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On Quaternion-Gaussian Third-order Jacobsthal Polynomials

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ABSTRACT. In this paper, we define Gaussian third-order Jacobsthal quaternion polynomials and Gaussian third-order Jacobsthal-Lucas quaternion polynomials. We also investigate some properties of these quaternion polynomials.

1. INTRODUCTION

Gaussian number, investigated by Carl Gauss in 1832, is a complex number with integer coefficients. In 2013, Aşci and Gürel introduced in [1] the concept of complex Jacobsthal numbers (called Gaussian Jacobsthal numbers) and its generalization Gaussian Jacobsthal polynomials [2]. Then, Gaussian Jacobsthal numbers and generalized Gaussian Jacobsthal numbers are studied by many authors. Some example of these studies can be found in [10, 11, 12, 13], among others.

The nth Gaussian third-order Jacobsthal number is defined by the relation

$$JG_n^{(3)} = JG_{n-1}^{(3)} + JG_{n-2}^{(3)} + 2JG_{n-3}^{(3)}, \quad n \geq 3,$$

with initial conditions $JG_0^{(3)} = 0$, $JG_1^{(3)} = 1$ and $JG_2^{(3)} = 1 + i$ (see [5]).

It is easy to see that $JG_n^{(3)} = J_n^{(3)} + iJ_{n-1}^{(3)}$ and $J_{-1}^{(3)} = 0$, where $J_n^{(3)}$ is the nth third-order Jacobsthal number defined recursively by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$ with $J_0^{(3)} = 0$ and $J_1^{(3)} = J_2^{(3)} = 1$.

Similarly, the nth Gaussian modified third-order Jacobsthal number is defined by the relation

$$KG_n^{(3)} = KG_{n-1}^{(3)} + KG_{n-2}^{(3)} + 2KG_{n-3}^{(3)}, \quad n \geq 3,$$

with initial conditions $KG_0^{(3)} = 3 - \frac{1}{2}i$, $KG_1^{(3)} = 1 + 3i$ and $KG_2^{(3)} = 3 + i$.

It is clear that $KG_n^{(3)} = K_n^{(3)} + iK_{n-1}^{(3)}$ and $K_{-1}^{(3)} = -\frac{1}{2}$, where $K_n^{(3)}$ is the nth modified third-order Jacobsthal number defined recursively by $K_n^{(3)} = K_{n-1}^{(3)} + K_{n-2}^{(3)} + 2K_{n-3}^{(3)}$ with $K_0^{(3)} = 3$, $K_1^{(3)} = 1$ and $K_2^{(3)} = 3$. For more details on third-order Jacobsthal and modified third-order Jacobsthal numbers, see [3, 7].

Furthermore, the third-order Jacobsthal polynomials studied by Morales in [8] are defined by the relation

$$J_n^{(3)}(x) = (x-1)J_{n-1}^{(3)}(x) + (x-1)J_{n-2}^{(3)}(x) + xJ_{n-3}^{(3)}(x), \quad n \geq 3,$$

with initial conditions $J_0^{(3)}(x) = 0$, $J_1^{(3)}(x) = 1$ and $J_2^{(3)}(x) = x - 1$, for any real variable x such that $x^3 \neq 1$.

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The modified third-order Jacobsthal polynomials are defined by the relation

$$K_n^{(3)}(x) = (x-1)K_{n-1}^{(3)}(x) + (x-1)K_{n-2}^{(3)}(x) + xK_{n-3}^{(3)}(x), \quad n \geq 3,$$

with initial conditions $K_0^{(3)}(x) = 3$, $K_1^{(3)}(x) = x-1$ and $K_2^{(3)}(x) = x^2 - 1$.

In [6], Morales studied Gaussian third-order Jacobsthal polynomials as follows:

The Gaussian third-order Jacobsthal polynomials are defined by the relation

$$JG_n^{(3)}(x) = (x-1)JG_{n-1}^{(3)}(x) + (x-1)JG_{n-2}^{(3)}(x) + xJG_{n-3}^{(3)}(x), \quad n \geq 3, \tag{1.1}$$

with initial conditions $JG_0^{(3)}(x) = 0$, $JG_1^{(3)}(x) = 1$ and $JG_2^{(3)}(x) = x - 1 + i$.

Moreover, it is easy to see that $JG_n^{(3)}(x) = J_n^{(3)}(x) + iJ_{n-1}^{(3)}(x)$. Setting $x = 2$ in the Eq. (1.1), the Gaussian third-order Jacobsthal number $JG_n^{(3)}$ can be obtained.

The Binet-type formula for the Gaussian third-order Jacobsthal polynomials is given by

$$JG_n^{(3)}(x) = \frac{x^n(x+i)}{x^2+x+1} - \frac{\omega_1^n(\omega_1+i)}{(x-\omega_1)(\omega_1-\omega_2)} + \frac{\omega_2^n(\omega_2+i)}{(x-\omega_2)(\omega_1-\omega_2)}, \tag{1.2}$$

where ω_1 and ω_2 are the roots of the equation $\lambda^2 + \lambda + 1 = 0$.

The Gaussian modified third-order Jacobsthal polynomials are defined by the relation

$$KG_n^{(3)}(x) = (x-1)KG_{n-1}^{(3)}(x) + (x-1)KG_{n-2}^{(3)}(x) + xKG_{n-3}^{(3)}(x), \quad n \geq 3, \tag{1.3}$$

with initial conditions $KG_0^{(3)}(x) = 3 + \left(\frac{1}{x} - 1\right)i$, $KG_1^{(3)}(x) = x - 1 + 3i$ and $KG_2^{(3)}(x) = x^2 - 1 + (x-1)i$.

Furthermore, it is clear that $KG_n^{(3)}(x) = K_n^{(3)}(x) + iK_{n-1}^{(3)}(x)$. Setting $x = 2$ in the Eq. (1.3), the Gaussian modified third-order Jacobsthal number $KG_n^{(3)}$ can be obtained.

The Binet-type formula for the Gaussian modified third-order Jacobsthal polynomials is given by

$$KG_n^{(3)}(x) = x^{n-1}(x+i) + \omega_1^{n-1}(\omega_1+i) + \omega_2^{n-1}(\omega_2+i), \tag{1.4}$$

where ω_1 and ω_2 are same as defined in Eq. (1.2).

Quaternions, four-dimensional hyper-complex numbers, introduced by Hamilton in 1843. These numbers have found widespread application in quantum physics, computer graphics, robotics and signal processing.

A quaternion Q is of the form

$$Q = Q_0 + Q_1 i + Q_2 j + Q_3 k,$$

where Q_0, Q_1, Q_2, Q_3 are real numbers, and i, j, k are quaternionic units which satisfy the equalities

$$i^2 = j^2 = k^2 = ijk = -1, \; ij = k = -ji, \; jk = i = -ki, \; ki = j = -ik. \tag{1.5}$$

The set of all quaternions denoted by $\mathbb{R}[i,j,k]$ is a non-commutative associative algebra over the real numbers $\mathbb{R}$. For a survey on quaternions, we refer the reader to [14].

In [9], Morales defined the third-order Jacobsthal quaternions as

$$JQ_n^{(3)} = J_n^{(3)} + J_{n+1}^{(3)} i + J_{n+2}^{(3)} j + J_{n+3}^{(3)} k, \quad n \geq 0,$$

here $J_n^{(3)}$ is the nth third-order Jacobsthal number and i, j, k are quaternionic units which satisfy the rules (1.5).

The main objective of this paper is to define and study Gaussian third-order Jacobsthal and modified third-order Jacobsthal quaternion polynomials in the sense of [4, 15].

We shall give recurrence relations, Binet's formulas, generating functions and summation formulas involving these quaternion polynomials.

2. The Gaussian third-order Jacobsthal quaternion polynomials

In this section, we first give the definitions of Gaussian third-order Jacobsthal quaternion polynomials and Gaussian modified third-order Jacobsthal quaternion polynomials. We then obtain some results for these special quaternion polynomials.

The Gaussian third-order Jacobsthal quaternion polynomials $\{JGQ_n^{(3)}\}_{n\geq 0}$ and Gaussian modified third-order Jacobsthal quaternion polynomials $\{KGQ_n^{(3)}\}_{n\geq 0}$ are defined by

$$JGQ_n^{(3)}(x) = JG_n^{(3)}(x) + JG_{n+1}^{(3)}(x)i + JG_{n+2}^{(3)}(x)j + JG_{n+3}^{(3)}(x)k \tag{2.6}$$

and

$$KGQ_n^{(3)}(x) = KG_n^{(3)}(x) + KG_{n+1}^{(3)}(x)i + KG_{n+2}^{(3)}(x)j + KG_{n+3}^{(3)}(x)k, \tag{2.7}$$

respectively, $JG_n^{(3)}(x)$ is the nth Gaussian third-order Jacobsthal polynomial, $KG_n^{(3)}(x)$ is the nth Gaussian modified third-order Jacobsthal polynomial.

It is easy to see that the nth Gaussian third-order Jacobsthal quaternion polynomial is defined recursively by

$$JGQ_n^{(3)}(x) = (x-1)JGQ_{n-1}^{(3)}(x) + (x-1)JGQ_{n-2}^{(3)}(x) + xJGQ_{n-3}^{(3)}(x), \tag{2.8}$$

with initial conditions

$$\begin{aligned} JGQ_0^{(3)}(x) &= i + (x^2 - x + 1)k \\ JGQ_1^{(3)}(x) &= (x-1)i + (x^3 - x^2 + x)k \\ JGQ_2^{(3)}(x) &= (x^2 - x + 1)i + (x^4 - x^3 + x^2 - 1)k. \end{aligned}$$

Similarly, the nth Gaussian modified third-order Jacobsthal quaternion polynomial is defined recursively by

$$KGQ_n^{(3)}(x) = (x-1)KGQ_{n-1}^{(3)}(x) + (x-1)KGQ_{n-2}^{(3)}(x) + xKGQ_{n-3}^{(3)}(x), \tag{2.9}$$

with initial conditions

$$\begin{aligned} KGQ_0^{(3)}(x) &= \left(x + \frac{1}{x} - 2\right) i + (x^3 + x + 1)k \\ KGQ_1^{(3)}(x) &= (x^2 + 2)i + (x^4 + x^2 - 2)k \\ KGQ_2^{(3)}(x) &= (x^3 + x + 1)i + (x^5 + x^3 + 1)k. \end{aligned}$$

It must be noted that if we set $x = 2$ in Eqs. (2.6) and (2.8), we obtain Gaussian third-order Jacobsthal quaternions

$$JGQ_n^{(3)} = JGQ_{n-1}^{(3)} + JGQ_{n-2}^{(3)} + 2JGQ_{n-3}^{(3)},\ n \geq 3,$$

with initial conditions $JGQ_0^{(3)} = i + 3k$, $JGQ_1^{(3)} = i + 6k$, $JGQ_2^{(3)} = 3i + 11k$, and if we set $x = 2$ in Eqs. (2.7) and (2.9), we obtain Gaussian modified third-order Jacobsthal quaternions

$$KGQ_n^{(3)} = KGQ_{n-1}^{(3)} + KGQ_{n-2}^{(3)} + 2KGQ_{n-3}^{(3)},\ n \geq 3,$$

with initial conditions $KGQ_0^{(3)} = \frac{i}{2} + 11k$, $KGQ_1^{(3)} = 6 + 18k$ and $KGQ_2^{(3)} = 11i + 41k$.

Let ω_1 and ω_2 be the roots of the characteristic equation $\lambda^2 + \lambda + 1 = 0$ on the recurrence relation (2.8) of Gaussian third-order Jacobsthal quaternion polynomials. These roots satisfy the following rules $\omega_1 + \omega_2 = -1$, $\omega_1\omega_2 = 1$ and $\omega_1^3 = \omega_2^3 = 1$.

For simplicity of notation, let

$$Z_{n+1}(x) = \frac{1}{\omega_1 - \omega_2}\left((\omega_2 - x)\omega_1^{n+1} - (\omega_1 - x)\omega_2^{n+1}\right), \tag{2.10}$$
$$Y_n = \omega_1^n + \omega_2^n.$$

Then, we can write

$$J_n^{(3)}(x) = \frac{1}{x^2+x+1}\left(x^{n+1} + Z_{n+1}(x)\right)$$

and

$$K_n^{(3)}(x) = x^n + Y_n.$$

Furthermore, we have $Z_{n+1}(x) = -Z_n(x) - Z_{n-1}(x)$, $Z_1(x) = -x$ and $Z_2(x) = x + 1$.

We now give the Binet's formulas for the Gaussian third-order Jacobsthal and modified third-order Jacobsthal quaternion polynomials in the following theorem.

Theorem 2.1. *The Binet's formulas for the Gaussian third-order Jacobsthal and modified third-order Jacobsthal quaternion polynomials are given by*

$$JGQ_n^{(3)}(x) = \frac{1}{x^2+x+1}\left\{\begin{array}{c} JGQ_2^{(3)}(x)\left(x^n + Z_n(x)\right) \\ +JGQ_1^{(3)}(x)\left(x^n - xZ_n(x) - Z_{n-1}(x)\right) \\ +JGQ_0^{(3)}(x)\left(x^n + xZ_{n-1}(x)\right) \end{array}\right\}$$

and

$$KGQ_n^{(3)}(x) = \frac{1}{x^2+x+1}\left\{\begin{array}{c} KGQ_2^{(3)}(x)\left(x^n + Z_n(x)\right) \\ +KGQ_1^{(3)}(x)\left(x^n - xZ_n(x) - Z_{n-1}(x)\right) \\ +KGQ_0^{(3)}(x)\left(x^n + xZ_{n-1}(x)\right) \end{array}\right\}$$

with $Z_n(x)$ as in Eq. (2.10).

Proof. From the general solution for the recurrence relation and using initial conditions the desired results can be obtained easily. □

Note that we can also write the Binet's formula $JGQ_n^{(3)}(x)$ as follows:

$$JGQ_n^{(3)}(x) = J_{n-1}^{(3)}(x)JGQ_2^{(3)}(x) + \left[J_n^{(3)}(x) - (x-1)J_{n-1}^{(3)}(x)\right]JGQ_1^{(3)}(x) + xJ_{n-2}^{(3)}(x)JGQ_0^{(3)}(x),$$

where $J_n^{(3)}(x)$ is the nth third-order Jacobsthal polynomial.

If we set $x = 2$ in $JGQ_n^{(3)}(x)$ using Theorem 2.1, we obtain the Binet's formula for the Gaussian third-order Jacobsthal quaternions as follow:

$$JGQ_n^{(3)} = J_{n-1}^{(3)}JGQ_2^{(3)} + \left(J_n^{(3)} - J_{n-1}^{(3)}\right)JGQ_1^{(3)}(x) + 2J_{n-2}^{(3)}JGQ_0^{(3)},$$

where $J_n^{(3)}$ is the nth third-order Jacobsthal number.

Now, the ordinary generating functions and exponential generating functions for the Gaussian third-order Jacobsthal quaternion polynomials and Gaussian modified third-order Jacobsthal quaternion polynomials are given in the following results, respectively.

Theorem 2.2. *The ordinary generating functions for the Gaussian third-order Jacobsthal quaternion polynomials and Gaussian modified third-order Jacobsthal quaternion polynomials are given by*

$$g(J;t) = \frac{i + (x^2-x+1)k + \left((x^2-x+1)k\right)t + \left(i + (x^2-x)k\right)t^2}{1 - (x-1)t - (x-1)t^2 - xt^3} \tag{2.11}$$

and

$$g(K;t)=\frac{\left\{\begin{array}{c}\left(x+\frac{1}{x}-2\right)i+(x^3+x+1)k+\left(\left(3x+\frac{1}{x}-1\right)i+(x^3-1)k\right)t\\ +\left(\left(x+\frac{1}{x}\right)i+(x^3+2x)k\right)t^2\end{array}\right\}}{1-(x-1)t-(x-1)t^2-xt^3}, \tag{2.12}$$

respectively.

Proof. Let $g(J;t)$ be the generating function for the Gaussian third-order Jacobsthal quaternion polynomials. Then, we write

$$g(J;t)=\sum_{n=0}^{\infty}JGQ_n^{(3)}(x)t^n=JGQ_0^{(3)}(x)+JGQ_1^{(3)}(x)t+JGQ_2^{(3)}(x)t^2+\cdots \tag{2.13}$$

Multiplying the Eq. (2.13) with $-(x-1)t$, $-(x-1)t^2$ and $-xt^3$, respectively, we get

$$-(x-1)tg(J;t)=-(x-1)JGQ_0^{(3)}(x)t-(x-1)JGQ_1^{(3)}(x)t^2-\cdots, \tag{2.14}$$

$$-(x-1)t^2g(J;t)=-(x-1)JGQ_0^{(3)}(x)t^2-(x-1)JGQ_1^{(3)}(x)t^3-\cdots \tag{2.15}$$

and

$$-xt^3g(J;t)=-xJGQ_0^{(3)}(x)t^3-xJGQ_1^{(3)}(x)t^4-\cdots. \tag{2.16}$$

Then, adding the Eqs. (2.13), (2.14), (2.15) and (2.16), we obtain

$$\begin{aligned}&\left[1-(x-1)t-(x-1)t^2-xt^3\right]g(J;t)\\ &=JGQ_0^{(3)}(x)+\left[JGQ_1^{(3)}(x)-(x-1)JGQ_0^{(3)}(x)\right]t\\ &\quad+\left[JGQ_2^{(3)}(x)-(x-1)JGQ_1^{(3)}(x)-(x-1)JGQ_0^{(3)}(x)\right]t^2\\ &\quad+\sum_{n=3}^{\infty}\left[JGQ_n^{(3)}(x)-(x-1)JGQ_{n-1}^{(3)}(x)-(x-1)JGQ_{n-2}^{(3)}(x)-xJGQ_{n-3}^{(3)}(x)\right]t^n.\end{aligned}$$

From the Eq. (2.8), we get

$$\begin{aligned}&(1-(x-1)t-(x-1)t^2-xt^3)g(J;t)\\ &\qquad=JGQ_0^{(3)}(x)+\left[JGQ_1^{(3)}(x)-(x-1)JGQ_0^{(3)}(x)\right]t\\ &\qquad\quad+\left[JGQ_2^{(3)}(x)-(x-1)JGQ_1^{(3)}(x)-(x-1)JGQ_0^{(3)}(x)\right]t^2.\end{aligned}$$

Using the initial conditions $JGQ_0^{(3)}(x)$, $JGQ_1^{(3)}(x)$ and $JGQ_2^{(3)}(x)$, we have

$$\begin{aligned}(1-(x-1)t-(x-1)t^2-xt^3)g(J;t)&=i+(x^2-x+1)k+\left(x^2-x+1\right)t\\ &\quad+\left(i+(x^2-x)k\right)t^2\end{aligned}$$

which completes the proof of the first statement.

The second statement of the theorem can be proved in a similar manner. □

If we set $x=2$ in Eq. (2.11), we obtain the ordinary generating function for the Gaussian third-order Jacobsthal quaternions as follow:

$$g(J;t)=\frac{i+3k+(3k)t+(i+2k)t^2}{1-t-t^2-2t^3}.$$

Furthermore, if we set $x=2$ in Eq. (2.12), we obtain the ordinary generating function for the Gaussian modified third-order Jacobsthal quaternions as follow:

$$g(K;t)=\frac{\frac{i}{2}+11k+\left(\frac{9i}{2}+7k\right)t+\left(\frac{5i}{2}+12k\right)t^2}{1-t-t^2-2t^3}.$$

Theorem 2.3. *The exponential generating functions for the Gaussian third-order Jacobsthal quaternion polynomials and Gaussian modified third-order Jacobsthal quaternion polynomials are given by*

$$J(t)=\frac{1}{\Phi(x)}\left\{\begin{array}{c}\left[JGQ_2^{(3)}(x)+JGQ_1^{(3)}(x)+JGQ_0^{(3)}(x)\right]e^{xt}\\ +\frac{1}{\omega_1-\omega_2}\left[(\omega_2-x)\Psi_J(\omega_2)e^{\omega_1 t}-(\omega_1-x)\Psi_J(\omega_1)e^{\omega_2 t}\right]\end{array}\right\}$$

and

$$K(t)=\frac{1}{\Phi(x)}\left\{\begin{array}{c}\left[KGQ_2^{(3)}(x)+KGQ_1^{(3)}(x)+KGQ_0^{(3)}(x)\right]e^{xt}\\ +\frac{1}{\omega_1-\omega_2}\left[(\omega_2-x)\Psi_K(\omega_2)e^{\omega_1 t}-(\omega_1-x)\Psi_K(\omega_1)e^{\omega_2 t}\right]\end{array}\right\},$$

respectively. Further, $\Phi(x)=x^2+x+1$ *and*

$$\Psi_J(\omega)=JGQ_2^{(3)}(x)-(x+\omega)JGQ_1^{(3)}(x)+x\omega JGQ_0^{(3)}(x),$$
$$\Psi_K(\omega)=KGQ_2^{(3)}(x)-(x+\omega)KGQ_1^{(3)}(x)+x\omega KGQ_0^{(3)}(x).$$

Proof. Using $\Phi(x)=x^2+x+1$, the Binet's formula in Theorem 2.1 of the Gaussian third-order Jacobsthal quaternion polynomials and Eq. (2.10), we have

$$\begin{aligned}
&\Phi(x)\sum_{n=0}^{\infty}JGQ_n^{(3)}(x)\frac{t^n}{n!}\\
&=JGQ_2^{(3)}(x)\sum_{n=0}^{\infty}\left(x^n+Z_n(x)\right)\frac{t^n}{n!}\\
&+JGQ_1^{(3)}(x)\sum_{n=0}^{\infty}\left(x^n-xZ_n(x)-Z_{n-1}(x)\right)\frac{t^n}{n!}\\
&+JGQ_0^{(3)}(x)\sum_{n=0}^{\infty}\left(x^n+xZ_{n-1}(x)\right)\frac{t^n}{n!}\\
&=\left[JGQ_2^{(3)}(x)+JGQ_1^{(3)}(x)+JGQ_0^{(3)}(x)\right]\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\\
&+\frac{1}{\omega_1-\omega_2}\left[(\omega_2-x)\Psi_J(\omega_2)\sum_{n=0}^{\infty}\frac{(\omega_1 t)^n}{n!}-(\omega_1-x)\Psi_J(\omega_1)\sum_{n=0}^{\infty}\frac{(\omega_2 t)^n}{n!}\right],
\end{aligned}$$

where $\Psi_J(\omega)=JGQ_2^{(3)}(x)-(x+\omega)JGQ_1^{(3)}(x)+x\omega JGQ_0^{(3)}(x)$. Thus, the proof of the first statement is completed. The second statement can be proved using the Binet's formula of the Gaussian modified third-order Jacobsthal quaternion polynomials in a similar manner. □

Theorem 2.4. *For* $n\geq 0$, *the following identities hold*

$$\sum_{r=o}^{n}JGQ_r^{(3)}(x)=\frac{1}{3(x-1)}\left\{\begin{array}{c}JGQ_{n+2}^{(3)}(x)-(x-2)JGQ_{n+1}^{(3)}(x)\\ +xJGQ_n^{(3)}(x)-JGQ_2^{(3)}(x)\\ +(x-2)JGQ_1^{(3)}(x)+(2x-3)JGQ_0^{(3)}(x)\end{array}\right\}\tag{2.17}$$

and

$$\sum_{r=o}^{n}KGQ_r^{(3)}(x)=\frac{1}{3(x-1)}\left\{\begin{array}{c}KGQ_{n+2}^{(3)}(x)-(x-2)KGQ_{n+1}^{(3)}(x)\\ +xKGQ_n^{(3)}(x)-KGQ_2^{(3)}(x)\\ +(x-2)KGQ_1^{(3)}(x)+(2x-3)KGQ_0^{(3)}(x)\end{array}\right\}.\tag{2.18}$$

Proof. Using Eq. (2.8) and initial conditions of $JGQ_n^{(3)}(x)$, we obtain

$$\begin{aligned}\sum_{r=0}^{n} JGQ_r^{(3)}(x) &= JGQ_0^{(3)}(x) + JGQ_1^{(3)}(x) + JGQ_2^{(3)}(x) + \sum_{r=3}^{n} JGQ_r^{(3)}(x)\\ &= JGQ_0^{(3)}(x) + JGQ_1^{(3)}(x) + JGQ_2^{(3)}(x)\\ &\quad + (x-1)\sum_{r=3}^{n} JGQ_{r-1}^{(3)}(x) + (x-1)\sum_{r=3}^{n} JGQ_{r-2}^{(3)}(x) + x\sum_{r=3}^{n} JGQ_{r-3}^{(3)}(x)\\ &= JGQ_0^{(3)}(x) + JGQ_1^{(3)}(x) + JGQ_2^{(3)}(x)\\ &\quad + (x-1)\sum_{r=2}^{n-1} JGQ_r^{(3)}(x) + (x-1)\sum_{r=1}^{n-2} JGQ_r^{(3)}(x) + x\sum_{r=0}^{n-3} JGQ_r^{(3)}(x)\\ &= (3x-2)\sum_{r=0}^{n} JGQ_r^{(3)}(x)\\ &\quad + JGQ_2^{(3)}(x) - (x-2)JGQ_1^{(3)}(x) - (2x-3)JGQ_0^{(3)}(x)\\ &\quad - JGQ_{n+2}^{(3)}(x) + (x-2)JGQ_{n+1}^{(3)}(x) - xJGQ_n^{(3)}(x).\end{aligned}$$

Further,

$$\begin{aligned}3(x-1)\sum_{r=0}^{n} JGQ_r^{(3)}(x) &= JGQ_{n+2}^{(3)}(x) - (x-2)JGQ_{n+1}^{(3)}(x) + xJGQ_n^{(3)}(x)\\ &\quad - JGQ_2^{(3)}(x) + (x-2)JGQ_1^{(3)}(x) + (2x-3)JGQ_0^{(3)}(x).\end{aligned}$$

Then, the first result is completed. The second statement of the theorem (2.18) can be proved in a similar manner. □

If we set $x = 2$ in Eq. (2.17), we obtain the summation formula for the Gaussian third-order Jacobsthal quaternions as follow:

$$\sum_{r=o}^{n} JGQ_r^{(3)} = \frac{1}{3}\left(JGQ_{n+2}^{(3)} + 2JGQ_n^{(3)} - 2i - 8k\right).$$

If we set $x = 2$ in Eq. (2.18), we obtain the summation formula for the Gaussian modified third-order Jacobsthal quaternions as follow:

$$\sum_{r=o}^{n} KGQ_r^{(3)} = \frac{1}{3}\left(KGQ_{n+2}^{(3)} + 2KGQ_n^{(3)} - \frac{21}{2}i - 30k\right).$$

Theorem 2.5. *Let $n \geq 2$ be any integer. Then,*

$$KGQ_n^{(3)}(x) = (x-1)JGQ_n^{(3)}(x) + 2(x-1)JGQ_{n-1}^{(3)}(x) + 3xJGQ_{n-2}^{(3)}(x). \tag{2.19}$$

Proof. To prove Eq. (2.19), we use induction on n. Let $n = 2$, we get

$$\begin{aligned}&(x-1)JGQ_2^{(3)}(x) + 2(x-1)JGQ_1^{(3)}(x) + 3xJGQ_0^{(3)}(x)\\ &\quad = (x-1)(x^2-x-1)i + 2(x-1)(x-1)i + 3xi\\ &\qquad + (x-1)(x^4-x^3+x^2-1)k + 2(x-1)(x^3-x^2+x)k + 3x(x^2-x+1)k\\ &\quad = (x^3+x+1)i + (x^4-x^3+x^2-1)k = KGQ_2^{(3)}(x).\end{aligned}$$

Let us assume that $KGQ_{m}^{(3)}(x)=(x-1)JGQ_{m}^{(3)}(x)+2(x-1)JGQ_{m-1}^{(3)}(x)+3xJGQ_{m-2}^{(3)}(x)$ is true for all values m less than or equal $n\geq 2$. Then, we have

$$\begin{aligned}
KGQ_{m+1}^{(3)}(x) &= (x-1)KGQ_{m}^{(3)}(x)+(x-1)KGQ_{m-1}^{(3)}(x)+xKGQ_{m-2}^{(3)}(x)\\
&= (x-1)\left[(x-1)JGQ_{m}^{(3)}(x)+2(x-1)JGQ_{m-1}^{(3)}(x)+3xJGQ_{m-2}^{(3)}(x)\right]\\
&\quad +(x-1)\left[(x-1)JGQ_{m-1}^{(3)}(x)+2(x-1)JGQ_{m-2}^{(3)}(x)+3xJGQ_{m-3}^{(3)}(x)\right]\\
&\quad +x\left[(x-1)JGQ_{m-2}^{(3)}(x)+2(x-1)JGQ_{m-3}^{(3)}(x)+3xJGQ_{m-4}^{(3)}(x)\right]\\
&= (x-1)\left[(x-1)JGQ_{m}^{(3)}(x)+(x-1)JGQ_{m-1}^{(3)}(x)+xJGQ_{m-2}^{(3)}(x)\right]\\
&\quad +2(x-1)\left[(x-1)JGQ_{m-1}^{(3)}(x)+(x-1)JGQ_{m-2}^{(3)}(x)+xJGQ_{m-3}^{(3)}(x)\right]\\
&\quad +3x\left[(x-1)JGQ_{m-2}^{(3)}(x)+(x-1)JGQ_{m-3}^{(3)}(x)+xJGQ_{m-4}^{(3)}(x)\right]\\
&= (x-1)JGQ_{m+1}^{(3)}(x)+2(x-1)JGQ_{m}^{(3)}(x)+3xJGQ_{m-1}^{(3)}(x),
\end{aligned}$$

as desired. □

We now define the matrices $\mathcal{J}$ and $\mathcal{Q}(J)$ as follows:

$$\mathcal{J}=\left[\begin{array}{ccc} x-1 & x-1 & x\\ 1 & 0 & 0\\ 0 & 1 & 0\end{array}\right]$$

and

$$\mathcal{Q}(J)=\left[\begin{array}{ccc} JGQ_{3}^{(3)} & JGQ_{4}^{(3)}-(x-1)JGQ_{3}^{(3)} & xJGQ_{2}^{(3)}\\ JGQ_{2}^{(3)} & JGQ_{3}^{(3)}-(x-1)JGQ_{2}^{(3)} & xJGQ_{1}^{(3)}\\ JGQ_{1}^{(3)} & JGQ_{2}^{(3)}-(x-1)JGQ_{1}^{(3)} & xJGQ_{0}^{(3)}\end{array}\right]$$

Theorem 2.6. *For $n\geq 2$, we have*

$$\mathcal{Q}(J)\mathcal{J}^{n-2}=\left[\begin{array}{ccc} JGQ_{n+1}^{(3)} & JGQ_{n+2}^{(3)}-(x-1)JGQ_{n+1}^{(3)} & xJGQ_{n}^{(3)}\\ JGQ_{n}^{(3)} & JGQ_{n+1}^{(3)}-(x-1)JGQ_{n}^{(3)} & xJGQ_{n-1}^{(3)}\\ JGQ_{n-1}^{(3)} & JGQ_{n}^{(3)}-(x-1)JGQ_{n-1}^{(3)} & xJGQ_{n-2}^{(3)}\end{array}\right].$$

Proof. The result can be obtained easily using the mathematical induction on n. □

3. Conclusions and future studies

In this paper, we study the Gaussian third-order Jacobsthal and modified third-order Jacobsthal quaternion polynomials. We give some results including recurrence relations, Binet's formulas, generating functions and summation formulas for these complex quaternion polynomials. It must be noted that for $x=2$, the results for the Gaussian third-order Jacobsthal quaternion polynomials and Gaussian modified third-order Jacobsthal quaternion polynomials given in this study correspond to the Gaussian third-order Jacobsthal quaternions and Gaussian modified third-order Jacobsthal quaternions, respectively.

By applying these theoretical results to Gaussian third-order Jacobsthal numbers and Gaussian third-order Jacobsthal polynomials, we provided illustrative examples that not only validated the accuracy of our findings but also reinforced the broader applicability of the derived results. Suggestions for future research:

(1) Application to other polynomial and number sequences: Future studies could explore the application of Gaussian quaternion polynomials to other special sequences, such as Gaussian Tribonacci numbers and Gaussian Tribonacci polynomials, to investigate whether similar properties and patterns emerge.
(2) Generalization to higher dimensions: Extending the Gaussian quaternion polynomials framework to dual numbers or hyper-dual numbers could provide deeper insights and broader applications in fields like derivative calculations and multibody kinematics.
(3) Connections with graph theory: The Gaussian quaternion matrices could be studied in the context of graph theory, particularly for analyzing adjacency or Laplacian matrices of structured graphs. A matrix representation for the case of Gaussian Fibonacci quaternion polynomials is presented in [4, Theorem 2.4]

By building on the foundations laid in this work, further research can uncover additional properties and applications of Gaussian third-order Jacobsthal quaternion polynomials and matrices, cementing their role in both theoretical studies and applied mathematics.

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