d\beta + \int_{\frac{b_h k}{h+1}-x}^{\frac{b_h(k+1)}{h+1}-x} M(h,k,x,\beta)d\beta \right) \\
=& \left(\frac{b_h k}{h+1} - x\right) + \frac{\sqrt{h+1}}{\sqrt{b_h}} \int_{\frac{b_h k}{h+1}-x}^{\frac{b_h(k+1)}{h+1}-x} \sqrt{\frac{b_h(k+1)}{h+1} - x - \beta d\beta} \\
=& \left(\frac{b_h k}{h+1} - x\right) + \frac{\sqrt{h+1}}{\sqrt{b_h}} \int_{\frac{b_h k}{h+1}-x}^{\frac{b_h(k+1)}{h+1}-x} \sqrt{\eta} d\eta \\
=& \left(\frac{b_h k}{h+1} - x\right) + \frac{2}{3}\frac{b_h}{h+1} \\
=& \frac{h}{h+1}\left(\frac{b_h k}{h} - x\right) + \frac{2b_h - 3x}{3(h+1)} \\
\leq& \frac{h}{h+1}\left|\frac{b_h k}{h} - x\right| + \frac{2b_h}{3(h+1)} \\
\leq& \left|\frac{b_h k}{h} - x\right| + \frac{2b_h}{3(h+1)} < \left|\frac{k b_h}{h} - x\right| + \frac{2b_h}{3h}
\end{aligned}$$

Case 3. Since $\mid t-x \mid = x-t$, denoting $M(h,k,x,\beta) := \mu\left(\{t \in I_{h,k}; t \leq x-\beta\}\right)$ and reasoning as in the case (2), we get

$$\begin{aligned}
C_{h,k}(x) =& \frac{\sqrt{h+1}}{\sqrt{b_h}} \int_0^\infty M(h,k,x,\beta)d\beta \\
=& \frac{\sqrt{h+1}}{\sqrt{b_h}} \int_0^{x-\frac{b_h k}{h+1}} M(h,k,x,\beta)d\beta \\
=& \frac{\sqrt{h+1}}{\sqrt{b_h}} \left(\int_0^{x-\frac{b_h(k+1)}{h+1}} M(h,k,x,\beta)d\beta + \int_{x-\frac{b_h(k+1)}{h+1}}^{\frac{x-b_h k}{h+1}} M(h,k,x,\beta)d\beta \right)
\end{aligned}$$

$$= \left(x - \frac{b_h(k+1)}{h+1}\right) + \frac{\sqrt{h+1}}{\sqrt{b_h}} \int_0^{\frac{b_h}{h+1}} \sqrt{\eta} d\eta$$
$$= \left(x - \frac{b_h(k+1)}{h+1}\right) + \frac{2b_h}{3(h+1)} = \frac{h}{h+1}\left(x - \frac{kb_h}{h}\right) + \frac{3x - b_h}{3(h+1)}$$
$$\leq \frac{h}{h+1}\left|x - \frac{kb_h}{h}\right| + \frac{b_h}{h+1} \leq \left|x - \frac{kb_h}{h}\right| + \frac{b_h}{h+1} < \left|x - \frac{kb_h}{h}\right| + \frac{b_h}{h}$$

Collecting the estimates in the three cases (1), (2) and (3), then we obtain

$$C_{h,k}(x) \leq \left|\frac{kb_h}{h} - x\right| + \frac{b_h}{h}, \forall x \in [0, b_h]$$

which immediately implies

$$CK_{h,\mu}(\varphi_x)(x) \leq \sum_{k=0}^{n} s_{h,k}(x) \left[\left|\frac{kb_h}{h} - x\right| + \frac{b_h}{h}\right]$$
$$\leq \frac{\sqrt{x(b_h - x)}}{\sqrt{h}} + \frac{b_h}{h},$$

utilizing the estimate (3.1) from Theorem 3.1, we get the required estimate. In the final row of inequalities, we applied the established inequality $C_h(\varphi_x)(x) \leq \frac{\sqrt{x(b_h-x)}}{\sqrt{h}}$, when C_h denotes the classical Bernstein-Chlodowsky polynomials. □

Example 3.2. *The Choquet integral is recognized for providing the right-skewed average for convex functions, particularly when $\mu(E) = \sqrt{m(E)}$, the square root of the Lebesgue measure. The selected test function is $g(x) = e^x$, which is positive and convex, therefore illustrating the advantages of the Choquet integral.*

Indeed, while the classical Lebesgue mean is

$$M_{norm}(a, b) = \frac{e^a + e^b}{2}$$

the Choquet mean with capacity $\mu = \sqrt{m}$ is approximately

$$M_{choq}(a, b) = \frac{e^a + e^b + e^{\frac{a+b}{2}}}{3}$$

At this point, due to the convexity of e^x, we have $M_{choq}(a, b) \geq M_{norm}(a, b)$, which provides a closer approximation to the function's orijinal behavior.

Consequently, the expression

$$\mathrm{CK}_{h,\Gamma_{h,x}}(e^x)(x) = \sum_{k=0}^{h} s_{h,k}(x)\, M_{choq}\big(I_{k,h}\big),$$

converges uniformly to the function e^x when h increases.

Remark 3.5. *This example clearly demonstrates the improvement provided by the Choquet integral for convex functions. While the classical Kantorovich average exhibits a tendency toward underestimation, the Choquet average reduces underestimation by evaluating around $a + \frac{2}{3}(b-a)$. Thus, it has been confirmed by numerical results and graphs that the operator $\mathrm{CK}_{h,\Gamma_{h,x}}(e^x)(x)$ approaches the function e^x more quickly, especially for small values of h.*

Example 3.3 (Numerical Approaches). *The table below shows the numerical values and errors of the Choquet BC-Kantorovich operator for the function $g(x) = e^x$ on the interval $[0, 0.4]$.*

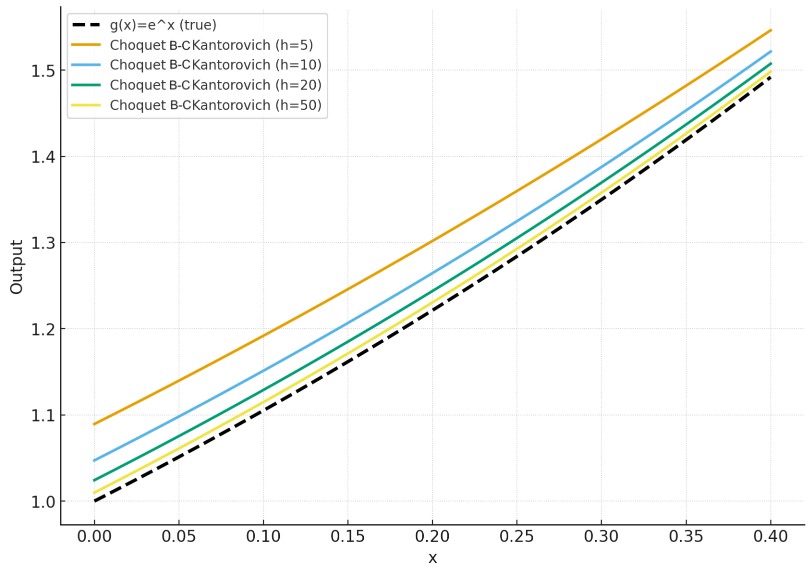

FIGURE 1. $CK_{h,\Gamma_{h,x}}(g)(x)$ approximation to e^x on $[0, 0.4]$

TABLE 1. Approach values

x	e^x (true)	h=5	h=10	h=20	h=50
0.00	1.00000	1.08942	1.04722	1.02429	1.00988
0.05	1.05127	1.13972	1.09814	1.07542	1.06111
0.10	1.10517	1.19186	1.15127	1.12897	1.11488
0.15	1.16183	1.24589	1.20670	1.18504	1.17131
0.20	1.22140	1.30186	1.26452	1.24376	1.23055
0.25	1.28403	1.35983	1.32483	1.30523	1.29271
0.30	1.34986	1.41984	1.38771	1.36958	1.35795
0.35	1.41907	1.48195	1.45326	1.43694	1.42641
0.40	1.49182	1.54622	1.52159	1.50744	1.49826

TABLE 2. Error values

x	$\lvert h=5-true \rvert$	$\lvert h=10-true \rvert$	$\lvert h=20-true \rvert$	$\lvert h=50-true \rvert$
0.00	0.08942	0.04722	0.02429	0.00988
0.05	0.08845	0.04687	0.02415	0.00984
0.10	0.08669	0.04610	0.02380	0.00971
0.15	0.08406	0.04486	0.02321	0.00948
0.20	0.08046	0.04312	0.02236	0.00914
0.25	0.07580	0.04080	0.02120	0.00869
0.30	0.06998	0.03785	0.01972	0.00809
0.35	0.06289	0.03419	0.01787	0.00735
0.40	0.05440	0.02976	0.01561	0.00643

In [6], the max-product operators of the Bernstein–Kantorovich type, beginning with classical linear operators were introduced and studied. In [4], the max-product Bernstein-Kantorovich-Choquet operators, with regard to $\Gamma_{h,x} = \{\mu_{h,k,x}\}_{k=0}^{h}$ is defined as

$$K^{(M)}_{h,\Gamma_{h,x}}(g)(x) = \frac{\bigvee_{k=0}^{h} \binom{h}{k} x^k (1-x)^{h-k} \frac{(C)\int_{k/(h+1)}^{(k+1)/(h+1)} g(t)\, d\mu_{h,k,x}(t)}{\mu_{h,k,x}([k/(h+1),(k+1)/(h+1)])}}{\bigvee_{k=0}^{h} \binom{h}{k} x^k (1-x)^{h-k}}.$$

Now, we give the definition of Bernstein-Chlodowsky-Kantorovich operators of Choquet types and investigate some approximation properties. Firstly, let us introduce the definition of Bernstein-Chlodowsky-Kantorovich-Choquet operators as follows

$$CK^{(M)}_{h,\Gamma_{h,x}}\left(g\right)\left(x\right)=\frac{\bigvee_{k=0}^{h}\binom{h}{k}\left(\frac{x}{b_h}\right)^{k}(1-\frac{x}{b_h})^{h-k}\frac{(C)\int_{b_h k/(h+1)}^{b_h(k+1)/(h+1)}g(t)d\mu_{h,k,x}(t)}{\mu_{h,k,x}([b_h k/(n+1),b_h(k+1)/(h+1)])}}{\bigvee_{k=0}^{h}\binom{h}{k}\left(\frac{x}{b_h}\right)^{k}(1-\frac{x}{b_h})^{h-k}}.$$

If we assume that $g: I\to\mathbb{R}_+$ is a $\mathcal{B}_I$-measurable function, bounded on $I=[0,b_h]$ for $CK^{(M)}_{h,\Gamma_{h,x}}\left(g\right)\left(x\right)$ then this operator is well-defined. If $\mu_{h,k,x}=m$ for all h,k and x, where m is the Lebesgue measure, the previous operators convert into the max-product Kantorovich-type operators, as discussed in [7].

Let us now present the first result of this section, denoted by $L_h^{(M)}(g)$ for simplicity, for any of the operators $CK^{(M)}_{h,\Gamma_{h,x}}\left(g\right)$.

Theorem 3.2. *Let $I=[0,b_h]$, and let $C_+^b(I)$ denote the space of all bounded, continuous, and non-negative real-valued functions defined on I. Then, for every $g\in C_+^b(I)$, $x\in I$, and $h\in\mathbb{N}$, the following inequality holds:*

$$\left|L_h^{(M)}(g)(x)-g(x)\right|\le 2\,\omega_1\left(g;\,L_h^{(M)}(\varphi_x)(x)\right),\quad \text{where } \varphi_x(t)=|t-x|. \tag{3.5}$$

Here, $\omega_1(g;\delta)$ denotes the modulus of continuity of first order, defined by:

$$\omega_1(g;\delta)=\sup\left\{|g(t)-g(s)|\,:\,t,s\in I,\,|t-s|\le\delta\right\}.$$

Proof. For $x\in I$, $h,k\in\mathbb{N}$, now we can consider about $T_{h,k,x}: C_+^b(I)\to\mathbb{R}_+$ described by

$$T_{h,k,x}(f)=C\int_{I_{k,h}}g(t)d\mu_{h,k,x}(t)/\mu_{h,k,x}(I_{k,n}),\ g\in C_+^b(I),$$

where $I_{k,h}=\left[\frac{b_h k}{h+1},\frac{b_h(k+1)}{h+1}\right]$ for $CK_{h,\Gamma_{h,x}}\left(g\right)\left(x\right)$.

Let us denote

$$s_{h,k}(x)=\binom{h}{k}\left(\frac{x}{b_h}\right)^{k}\left(1-\frac{x}{b_h}\right)^{h-k}.$$

Taking into account the properties of the Choquet integral in Remark 2.1, we can say that $T_{h,k,x}$ is positively homogeneous, sublinear, and monotonically increasing, multiplying it by $p_{n,k}(x)$, passing to supremum after k and finially dividing by

$$\bigvee_{k=0}^{h}s_{h,k}(x),$$

we immediately get that

$$L_h^{(M)}(g)$$

shares the same properties, that is,

$$L_h^{(M)}(\lambda g)=\lambda L_h^{(M)}(g),$$

$$L_h^{(M)}(g+l)\le L_h^{(M)}(g)+L_h^{(M)}(l),$$

$g\le l$ on I implies $L_h^{(M)}(g)\le L_h^{(M)}(l)$ on I, for all $\lambda\ge 0$, $g,l\in C_+^b(I)$, $h\in\mathbb{N}$, $x\in I$. Then we get that

$$\left|L_h^{(M)}(g)(x)-L_h^{(M)}(l)(x)\right|\le L_h^{(M)}(|g-l|)(x). \tag{3.6}$$

Indicating $e_0(t) = 1$, for all $t \in I$, since obviously $L_h(e_0)(x) = 1$, for all $x \in I$ and considering the property in Remark 2.1 for any fixed x, we obtain

$$\begin{aligned}\left|L_h^{(M)}(g)(x) - g(x)\right| &= \left|L_h^{(M)}(g(t))(x) - L_h^{(M)}(g(x))(x)\right| \\ &\leq L_h^{(M)}\left(|g(t) - g(x)|\right)(x).\end{aligned} \tag{3.7}$$

Considering the characteristics of the modulus of continuity, for all $t, x \in I$ and $\delta > 0$, we obtain

$$| g(t) - g(x) | \leq \omega_1\left(g; \| t - x \|\right)_I \leq \left[\frac{1}{\delta} \| t - x \| + 1\right] \omega_1\left(g; \delta\right)_I . \tag{3.8}$$

Now from (3.7) and applying $L_h^{(M)}$ to (3.8), by the properties of $L_h^{(M)}$ given after the inequality (3.6), we immediately obtain

$$\left|L_h^{(M)}(g)(x) - g(x)\right| \leq \left[\frac{1}{\delta} L_h^{(M)}\left(\varphi_x\right)(x) + 1\right] \omega_1\left(g; \delta\right)_I .$$

Choosing here $\delta = L_h^{(M)}\left(\varphi_x\right)(x)$, we obtained the desired estimate.

□

Corollary 3.2. *Let $g \in C_+^b([0, b_h])$, $x \in [0, b_h]$, and $h \in \mathbb{N}$. Assume that the monotone set function μ is given by $\mu := \sqrt{m}$, where m denotes the Lebesgue measure on $[0, b_h]$.*

Then, the following inequality holds:

$$\left|CK_{h,\mu}^{(M)}(g)(x) - g(x)\right| \leq 2\,\omega_1\left(g;\, 6\frac{b_h}{\sqrt{h+1}} + \frac{1}{h}\right)_{[0,b_h]} .$$

Proof. From Remark 2.1, $\mu = \sqrt{m}$ is a monotone and submodular set function. Additionally, it is clear that μ is strictly positive. Indicate that $s_{h,k}(x) = \binom{h}{k}\left(\frac{x}{b_h}\right)^k\left(1 - \frac{x}{b_h}\right)^{h-k}$. To estimate $CK_{h,\mu}^{(M)}(\varphi_x)(x)$, let us indicate

$$C_{h,k}(x) = \frac{(C)\int_{b_h k/(h+1)}^{b_h(k+1)/(h+1)} | t - x | \, d\mu_{h,k,x}(t)}{\mu_{h,k,x}\left(\left[\frac{b_h k}{h+1}, \frac{b_h(k+1)}{h+1}\right]\right)}.$$

As demonstrated in the proof of Corollary 3.1, we obtain

$$C_{h,k}(x) \leq \left|\frac{kb_h}{h} - x\right| + \frac{b_h}{h}$$

which immediately implies

$$\begin{aligned}CK_{h,\mu}^{(M)}(\varphi_x)(x) \leq& \frac{\bigvee_{k=0}^{h} s_{h,k}(x)\left(\left|\frac{kb_h}{h} - x\right| + \frac{b_h}{h}\right)}{\bigvee_{k=0}^{h} s_{h,k}(x)} \\ \leq& \frac{\bigvee_{k=0}^{h} p_{h,k}(x)\left(\left|\frac{kb_h}{h} - x\right|\right)}{\bigvee_{k=0}^{h} s_{h,k}(x)} + \frac{b_h}{h}\end{aligned}$$

where the estimate has been applied (see [8])

$$\frac{\bigvee_{k=0}^{h} s_{h,k}(x)\left(\left|\frac{kb_h}{h} - x\right|\right)}{\bigvee_{k=0}^{h} s_{h,k}(x)} \leq \frac{6b_h}{\sqrt{h+1}}.$$

The expected result is now also obtained by applying the estimate (3.5) in Theorem 3.2.

□

4. Conclusions

Recent research has introduced two novel approaches for function approximation: max-product operators and Choquet integral operators, introduce nonlinear approximation operators which produce better estimates compared to linear methods.The max-product operators are formally associated with conventional linear and positive operators by substituting the sum in their expressions with the maximum (supremum). The Choquet integral operators are characterized by the substitution of the classical linear integral with the nonlinear Choquet integral in the expressions of the integral operators. This methodology has prospective applications in statistical mechanics, potential theory, co-operative games, decision-making under risk and uncertainty, finance, economics, and insurance.We present the Choquet integral in connection with Bernstein–Chlodowsky-Kantorovich operators and establish quantitative estimates for uniform and pointwise approximation utilizing these operators. The maximum product variant of these operators is defined and their approximation characteristics are examined.

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